AGRICULTURAL PRODUCTION FUNCTIONS

Earl O. Heady

Professor of Economics
and C. F. Curtiss Distinguished Professor of Agriculture,
Iowa State University

John L. Dillon

Research Officer,
Commonwealth Scientific and Industrial Research Organization,
Melbourne, Australia

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Preface

The application of formal production function concepts in agricultural research is a relatively recent development. The area of analysis was initiated by W. J. Spillman and other pioneer economists and physical scientists in agriculture. However, economists and physical scientists fairly well "went their own way" and several decades went by with little co-ordinated research in the prediction of physical production and in economic application of the results. However, with increased commercialization of agriculture and greater economic literacy of farm operators there is need for design of physical and biological research so that the results can have greater economic interpretation and application.

This text, considered as one in agricultural science rather than purely in economics, summarizes certain concepts and methods relating to the prediction and use of agricultural production functions. It also reports, in summary form, the results from selected production function studies. Emphasis is on concepts, principles, and methodological results. Practical application is considered to be a second step in communicating principles and prediction for farmer use. However, a few illustrative examples are included to indicate how complex data and concepts can be interpreted and presented for practical use.

The studies reported are largely those resulting from co-operative research efforts at Iowa State University over the past decade. These studies include an important group of products and are drawn together in this monograph for use of other scientists in economic and biological phases of agriculture. Numerous studies represent pioneering efforts in predicting production surfaces and in adapting them for economic interpretations. The designs explained are not suggested as optimum for the purposes, but are simply those which appeared to be appropriate at the time a particular experiment was initiated or which were consistent with the time and research funds available. Predictions for some of these experiments, if designs such as those summarized in Chapter 5 might have been used, could have been improved. Subsequent research has employed these designs or modifications of them. Functional forms are not presented as those most appropriate under all environmental conditions. Rather, they are used to illustrate the types of relationships and recommendations which result when particular forms of

functions are employed, or serve most efficiently under a given set of conditions.

The studies of empirical production functions relating to crops, livestock, and poultry were possible only because of the scientific interest, knowledge, and professional abilities of Damon V. Catron, John T. Pesek, Stanley Balloun, and Norman L. Jacobson. Along with Gordon C. Ashton, Roger Woodworth, Solomon Bloom, Vaughn C. Speer, John A. Schnittker, William G. Brown, Robert McAlexander, Dean E. McKee, Joseph Stritzel, John P. Doll, Gerald W. Dean, Harold O. Carter, C. C. Culbertson, Owen W. McCarthy and R. P. Nicholson, these persons should be considered as co-authors of this monograph. We have only drawn together the numerous studies conducted by these persons, adding a few chapters of interest to those conducting similar studies and listing a collection of reading pertinent to agricultural production functions. Accordingly, authorship of the persons named above is recognized in the appropriate chapters and we should more properly be considered as editors of this monograph. Too, relative to the array of farm-firm production function estimates presented in the final chapter, we are grateful to Takashi Takayama, K. S. Suryanarayana, W. Darcovich, T. Godsell, G. D. Agrawal, Y. Wang, George Mason, and especially, Lennart Hjelm, Eje Sandqvist, and Yair Mundlak, for supplementary information willingly given.

The authors hope that this text will serve to stimulate more and improved research in agricultural production functions. Too, we hope that it serves as foundation for further co-operative effort among personnel of biological, physical, and economic sciences. The concepts, principles, and quantities presented are those relevant both for greater scientific knowledge and more efficient and practical use of certain agricultural research.

In drawing together the set of studies conducted at Iowa State University, we do not depreciate or overlook those which have been completed or are underway at other research organizations and institutions. Time, space, and publication costs posed restraints which could not be eliminated at the time of preparing this monograph.

Earl O. Heady

John L. Dillon

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Contents

CHAPTER 1

Development of Production Function Studies

THIS BOOK deals with agricultural production functions. Its purpose is to summarize certain concepts, empirical methods, and quantitative researches which relate to or have been derived for farm production functions. It covers both physical production functions based on experiments with crops and livestock and firm production functions based on cross-sectional or time series samples. In particular, it emphasizes research completed in these areas at Iowa State University.

USE OF PRODUCTION FUNCTION CONCEPTS

The production function is a concept in physical and biological science. However, it was largely developed and, until recently, used mainly by economists. Historically, refinements in concepts relating to production functions grew out of economics probably because of the following reasons. (1) The nature of production functions is important in economic development and in determining the extent to which national products can be increased from given resource stocks. (2) The magnitude of production coefficients serve as the base for determining optimum patterns of international or interregional trade. (3) The concept is basic to certain theories in the functional distribution of income. The conditions under which a total output can be imputed to the factors from which it is produced with the product just exhausted depends on the nature of the production function. (4) The production function provides half or one of two general categories of the data needed in determining or specifying the use of resources and the pattern of outputs which maximize firm profits. (5) The algebraic nature of supply functions rests, in large part, upon the nature of the production function.

Research workers in the physical and biological sciences of land-grant colleges and the United States Department of Agriculture have long conducted research providing information on the nature of agricultural production functions. However, historically, this research was designed and conducted somewhat apart from the formal concept of production functions represented by regression equations. More typically, research was designed on the basis of discrete phenomena

wherein two or a few treatments were used to provide point estimates of crop or livestock output resulting from input of factors (materials representing treatments). In some instances, although not designed for these purposes, the data were sufficient for deriving simple regression equations or input-output curves. More frequently, the experimental designs and statistical procedures used have only allowed indication of whether mathematically significant differences exist between the yield or output level of two or three discrete treatments or input levels. From these differences could be computed the relative profitability of the few treatments or inputs. However, it was generally impossible to apply refined economic principles in determining the most profitable level of output and input, or the most profitable combination of inputs for a specified output.

These designs and approaches have proved useful in the past and may continue to do so under certain conditions. In many cases, research workers in biological fields have been concerned only with estimating the output from a specific quantity of new material which serves as an innovation. Here the goal of the research often has been to answer the question: Does the material or resource, used at any level whatsoever, give a response? Much early research on fertilizer fell in this framework. In some cases, the practice or treatment under consideration represents a resource or material of discrete and limitational nature. Artificial insemination is an example. Here there is no important question of "dosage" and a formal production function approach is inappropriate. For other materials used in production, the phenomena under consideration could have been estimated as a continuous function, but there was little need to so represent it. For example, fertilizer rates recommended to farmers, based on a few point estimates, have not always been as high as those which would maximize profits in the classical sense of a farm operating with unlimited capital under static conditions. However, because farmers operate in a decision-making framework of uncertainty and limited funds, they often have failed to use fertilizer inputs even as large as those recommended on the basis of trials which include only a few discrete treatment levels. A final reason might be given to justify experiments designed to give point estimates of yield or output from a few discrete input levels: the results provide points similar to the "straight line" segments assumed in linear programming. The optima selected within this framework of assumed physical relationships always fall "on the corners," represented by the point estimates. However, refinements allowing optima to fall between these "corners" may not be important where price uncertainty is so great that *ex poste* accuracy in decisions can never be attained.

There are many biological and physical areas, however, where continuous relationships are involved and the data lend themselves to formal production function analysis. Also, in many of these areas, recommendations to farmers could be made with greater economic meaning if the experimental design and statistical analysis were of a form to allow

prediction of the production functions involved. In the past, these procedures were not often used for several reasons. (1) The scientists conducting the research often have used criteria other than economics in interpreting their findings and in making recommendations to farmers. In the past, for example, the criteria used often have been rations which gave (a) the largest gain per pound of feed, (b) the greatest output per cow or hen, or (c) the fastest daily gain. The most profitable output or resource input is seldom identical with these maxima or minima. However, even where the objective is prediction of a physical maximum or minimum, the exact quantity can be estimated more accurately where the data, if of appropriate nature, are used to estimate the regression equations representing the production functions involved. Derivatives then can be computed and equated to zero, with the appropriate magnitude of input then derived. (2) The statistical methods serving as a guide for research were based on early biological procedures which supposed the data to be discrete phenomena most appropriate for point estimates. Early texts on experimental design and statistical analysis emphasized these methods as appropriate for experiments in biological sciences. (Early emphasis on continuous relationships and regression analysis, as characteristic of physical data representative of production functions, was made especially by econometricians.) (3) Many physical scientists have not been acquainted with production function concepts and the economic principles which define profit maximization or cost minimization. Since they were particularly concerned with conditions of profit maximization and competing economic alternatives within the farm or firm, economists have been concerned with marginal products, marginal rates of substitution, isoquants, and isoclines; quantities which are derived from continuous production functions.

In the past decade, however, an increasing number of physical and biological scientists in agriculture have become acquainted with production function concepts. They have become interested in interpreting basic relationships of nature accordingly and in using them to make recommendations to farmers. This increased interest and activity partly grows out of scientific advance and is parallel to that in the field of agricultural economics. Generally, new fields of research start with a less formal and precise set of concepts and models. Over the decades, continuous scientific investigation and thought provide refinement to the theories, concepts, and models which serve as the basis of designs in experiments and of principles in decision making. Pioneer agricultural economists, like other agricultural scientists, also were concerned with physical relationships which relate to production functions. Without a well-defined set of concepts, they derived principles termed "factors affecting farm profits." The "factors affecting farm profits" principles implicitly supposed certain conditions in respect to physical production functions. Generally, these early findings in agricultural economics served sufficiently in leading farmers to improve their farming operations. But these less formal principles also have,

with the advent of time and improved knowledge by both farmers and research workers, been replaced by more exact principles of profit maximization; principles which directly require knowledge of the relevant production and price quantities. Given recognition of these principles, both physical scientists and economists will have greater future interest in quantities derived from production functions. While the relationships represented by production functions are physical phenomena, economic principle is involved when recommendations to farmers, or decisions by them, relate to the quantity or mix of resources and products to be used and produced respectively. Accordingly, more cooperative research and educational activities can be expected among economists, agronomists, engineers, and animal scientists.

NEED FOR PRODUCTION FUNCTION RESEARCH

This text emphasizes production function estimates of the two following types: (a) biological functions derived from experiments where the plant is fixed at the magnitude of an animal or acre and (b) farm production functions derived from samples where plant size, as measured by "fixity" of a particular resource, is either fixed or variable. Perhaps the most appropriate use of biological functions is that of guiding farmers in their individual decisions. However, these same data can be of extreme use for certain purposes of policy and economic development. For example, a nation such as India may have limited facilities for producing or purchasing fertilizer materials. With a given amount of fertilizer for annual allocation to agriculture, how much should be distributed to various soil, geographic, and climatic areas? If the goal is to maximize the food product available from limited fertilizer resources, production functions derived for major regions and crops, with an indication of marginal coefficients, can provide a basis or guide for attaining the goal. Similarly, the same type of information can be used during wartime or during any other emergency period when material shortages exist. On the other hand, farm production functions of the type discussed in this text probably best serve for the diagnostic purposes pointed out later. In this vein, they can provide general guidance for farmers' decisions, credit policy formulation, readjustment of agricultural regions, etc. However, aside from studies based on highly refined samples, such as some mentioned in later chapters, they seldom can be used to indicate "exact equilibrium use" of resources by an individual farmer.

With the passage of time, a greater knowledge level of farmers, and an increased commercialization of agriculture, there is increased need for experimental designs and research in biological fields which lend themselves to estimation of production functions. Data can then be better adapted for economic interpretations and recommendations. An increasing number of farmers has knowledge of principles of profit maximization and wish to use fertilizer, feed, and similar resources in

a manner to approach formal economic optima. Not only does a greater proportion of individual farmers have this knowledge, but also highly specialized operations found in vegetable, cotton, grain, and large scale and integrated livestock and poultry farming generally have management personnel who seek out and use the appropriate economic principles and physical data. Under these highly commercial operations, management is not satisfied to stop with use of a quantity of fertilizer or a feed ration which is profitable. They are concerned with the *most* profitable quantity and mix of ingredients. Under large-scale operations, large profits may be sacrificed unless fertilizer is used at levels which give most profitable yields, or unless livestock rations are those which minimize costs for given gains.

The need for data in a form which allows the correct or appropriate economic recommendation is itself a sufficient reason why more biological research might well be based on production function concepts or models. But there also are other reasons why this approach is desirable. If data provided in this form came at great cost to society and farmers, the appropriateness of the experimental designs and statistical procedures implied might well be questioned. The important decision then hinges around this: Is the added gain from improved recommendations and decisions greater than the added costs of the research refinements? In many, perhaps the numerical majority of experiments, adaptations of designs to allow estimates of production functions or continuous input-output relationships have a very low marginal cost. Frequently the marginal cost may be zero. Hence, as long as the experiment is being conducted anyway, it might as well be designed to allow the appropriate representation of the physical relationships involved, and to allow application of appropriate economic principles. Adaptations to allow production function estimates ordinarily require few additional treatments, if the general range of information is to be the same in both cases. The goal of some research is to determine the magnitude of intra-treatment variance. But where this is not necessary, the added cost of adapting an experiment to allow regression estimate of the functional input-output relationship can be at zero cost. For example, a fertilizer experiment with three treatments replicated three times might be converted to one with nine different treatments with each representing a different fertilization level. The latter approach would not allow computation of error terms represented by variance within individual treatments, but it would allow estimate of the standard error of the regression coefficient and other probability criteria which are appropriate for choice and decision making.

Where design of experiments to allow production function analysis may result in slightly higher total costs in conducting experiments, the cost per unit of scientific information obtained will generally be lower. Formal estimation of production functions allows derivation of marginal products, physical maxima or minima, isoquants, isoclines, marginal rates of substitution, and other quantities which cannot be computed from experiments which imply discrete data and which are designed

only to provide point estimates. From this standpoint, production function analysis has scientific as well as practical importance. Wherever biological or physical relationships of nature are of continuous functional form it would appear that the interest of even pure science would be that of estimating them in this manner. With a large annual input of funds already devoted to biological or physical research in land-grant colleges, the United States Department of Agriculture, and other research organizations, it would seem desirable that more of this be diverted to alternative but more complete methods of estimating the basic relationships involved.

The amount of resources devoted to estimating farm and industry production functions is much smaller than that devoted to technical functions. The method is only one of many for providing knowledge of agricultural structure and the foundation on which choice and decision can be based. It is not a substitute for budgeting, programming, or planning procedures which provide more broader estimates of optimum resource use within a farm. It has important limitations as an empirical procedure for accomplishing these ends. Accordingly, it will never claim the absolute amount or proportion of research resources devoted to estimating purely physical quantities and relationships. Yet it does have some uses in assessing resource productivity in agriculture and for decision making by individual farmers and policy administrators.

INTERDISCIPLINARY CO-OPERATION IN PRODUCTION FUNCTION ESTIMATES

As mentioned previously, production functions derived for fertilization, irrigation, livestock feeding, and similar agricultural processes essentially are problems in technical sciences. However, most of the formal production function research completed to date has resulted from co-operative effort among scientists in physical and economic fields. This has held true perhaps largely because the technological models involved have had greatest conceptual use and refinement in economic analysis and because there has been some need for technical data better adapted for economic interpretation and use. In the future it is entirely likely that physical scientists will independently conduct most of these research projects. However, since much of the same data will have greater applicability in economic recommendations and choice, continued co-operative effort between scientists in physical and economic fields appears desirable. Accordingly, we include some notes relating to success of co-operative endeavors.

Biological Functions

Co-operative work on production functions among physical scientists and economists at Iowa State University has been more or less

spontaneous. Also, it has mostly been on an informal basis between individuals, rather than on the basis of detailed and formal interdepartmental agreements. In most cases, the work has been done entirely outside the realm of any type of joint project; the activity, research interest, and mutual trust of the individuals serving alone as the "joint ingredient." An important element of the success has hinged on the ability and interests of the physical scientists involved.

The first requirement for successful co-operative research is an understanding by the physical scientist of the analytical models which thus far have been emphasized in economics. The physical scientist will be more enthusiastic over use of these models as a basis for his experiments if he actually understands them. Although graduate study is a help, it is not necessary that the physical scientist have graduate courses in economics. The relevant number of concepts is small and can be acquired in a few informal seminars. Informal sessions with two or four people meeting four to six times often are more productive than formal seminars with a room full of people where much time is spent informing those on the "fringe of interest" who will not be engaged in the research. Insufficient time may be devoted to the interests of the scientist who will actually initiate an experiment.

Second, the persons involved need to be genuinely interested in the investigation. Successful co-operative research seldom comes about by administrative edict alone wherein staff members are simply informed that a joint project is to be put into effect. Informal arrangement of projects at Iowa State University has been possible because all parties have had a genuine interest in their work. Generally, persons in relevant departments who promise to have an interest in the approach get together and go over the concepts and possible designs. Then, if all see sufficient mutual interest in the approach, they get in contact with the proper administrative officials.

A statement by administrators of the need for the research may facilitate co-operative effort, but will not assure highly productive co-operative research. The real stimulus for scientific integration must come from the individual research workers. They must see the need for the prospective product; they must be willing to spend several hours in informal seminars, prior to the actual initiation of the empirical work, in learning and understanding each other's concepts, models, and laws. The economist has as much to learn about the biological phenomena and the experimental difficulties as the physical scientist has to learn about the economic concepts. Frequently the agronomist or animal scientist already has a grasp of the basic relationships involved, but has not formalized them into the theoretical skeleton used by the economist.

Third, the people involved must not be of the "lone-wolf" type. Co-operative research requires that some people "run in packs." We do, of course, need lone-wolf research. Most important fundamental discoveries come from the minds and imaginations of individuals. Seldom can two individuals be put together and give birth to a joint idea.

Concepts are usually formed in each individual mind. Most discoveries in fundamental science will continue to be born in the minds of individuals. However, in the case of applied research relating to eventual farm recommendation, the manager generally makes his choice of practices or resource use patterns from more than one field of science. Problems or managerial decisions of any magnitude always involve phenomena from two or more fields. Applied research needs to be planned accordingly.

A fourth requirement is scientific objectivity. One scientist must not have a special interest in one kind of finding. Results, whether negative or positive to the cause of one scientist, should be accepted objectively by both. If each wants to prove his predetermined notions, he may not be willing to accept outcomes in terms of probabilities.

A fifth requirement is to establish a satisfactory division of labor. In some cases the largest input of time and experimental funds is in conducting the experiment. In other cases, the analysis of the data is more time-consuming. Where the largest burden is in conducting the experiment, funds and personnel should be provided accordingly. The economist may need to help provide funds or personnel for use by the project leader from the other department. Generally, a division of labor between conducting the experiment and processing of the data, with joint effort on interpretation and presentation of results, may be satisfactory.

A final requirement for successful co-operative research is that the objectives of the two research workers be sufficiently complementary to allow for an integrated approach. Otherwise a basis does not exist for a co-operative project. Interests, however, do not need to be identical. An animal scientist might have prior interest in digestibility of feed materials fed in different proportions, while the economist has prior interest in estimating marginal productivities of feeds. In a case such as this, interests may be sufficiently complementary that the same design and experimental layout can accomplish both ends. In other cases, objectives of research may be partly complementary and partly competitive and the research workers will need to determine which of the competitive phases will be given priority.

Farm Functions

The discussion above has centered on physical production functions. Similar considerations of co-operative effort are not required for farm production functions since they generally are conducted by economists alone. However, there are important cases where the economist might best seek out the aid and co-operation of the physical scientist even for these types of studies. In a physical sense, what input categories can be aggregated? Over what range of soil types should the sample extend if a single production function, rather than a hybrid of several different production functions, is to be estimated? These physical questions are,

in fact, some of the more important ones which need to be answered for efficient estimates of farm production function. In the future, perhaps, as physical scientists assume more largely the role of estimating physical production functions, the need for economists to request the aid of agronomists, engineers, and animal scientists will be greater than the opposite.

HISTORIC SUMMARY

The amount of resources devoted to predicting production functions has increased greatly in the last decade. The concept is no longer simply one for the classroom, with the quantities themselves being unknown. Great progress has been made in estimating productivity coefficients for both physical and farm units. And while progress in putting these into practical use may have lagged somewhat behind the rate of derivation, they are coming into increased use for farmer decisions and economic policy projections. Still it should not be implied that concepts of production function forms and derivation of coefficients are only of recent origin. Some of the early articles and empirical studies on production functions had lasting effects on subsequent thought and research.

A summary review of historic developments in production function estimates follows. No attempt has been made to review the details of all studies and statements which, over time, have had direct or indirect bearing on subsequent empirical analysis. Those reviewed are mainly the ones which fall in a classical category, either in respect to the algebraic forms which they propose or the empirical procedures implied. Studies reviewed are mainly those in which the functional or algebraic nature of production relationships has been estimated or the magnitude of coefficients has been interpreted. This procedure, used because of space limitations, excludes mention of a large number of studies in which certain related quantities, ordinarily using different empirical procedures, have been provided in agricultural experiments, farm management and industry studies. The period reviewed ends approximately at the period of World War II.

The review is not limited to functions derived for agricultural production. Review is made of the early industry studies by Douglas and associates because these gave impetus to estimation of farm functions. They also gave rise to questions and discussions of methodology which are pertinent in estimation of agricultural production functions. Hence, the more important methodological discussions also are summarized.

Physical Production Functions

We first review studies relating to physical production functions of technical units in agriculture. This procedure is followed because some of these studies predate those made for firms and industries.

Justus Von Liebig's "law of the minimum" was the first attempt to define the fundamental relationship between fertilizer or nutrient inputs and crop yields.[1] Von Liebig stated that a soil containing all nutrients necessary for plant growth except one is barren for all crops for which the lacking nutrient is indispensable. He believed that crop yields were proportional to the amount of nutrients supplied to or provided from the soil, and that when all soil nutrients are present in sufficient supply, addition of one or more would not increase yield. While Von Liebig did not suggest algebraic equations representing his concept of the crop production function, the algebraic form implied is somewhat obvious. Each nutrient would serve as a limitational factor to the others. The production surface would reduce to a simple "knife's edge" with a constant slope to the maximum per acre yield. In general, this type of crop production function has been disproved. However, as data presented later in this test show, most crop production functions probably have a "Von Liebig point" where all nutrients are limitational and addition of any one will not increase yield. This "Von Liebig point" is at the maximum or peak of the production surface where the isoclines converge and the isoquant defining this output reduces to a single point. Baule suggests that Von Liebig's law of the minimum supposed that plants took up nutrients only in a given ratio and that yield would vary directly with the quantity of nutrient available in smallest supply.[2] Bondorff and Plessing interpreted his law as represented by the algebraic form[3]

$$y = ax \tag{1.1}$$

where y is yield response, x is the quantity of the nutrient, and a is the constant or coefficient defining the transformation ratio. Boresch proposed Von Liebig's law to be represented as:[4]

$$y = c + ax \tag{1.2}$$

where y is total yield, c is yield level in absence of applications of x, the nutrient added. Extension of these concepts to more than one nutrient would provide, in the case of equation 1.1, the function $y = ax_1 x_2 x_3$ for three nutrients and yield would be proportional to input of any one fertilizer resource. Yield would be zero in absence of any one. However, this algebraic form would not directly define the yield maximum discussed by Von Liebig.

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[1] Liebig, Justus von. Die grundsätze der agricultur - chemie mit rücksicht auf die in England angestellten untersuchungen. Friedrich Viewig und Sohn. Braunschweig. 1855.

[2] Baule, B. Zu Mitscherlichs gesetz der physiologischen beziechungen. Landw. Jahrb., 51: 363-85.

[3] Bondorff, K. A. Det kvantitative porhold mellem planternes ernaering og stofproduktion. Den Kongelige Veterinaer - og Landbokejskale Aarskrift. 1924. Pp. 293-336; and Plessing, H. C. Udbyttekurver med saerlight henblik pad matematisk formulering of landrugets udbyttelov. Nordisk Jordbrugsfarskning. Vol. 25: 399-424.

[4] Boresch, K. Über ertragsgesetze bei pflanzen. Ergebnisse Biologie, 4: 130-204.

The first attempt to define the algebraic nature of the fertilizer-crop production function was that of Mitscherlich in 1909.[5] He, perhaps, was the first agriculturist to suggest a nonlinear production function relating nutrient input and crop output. With the aid of Baule, a mathematician, he proposed the equation

$$\log A - \log (A - y) = cx \tag{1.3}$$

to explain fertilizer response allowing diminishing marginal productivity. In this equation, A is total yield when the nutrient, x, is not deficient (i.e., A is the maximum yield attainable from addition of x) and c is a proportionality constant, defining the rate at which marginal yields decline. Mitscherlich proposed the coefficient c was a constant for all crops, unaffected by type of crop, climate, or other environmental factors. A debate on these theories was conducted by Briggs, Rippel, and Mitscherlich.[6] One objection to the Mitscherlich equation was that it did not allow negative marginal products, or diminishing total yields. Accordingly, Mitscherlich made an adaptation to allow this condition.[7] This equation can be stated as

$$y = (1 - 10^{-cx})(10^{-kx^2})(10^{c}) \tag{1.4}$$

where k is a "damage factor" due to excessive magnitudes of x.

Working independently and without knowledge of this development, Spillman proposed an exponential yield equation similar to that of Mitscherlich.[8] In main form, Spillman's total yield equation is

$$Y = M - AR^{x} \tag{1.5}$$

where M is the maximum total yield attainable by increasing the nutrient input x, A is a constant defining the maximum response (the sum of marginal yields) attainable from use of x, and R is the coefficient defining the ratio by which marginal productivity of x declines. He developed this equation after examining results from fertilizer experiments on tobacco in North Carolina. In contrast to Mitscherlich, he believed that the constants in the production function varied with environmental conditions. In the above equation, total attainable output can be defined as $M = y_0 + y$ where y_0 is total yield when $x = 0$ and y is the addition to yield corresponding to various nonzero values of x.

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[5] Mitscherlich, R. Das gesetz des minimums und das gesetz des abnehmenden bodenertrages. Landw. Jahrb., 38: 537-52.

[6] Briggs, G. E. Plant yield and the intensity of external factors — Mitscherlich's "Wirkungs gesetz." Ann. Bot. 39: 475-502; and Rippel, A. Zwei experimentelle widerlegungen des Mitscherlich — Bauleschen wirkungsgesetzes der wachstumsfaktoren. Zeits Planzen Bung. und Boden, Series A, 8: 65-76.

[7] Mitscherlich, E. A. Second approximation of the law of action. Zeits. Pflanzen Bung. und Boden, Series A, 12: 373-82.

[8] Spillman, W. J. Application of the law of diminishing returns to some fertilizer and feed data. Jour. Farm Econ., 5: 36-52.

Hence, we can define the output forthcoming in the absence of the variable nutrient as $y_0 = M - A$ while the addition to yield or response is

$$y = A(1 - R^x) \tag{1.6}$$

with the latter defining the magnitude of incremental yields associated with various values of x. Using these components, total yield is $Y = y_0 + y = M - A + A(1 - R^x) = M - AR^x$, the quantity indicated in equation 1.5. The term $1 - R^x$ can be defined as a "percentage sufficiency" quantity. If x is zero, response also will be zero. As x increases in magnitude, the fraction $1 - R^x$, or the percentage represented by it, declines and the response approaches the mathematical limit, A. Hence, the input-output curve is asymptotic to A when measurement is of response to a variable factor and to M when measurement is of response to variable plus output forthcoming from fixed factors.

Baule extended the Mitscherlich equation to include n variables.[9] His equation was based on the "percentage sufficiency" concept for individual nutrients. Based on Baule's work, Spillman developed the equation

$$y = A(1 - R^x)(1 - R^z) \tag{1.7}$$

where y is output from variable inputs, A is the maximum increase from use of variable resource, R is the ratio defining marginal products, with the unit of variables defined to make R = .8, and x and z are quantities of two variable inputs. For this equation, the variable inputs serve as limitational factors at zero magnitude of either, since $1 - R^0 = 0$. Again, the function allows for diminishing marginal productivity of any resource but does not allow for negative marginal products; the resulting surface approaching a height asymptotic to A.

Pfeiffer and Frölich are reported to have used a single-variable polynomial of quadratic form in 1912, in relating total crop yield to supply of nitrogen. Panse used an equation of this type to fit response data for cotton while Sukhatme used it for rice.[10] Briggs suggested the use of the hyperbola function of the general form[11]

$$y = \frac{(x + b)E}{x + b + h} \tag{1.8}$$

where E is a maximum yield, b is the quantity of x initially in the soil, and h is the optimal supply of the input. Somewhat similar hyperbolic-type equations were suggested by Boresch, Balmukand, Baule and

[9] Baule, B. Zu Mitscherlich's gesetz der physiologischen bezichungen. Landw. Jahrb., 51: 363-85.

[10] Panse, V. G., et al. Co-ordinated manurial trials on rainfed cotton in peninsular India. Indian Jour. Agr. Sci., 21: 113-35, and Sukhatme, P. V. Economics of manuring. Indian Jour. Agr. Sci., 11: 325-37.

[11] Briggs, G. E. Plant yield and intensity of external factors — Mitscherlich's "Wirkungs gesetz." Ann. Bot., 39: 475-502.

Bondorff.[12] Thelau is quoted as suggesting an elliptical function to characterize response to fertilizer.[13] Other literature also could be cited, but the above represent the main early concepts of the algebraic nature of crop response to fertilizer.

E. M. Crowther and F. Yates analyzed all 1 year fertilizer experiments conducted in Great Britain between 1900 and 1941 and similar experiments conducted in Europe as a basis for formulation of wartime fertilizer policy.[14] They emphasized that final conclusions on fertilizer response must be based on a series of experiments conducted in different years, on different crops, and under varying farm and soil situations. The function used for certain of the estimates was $y = y_0 + d(1 - 10^{-kx})$, a modification of the Mitscherlich formula, where y_0 is yield without fertilizer, d is the limit in response, x is amount of nutrient applied, and k is a constant value for each nutrient. The average k values determined for the British experiments were 1.1 for nitrogen, .8 for phosphorus, .8 for potash, and .04 for manure. The authors seem to suggest that these k values are constants for different locations and conditions, as confirmed by analysis of additional experiments from Denmark and Sweden.

Soil scientists more often than animal scientists have been concerned with the functional nature of technical production functions. Discussion of production functions by soil scientists predated interests of agricultural economists. However, efforts of animal scientists in estimating production functions have been largely co-operative efforts with economists. A classical livestock production function study based on experimental feeding was published by Jensen *et al.* in 1942.[15] This study dealt with input-output relationships in milk production. The algebraic form used was that developed by Spillman for a single input category. The data came from a large-scale co-operative experiment between several land-grant colleges and the United States Department of Agriculture. The results of the study were used to indicate how feeding rates to maximize profits per cow should be varied from Haecker standards as prices of feed and milk vary. Marginal feed productivity was estimated for cows of both high and low producing capacity at stations with herds of different output levels. This study, made jointly by dairy scientists and agricultural economists, was important in helping to change technical views on the nature of the feed-milk production function. The 367 cows included in the experiments at nine stations received a variety of feeds in the form of grains, silage, hay, and pasture. These feeds were all converted to a total digestible nutrient

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[12] Boresch, K. Liber ertagsgesetz bei pflanzen. Erge. Biol., 4: 130-204; Balkumand, B. H. Studies in crop variation. Indian Jour. Agr. Sci., 18: 602-27; Boule, F. Liber die Weiternicklung. Zeit. Ahen Pflanzen, 96: 173-86; and Bondorff, F. (as reported in Plessing, Udbytte-kurver med saerlight henblik. Nordisk Jord., 25: 399-424).

[13] As quoted by Plessing, *ibid.*

[14] Crowther, E. M. and Yates, F. Fertilizer policy in wartime. The fertilizer requirements of arable crops. Empire Journal of Experimental Agriculture, 9: 77-98.

[15] Jensen, E., et. al. Input-output relationships in milk production. Tech. Bul. 815. USDA, Washington, D. C. 1941.

basis, a procedure completely satisfactory only if all feeds substitute at a constant marginal rate at the transformation ratio used, with the single feed input category measured in pounds of TDN. The authors did not provide the numerical magnitudes of the coefficients in the derived production functions but some of these can be estimated from the tabular and graphic materials presented. Since only grain was varied and cows generally were allowed to consume all the forage they desired, the TDN-milk transformation relationship is probably not a "true" production function in the theoretical sense. Instead, it more nearly expresses the relationship between TDN input and milk output along a stomach limit line such as explained by Heady.[16]

Another production function study in feeding levels also was originally inspired by considerations of farm profit maximization. Atkinson and Klein made a study of optimum marketing weight of hogs relative to feed transformation ratios and price relationships. The input-output or production function relationship derived relates weight per hog to feed intake measured as pounds of concentrates. The basic data are from swine feeding trials although the original experiments were not designed specifically for formal production function analysis. Numerous experiments from five land-grant colleges were aggregated for derivation of an average input-output curve involving a single feed variable. The several grains and protein supplements in the experiments were converted into a concentrate variable on the basis of weight. The single-variable algebraic function used also was of the type developed by Spillman. While the coefficients for the equation expressing hog gain relative to pounds of concentrate intake is not given, an algebraic function relating weight to age of hogs is published. It is

$$W = 588(.0097)^{.9923^{A}} \tag{1.9}$$

where W is hog weight in pounds and A is age in days from birth. The estimates derived in this study also were useful for wartime guides and policies relative to optimum marketing weights for hogs, and, hence, for structuring of administered price differentials for hogs of varying weights.

Another livestock study inspired by wartime food shortages and need for basic data in determining livestock feeding and price policy was conducted by Nelson.[17] This study between animal gain and feed intake for calves, yearlings, and 2 year old beef animals also was based on feeding experiments. Again, the experiments were conducted in a number of years prior to the analysis and were not designed specifically for production function analysis. A single-variable production function was derived for the several classes of cattle with the various grains, protein supplements, and forages converted to a single input

[16] Heady, Earl O. Economics of agricultural production and resource use. Prentice-Hall, Inc. New York. 1952. Pp. 156-57.

[17] Nelson, A. Relationship of feed consumed to food products produced by fattening cattle. Tech. Bul. 900. USDA, Washington, D. C. 1945.

category measured as total digestible nutrients. Estimates were derived for both live and dressed weight of cattle. The live weight regressions indicate diminishing marginal and average productivity of feed from the outset of the feeding period. The dressed weight regressions indicate, because dressing percentage of animals increases with weight, ranges of both increasing and decreasing marginal productivity of feed. Like the previous studies cited for hogs and dairy cattle, productivity coefficients cannot be specified for different feed categories, nor can marginal rates of substitution be specified for different feeds. The single-variable functions derived, adaptations of the Spillman function, were

$$\text{Calves:} \quad w = 1.446 - 1{,}049e^{-.00026f} \tag{1.10}$$

$$\text{Yearlings:} \quad w = 1{,}446 - 805e^{-.00028f} \tag{1.11}$$

$$\text{2 year:} \quad w = 1{,}446 - 610e^{-.00037f} \tag{1.12}$$

where w is live weight in pounds after start of the feeding period, e is the base of the natural system of logarithms, and f is feed intake per animal measured as pounds of total digestible nutrients. These live weight functions were transformed, on the basis of published data on dressing percentages, into quantities representing relationships between dressed weight and feed intake per animal.

Industry Functions

Numerous early general economists, such as Smith, Ricardo, and Malthus, suggested hypotheses about the general nature of the production function for the agricultural industry. Knut Wicksell was perhaps the first to suggest an algebraic form or physical nature for the agricultural production function.[18] He refined the notions of classical economists by stating that increasing returns to labor and capital are possible when fertilizer inputs also are applied to nutrient-short soils. He stated that the agricultural research stations should provide the necessary data for establishing the functions, but that they had not done so at that time. Wicksell indicated that agricultural output was clearly a mathematical function of the quantities of labor, land, and capital used and if these are expressed respectively as a, b, and c, with production per annum indicated as p, the production function can be expressed as

$$p = f(a, b, c) \; . \tag{1.13}$$

[18] Wicksell, Knut. Den "kritiska punkten" i lagen för jordbrukets aftagande produktivitet. Ekonomisk Tidskrift. 1916. Pp. 285-292.

However, he stated that this function is homogeneous of first degree, denoting constant returns to scale. Hence, if input of all factors is doubled, it becomes $f(2a, 2b, 2c) = 2p$, and output also is doubled. He expresses the belief that Rohtlieb,[19] in a previous paper, was also discussing the possibility of constant scale returns for factors increased in fixed proportions. However, Wicksell states, in argument with Rohtlieb, that if a simultaneous increase in the three factors leads to a proportion increase in output, then increase in one such as fertilizer cannot lead to the same kind of increase and, therefore, that classical economists were correct in their notions of diminishing agricultural productivity because they considered land to be fixed in quantity. In his terms, technical change simply gives rise to a new production function $\phi(a, b, c)$, which has constant scale returns for all factors increased by fixed proportion, but diminishing productivity for one factor increased relative to fixed quantities of the others. Here he is discussing the production function of a farm, but states that the production function, $F(A, B, C) = P$, for a nation need not be homogeneous of first degree. Possibilities of labor specialization, for example, in a colony as its population grows and more land is taken up, may allow increasing returns. Wicksell also implies that Rohtlieb's empirical analysis involves comparisons of different production functions, rather than of effects of different input levels for a given production function.

Actually, the equation which has come to be known as the Cobb-Douglas function traces to Wicksell. He stated, in a footnote, the function as

$$p = a^{\alpha} b^{\beta} c^{\gamma} \tag{1.14}$$

and, as in the early attempts of Douglas, said that α, β, and γ should sum to 1.0.

While the first general hypothesis about the algebraic nature of an industry production function was proposed for agriculture, the first empirical attempt was for nonfarm industries. Cobb and Douglas applied a function similar to Wicksell's to data for American manufacturing industries over the period 1899-1922.[20] Evidently, this was the first formal empirical production function fitted to time series data. The function fitted was of the form

$$P' = bL^{k} C^{1-k} \tag{1.15}$$

where P' was the predicted index of manufacturing output over the period, L was the index of employment in manufacturing industries, and C was the index of fixed capital in industry. The function derived from the time series data was

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[19] Rohtlieb, C. E. Om gränsen för jordbrukets intensifiering. Ekonomisk Tidskrift. 1916:189.

[20] Cobb, Charles W. and Douglas, Paul H. A theory of production. American Economic Review. Vol. 18, (March, supplement): 139-65. 1928.

(1.16) $$P' = 1.01L^{.75}C^{.25}.$$

Cobb and Douglas computed the marginal productivities of capital and labor which are respectively $.25PC^{-1}$ and $.75PL^{-1}$. They used these quantities to impute the shares of the actual total product to labor and capital during the period studied. On the basis of these quantities, they imputed the total product to capital and labor in the proportions .25P and .75P, respectively, where P is the actual index of production in any one year (as compared to P′ which is the index of production estimated from the derived function). The annual quantities so imputed from the empirical production function then were compared with estimated shares of the total manufacturing product actually distributed to labor and capital over the period. Cobb and Douglas also explored the possibilities that $k = .67$ and $1 - k = .33$ and decided that the derived values $k = .75$ and $1 - k = .25$ described the actual process of production in manufacturing in a fairly accurate manner. The value of R, the multiple correlation coefficient, for the function with these elasticities was .97.

They selected this function with its restraint that the sums of elasticities or regression coefficients should total 1.0 because they were interested in imputing the total product back to the two factor categories. With a sum of elasticities either greater or smaller than 1.0, the total product would have been respectively less or greater than the total amount imputed to the resources. Cobb and Douglas did not claim to have solved the laws of production or to have employed the most appropriate data. They stated that the form of function used was only one alternative and that formulas should be devised which would not require "constant relative contributions of each factor to total product but would allow for variations from year to year." Douglas indicated that one of his co-workers suggested a modification of the function as early as 1926.[21] He gave credit to Wilcox for suggesting the function

(1.17) $$P' = bR^{1-k-h}L^kC^h \quad \text{where} \quad R = (L^2 + C^2)^{.5}$$

and the sum of k and h is not forced to unity. In applying this to the data used for equation 1.16, Douglas obtained the values $b = 1.06$, $1 - k - h = -.146$, $k = .788$, and $h = .358$. However, Douglas and his associates did not use this specific function for later time series analyses.

Douglas and his associates used time series data to estimate several industry production functions of the form of equation 1.15 where the sum of the exponents are forced to unity. In all these cases, they were interested in the condition of equation 1.18 where the marginal products of labor and capital, $\frac{\delta P}{\delta L}$ and $\frac{\delta P}{\delta C}$ respectively,

(1.18) $$P = \frac{\delta P}{\delta L} \cdot L + \frac{\delta P}{\delta C} \cdot C$$

[21] Douglas, Paul H. The theory of wages. Macmillan Co., New York. 1934. P. 152.

multiplied by the quantities of labor and capital, L and C, would allow a sum equal to the total product, P. The following values were obtained for other sets of time series data.[22]

Country and Period	b	k	1-k
Massachusetts (1890-1926)	1.007	.74	.26
Victoria (1902-1929)	-	.71	.29
New South Wales (1901-1927)	1.018	.65	.35
New Zealand (1915-35)	-	.52	.48

In an article published in 1936, Edelberg concerned himself with a somewhat similar type of function in relation to time series data for industry.[23] The function he discussed was

$$P = cs^{\alpha}X^{\beta}L^{\gamma} \tag{1.19}$$

where L and X were respectively the rates at which land and labor are applied in production, s was the time or period of production and P was the rate of production. Capital letters were used to denote quantities for the economy, although he indicated that firm conditions might be represented by the same algebraic form and the same variables denoted by p, s, and 1. He obtained the following empirical functions over the period 1850-1910 for Great Britain and the United States where, like Cobb and Douglas, he assumed the function to have constant returns to scale in respect

$$\text{U.K.} \quad P = cs^{.33}L^{.07}X^{.93} \tag{1.20}$$

$$\text{U.S.} \quad P = cs^{.32}L^{.09}X^{.91} \tag{1.21}$$

to L and X but not in respect to s. He indicated that the original Cobb-Douglas function might be criticized for not including the time element and "circulating" capital. Edelberg was interested mainly in deriving labor demand functions.

Douglas and associates then relaxed the restraint that the sum of the elasticities in the production function should sum to unity and employed, at the suggestion of Durand,[24] the function

$$P = bL^{k}C^{j} \tag{1.22}$$

[22] Gunn, Grace T. and Douglas, Paul H. The production function for American manufacturing in 1919. American Economic Review, 31 (March): 67-80. 1941; Gunn, Grace T. and Douglas, Paul H. The production function for Australian manufacturing. Quarterly Journal of Economics, 56: 108-29. 1941; and Douglas, Paul H. The theory of wages. Macmillan Co., New York. 1934. Ch. 6-7.

[23] Edelberg, Victor. An econometric model of production and distribution. Econometrica, 4: 210-25. 1936.

[24] Durand, David. Some thoughts on marginal productivity with special reference to Professor Douglas' analysis. Journal of Political Economy, 45: 740-58. 1937.

where k and j could take any value. The resulting power function, which is linear in logs, has commonly come to be known as the *Cobb-Douglas function.* It has been applied in numerous production function studies for technical units and firms of agriculture. Douglas and associates applied it mainly to cross-sectional data where observations on indices of product, labor input, and capital investment were obtained from individual manufacturing industries. Statistics for some of these cross-sectional studies are as follows:[25]

Country, region, and year	Number of industries for observations	b	k	j
United States (1904)	336	2.08	.65	.31
United States (1909)	90	.90	.74	.32
United States (1914)	340	1.97	.61	.36
United States (1919)	556	2.39	.76	.25
Victoria (1910-11)	34	--	.74	.25
Victoria (1923-24)	38	--	.62	.30
Victoria (1927-28)	35	--	.59	.27
New South Wales (1933-34)	125	--	.65	.34
Australia (1934-35)	138	--	.64	.36

Bronfenbrenner and Douglas also computed separate cross-sectional production functions for several individual industries based on 1909 data.[26] These included:

Industry	Number of observations	b	k	j
Clothing-textiles	16	2.043	.978	-.070
Foods and beverages	12	.365	.715	.350
Metals and machinery	22	1.007	.714	.256

Verhulst estimated production functions from cross-sectional data for twenty-five French gas firms in the fourth quarter of 1945.[27] His purpose was to determine the nature of the production function of a gas firm in "the neighborhood of the point of equilbrium." His input variables included the net prime costs, x, of each firm and the index of capital charges of the same firms. His output, ρ, was represented by the quantity of gas sold. In estimating the production function in this

[25] See Gunn, Grace T. and Douglas, Paul H. The production function for American manufacturing in 1919. American Economic Review, 31 (March): 67-80. 1941; Gunn, Grace T. and Douglas, Paul H. The production function for Australian manufacturing. Quarterly Journal of Economics, 56: 108-29. 1941; Douglas, Paul H. The theory of wages. Macmillan Co., New York. 1934. Ch. 5-9; Daly, Patricia, Olson, Ernest, and Douglas, Paul H. The production function for manufacturing in the United States, 1904. Journal of Political Economy, 51: 61-65. 1943.

[26] Bronfenbrenner, M. and Douglas, Paul H. Cross-section studies in the Cobb-Douglas function. Journal of Political Economy, 47: 761-85. 1939.

[27] Verhulst, M. J. J. The pure theory of production applied to the French gas industry. Econometrica, 16: 295-308. 1948.

"neighborhood," the method of simultaneous equations outlined by Marschak and Andrews where the production function was the modified Cobb-Douglas type:[28]

$$\rho = ax^{\alpha'}y^{\alpha''} \tag{1.23}$$

His interest centered on methodological problems and no particular attempt was made to interpret the quantitative elasticities derived from the sample.

Criticisms of Industry Studies

Numerous criticisms of the functions applied by Douglas and his associates appeared in the literature after parameters were estimated from industry data on both a time series and cross-sectional basis. Some of these referred to inaccuracies in the available data but others were in terms of comparison of these results with those theorized for a firm. It has been pointed out that the resulting empirical functions, based on time series data wherein observations are drawn from individual years as aggregates for all industry or on cross-sectional data with observations representing aggregates of different industries, apply only to groups of industries and not to individual firms.

Durand initially criticized the form of the Cobb-Douglas function which required a sum of individual elasticities equal to unity.[29] He expressed doubt that the enterprise production function is homogeneous of first degree and indicated that the validity of this condition was yet to be established. If this condition actually held true, equilibrium output and prices could not be attained under pure competition. As indicated earlier, Douglas and associates then changed the function so that $k + j \neq 1.0$, to allow increasing, constant, or decreasing returns to scale. Durand also pointed out that the function (1) is not of the type ordinarily discussed in economic theory since it applies to groups of industries rather than to individual firms, and (2) is expressed in terms of undifferentiated labor and capital instead of specific resources actually employed. Durand also pointed out that methods of measuring depreciation and the fact that management inputs were not quantified would likely result in k and $k - 1$ values which did not bear the correct proportion to each other.

Bronfenbrenner and Douglas summarized other criticisms of the Cobb-Douglas functions as follows: Labor and capital are not on the same footing because the first is measured by quantity actually used in production while the second is measured by quantity available for production. The function cannot be statistically fitted in three dimensions, P, L, and C, because (1) the data are not sufficiently accurate, (2) the

[28] Marschak, J. and Andrews, W. H. Random simultaneous equations and the theory of production. Econometrica, 12: 143-205. 1944.

[29] Durand, *op. cit.*

functions were fitted to index numbers rather than to actual quantities of input and output, (3) the function as actually derived was only in the two directions, P/C and L/C, and (4), the minimization of deviations might, given the type of industry data, be in the direction of the L and C axes as appropriately as in the direction of the P axis. The relationships actually existing between P, L, and C in the time series data are simply functions of time as it relates to technological change, population growth, and economic development. Thus the original Cobb-Douglas function might not be a production function at all, but a spurious correlation of these three series with time.[30]

Bronfenbrenner and Douglas examined these criticisms, elaborating on some and attempting to disprove others. They suggested that the term "production function" need not apply to a particular aggregate of activity such as a single plant, but might be applied to an entire industry or other strata and aggregates of production. The criticism of capital measured in available quantity was dismissed as unimportant in years of prosperity and full employment. The limitation of observations measured in index numbers was eliminated in the cross-sectional studies since actual values of P, L, and C were used. The limitation of a function homogeneous of first degree was eliminated through modification of the original equation. The problem of multicollinearity or spurious correlations revolving around time were eliminated in use of cross-sectional data.

Menderhausen discussed Douglas' production function results largely from a statistical standpoint.[31] He found a high degree of multicollinearity between the variables in the time series data. He presented a three-dimensional figure to show that the product quantities, P, lie practically in a straight line over the input plane rather than forming a surface dispersed over the resource plane. Menderhausen illustrated that Douglas' three variates (log P, log L, and log C) can be represented as linear functions of time (t) as

$$\log P = \alpha t \qquad (\alpha = .0156) \tag{1.24}$$

$$\log L = \beta t \qquad (\beta = .0112) \tag{1.25}$$

$$\log C = \gamma t \qquad (\gamma = .0281) \tag{1.26}$$

and the differences in the trend-slope lines for P, L, and C are

$$\log P - \log C = \alpha - \gamma \tag{1.27}$$

$$\log L - \log C = \beta - \gamma \tag{1.28}$$

[30] Bronfenbrenner, M. and Douglas, Paul H. Cross-section studies in the Cobb-Douglas function. Journal of Political Economy, 47: 761-85. 1939.

[31] Menderhausen, Horst. On the significance of Professor Douglas' production function. Econometrica, 6: 143-53. 1938.

with the ratio $\frac{\alpha - \gamma}{\beta - \gamma}$ being equal to k of the Douglas time series function where k is simply a ratio of the two trend-slope differences. He showed that this ratio did, in fact, turn out to be .741, a quantity similar to the k value of .75 found by Douglas. Hence, he concludes that the k value in the Cobb-Douglas original function is simply an expression for the trend in technical development. He also mentioned that the measure of capital, C, over the time period analyzed represented the fixed capital present rather than that actually used, particularly in years of underemployment in industry. Tinbergen pointed out that Douglas' results would be perfectly acceptable if it could be assumed on a priori economic grounds that $k = 1 - j$ where k and j are respectively the derived elasticities for labor and capital.[32] He states that the problem is then reduced to one of simple correlation. Both Menderhausen and Tinbergen emphasized the time series estimates of Douglas gave rise to measurement difficulties. Menderhausen also emphasized that it was as relevant to minimize the deviations from the direction of the L and C axes, as well as from the P direction, and that the results were so different that no real credence could be given to the latter.

Several other economists have emphasized the difference between inter-firm, intra-firm, and inter-industry aspects of production functions such as those derived by Douglas. These studies were based on aggregate observations for groups of industries, rather than upon observations for individual firms as has been the case in more recent studies made for agriculture. Reder indicated that these industry functions differ conceptually in three ways from the production functions of economic theory: (1) In theory, the physical production function shows the functional relationship between the input quantities and the output of a firm. The functions derived by Douglas are the loci of input-output quantities of all firms used in the particular study. (2) The theoretical function is in terms of physical quantities while the functions fitted to industry data represent the value added in manufacturing to the physical inputs of labor and the value of plant and equipment owned. (3) The marginal value productivity of a factor is the partial derivative of total product in respect to the particular factor multiplied times the marginal revenue for this output.[33] The marginal value productivities derived from the Douglas functions can be derived directly, without multiplying by a corresponding marginal revenue times the marginal physical productivity, because of the method of measurement. According to Reder, the latter should be called inter-firm marginal productivities whereas the theoretical concepts refer to intra-firm productivities. Bronfenbrenner, in discussing the inter-firm aspects of these functions, attempts to show how results from inter-industry observations should be the same as those derived from inter-firm and

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[32] Tinbergen, J. Econometrics. Blakeston Co., New York. 1951. P. 123.

[33] Reder, M. W. An alternative interpretation of the Cobb-Douglas production function. Econometrica, 11: 259-64. 1943.

intra-firm observations in competitive industries.[34] He illustrated his propositions for competitive industries by drawing input-output curves representing individual firm production functions and associating these with the type of relationship derived in the classical Cobb-Douglas cross-sectional production function. He points out that, under competition, the slope of the production function under equilibrium should be the same between industries and firms. Smith pointed out some of the conditions necessary for fitting meaningful statistical production functions.[35] Some of his observations are applicable to functions derived from firm samples, as well as to industry observations or measurements. The optimum set of data would involve experiments with individual firms where labor and capital are combined in numerous proportions. With inability to conduct experiments of type, he indicated that the conditions of perfect competition and static equilibrium would serve as a substitute. However, Smith doubted that this assumption is relevant for industry observations of the type employed by Douglas and associates. If perfect competition did prevail, the production function of the firm would be of the same degree whether input-output relationships are expressed in physical or value terms and would differ only by a constant product representing price relationships. Under perfect competition and perfect factor markets, Smith stated, a long-run statistical function from cross-sectional data should allow estimates of the conventional intra-firm marginal value productivities. Under these conditions, the average and marginal value productivities should equal the factor prices paid by each firm. But under the conditions of imperfect markets expected in industry studies, Smith concluded that the degree of the value production function would be less than the degree of the physical production function. Smith points out several other ways in which the function derived from industry observations may differ from the theoretical physical production function. (1) Particular problems arise in measurement of capital input. Theoretically, the relevant input is annual capital use but capital investment ordinarily is used. If annual capital input always bore a constant ratio to investment in capital goods, the elasticities should be the same. (2) Firms in a cross-section study may employ different techniques, particularly due to fixed plants inherited from the past, and the long-run production functions so derived may represent "mongrels" or "hybrids." Some firm adjustments to price and other conditions are of long-run nature while some are of short-run nature and the differences among observations of this type cannot be separated conveniently. If short-run or long-run production functions were desired, the degree of adaptation in labor and capital would need to be limited in terms of concepts of the relevant period of time and change in plant structure. (3) The observations result from management decisions rather than experimentation or physical manipulation of resource mixes, to measure their effect on output.

[34] Bronfenbrenner, M. Production functions: Cobb-Douglas interfirm, intrafirm. Econometrica, 12: 35-40. 1944.

[35] Smith, Victor E. The statistical production function. Econometrica, 59: 543-62. 1945.

Accordingly, observations are available for only a small segment of the production surface and errors may arise due to changes in the function itself, to delayed or incomplete adjustments to price changes, and to errors on the part of management.

May stated conditions necessary for estimating a firm production function of technical or physical nature. The function must relate inputs and outputs in an individual productive unit. The variables of input and output must be scalars of definite units associated with observable and measurable quantities. There must be a scalar representing the input quantity of each factor, aggregates being excluded except in the trivial case where components of the aggregate are single valued functions of the aggregate. For fixed capital inputs there should be two variables, with one representing the stock of capital and the other its extent of use. The variables should be in physical rather than value units. Factors other than inputs should be excluded as variables and held fixed and include such considerations as time distribution of inputs over the calendar period, morale factors, intensity of effort, techniques, geographical environment, and others. He suggests that production functions might well be estimated for subprocesses of the over-all production process.[36]

Marschak and Andrews in 1944 discussed further the estimational procedures involved in deriving production functions such as those estimated by Douglas.[37] In this connection they proposed the possibility of using systems of equations in attempting to predict production functions, revenue functions, and factor demand functions. They outlined possible procedures in estimating an entire set of relevant equations, including the production function, for individual firms when data are drawn from an inter-firm sample where differences exist in production functions.

Contribution of Cobb-Douglas Functions

The discussion immediately above refers mainly to the types of industry functions fitted to time series or cross-sectional data by Douglas and his associates. In general, the data used in these types of studies have not been similar to those used for agricultural production functions. The latter have been based almost entirely on physical experiments and samples of firms. The literature surrounding the industry studies lists certain limitations which do not apply to agricultural production functions estimated from experimental and farm sample observations. However, some of the more technical points relating to inter-firm and intra-firm difficulties may have application in farm production functions. The literature relating to the Cobb-Douglas industry

[36] May, Kenneth. Structure of production function of the firm. Econometrica, 17: 186-87. 1949.

[37] Marschak, Jacob and Andrews, William H. Random simultaneous equations and the theory of production. Econometrica, 12: 143-205. 1944.

functions has been reviewed both to indicate (1) developments leading to estimates of value functions from economic observations, and (2) a few limitations which may also have application to farm value functions.

One of the main contributions of Cobb and Douglas, and also of Durand, to production function studies may have been development of the particular algebraic form of equation. It has been used widely because of its convenience in interpreting elasticities of production, because estimation of parameters involve fewer degrees of freedom than other algebraic forms which allow increasing or decreasing returns to scale, and because its use involves simple computations. It has not been used in agriculture in the original form requiring a sum of individual elasticities equal to 1.0 and it has seldom had use in estimating the functional distribution of the product among factors. The term Cobb-Douglas function might best be reserved for reference to the particular algebraic form. Unfortunately, however, this term is sometimes used synonymously with any production function estimated for firms or industries. Hence, an impression sometimes prevails that all of the limitations outlined above apply to value functions estimated from nonexperimental data. Obviously, all of the problems relating to multicollinearity in time series data, aggregation across industries, imperfect competition in the market, and others do not apply to functions derived from a sample of competitive firms such as those found in agriculture. Similarly, it is not necessary that a function fitted to a firm sample be of the algebraic form used by Douglas and his associates. Hence, in the chapters which follow, the term Cobb-Douglas function will refer to a particular algebraic form of equation, regardless of whether its parameters are estimated from firm industry or experimental observations. It will not be used, as is sometimes the convention, to indicate all production functions derived from nonexperimental observations.

Early Functions for Agricultural Enterprises and Firms

We now turn to a summary of early studies dealing with formal estimates of production functions from farm data. In contrast to studies reviewed previously, they are based on neither experimental data nor industry samples but on cross-sectional observations of enterprises or farms.

One of the first major attempts by agricultural scientists to fit production functions to farm data, in contrast to physical data, was published in 1924 by Tolley, Black, and Ezekiel.[38] The functions analyzed were largely single-variable physical relationships for enterprises. But unlike other physical production functions based on experimental

[38] Tolley, H. R., Black, J. D., and Ezekiel, M. J. B. Input as related to output in farm organization and cost of production studies. Tech. Bul. 1277. USDA, Washington, D. C. 1924.

Table 1.1. Daily Gains in Pounds per Head as a Function of Corn and Hay Input From Tolley, Black, and Ezekiel Study

Corn, Pounds per Day	Hay, Pounds per Day			
	8	12	16	20
10	1.61	1.81	1.98	2.13
15	1.96	2.16	2.33	2.48
20	2.27	2.47	2.64	2.79
25	2.41	2.61	2.78	

data, these were based on farm data, and although regression equations were not always computed, inputs analyzed included such categories as labor, fertilizer, and feed. Inputs were measured both in physical and monetary terms. A few linear regressions were derived relating total gain to total feed intake per hog. Data also were used to derive a simple hog production function which showed diminishing marginal productivity. They presented a "tabular production surface," showing daily gain for steers as a function of corn and hay input. The results are shown in Table 1.1. These estimates of intra-animal or intra-enterprise surfaces were derived from inter-farm or cross-sectional data. While they did not actually derive an algebraic production surface, they did present several two-dimension cost surfaces.

While such concepts as isoclines, isoquants, marginal rates of substitution, and production surface had not yet come into the literature, these early agricultural economists did come close to producing them in their analysis. They, along with Spillman, also used regression techniques. Hence, an interesting question arises: What caused the time relapse in estimation of farm and other agricultural production functions?

Firm functions

Perhaps the earliest production function study representative of estimates for farm firms was published by K. Kamiya of Tokyo University in 1941.[39] He analyzed data from a 1939 survey of paddy farms in the Tokoku and Seinan districts of Japan. The respective Cobb-Douglas functions fitted to these data are

$$P = 924T^{.73} L^{-.07} \tag{1.29}$$

$$P = 1{,}101T^{1.3} L^{-.53} \tag{1.30}$$

where P is receipts from rice production measured in *yen* (after deduction of expenses for fertilizers, seed and other materials), T is land area measured in *cho*, and L is labor in days. Data of both districts

[39] Kamiya, K. Productivity of labor. Journal of Rural Economics, 21(3): 22-44.

Table 1.2. Elasticities (Regression Coefficients) for 1939 Random Sample of Iowa Farms

Farm Group	Land	Labor	Equip-ment	Livestock and Feed	Miscellaneous Operating Expense	Sum of Elasticities
Northeast dairy	.2081*	-.0449	.0871†	.6136†	.0223†	.8862
Cash grain	.3425*	.1131	.0757†	.4115†	.0174†	.9502
Western meat	.1037†	.0107	.0888	.5511†	.0310†	.7853
Southern pasture	.1624*	-.0508	.1325	.4944†	.0438†	.7823
Eastern meat	.1918*	.2364	.0952	.4263*	.0245†	.9742
Crop	.2412*	-.0748	.0761†	.5294	.0166†	.9381
Hog	.0691*	.0209	.0965	.7449†	.0343†	.9657
Dual purposes and dairy	.1029	.0074	.0636	.6294†	.0178†	.8211
General	.1737†	.1192	.1586*	.4555†	.0302†	.9372
Special	.2970†	-.0070	.1097†	.3512†	.0379†	.7888
"Large"	.2814†	.0001	.1126†	.5378†	.0286†	.9605
"Small"	.2131†	.0482	.0843†	.4271†	.0324†	.8051
All farms	.2316†	.0282	.0844†	.4767†	.0317†	.8526

*Significant at the .05 level of probability.
†Significant at the .01 level of probability.

resulted in coefficients for labor which denote negative marginal productivities. Kamiya explained that the negative marginal productivities for labor might result from the fixity of family labor on Japanese farms and the size of farms. Under the family labor and land system, he proposed that the farmers actually purchase a large proportion of their own product through their labor. However, he also estimated that marginal productivity of labor would exist for farms with more than four *cho* of land in Tokoku district and more than three *cho* in Seinan district.

Evidently the first empirical estimates of production functions for agricultural firms in the United States were included in Iowa studies. Heady derived production functions for a 1939 random sample of 738 Iowa farms.[40] Functions were derived both for types of farms and areas of the state. In all cases the inputs were land, labor, power and equipment, livestock and feed, and operating expenses, all measured in dollars. Output was measured in dollar value of product. A Cobb-Douglas function without the restraint of constant returns to scale was employed. The computed elasticities are shown in Table 1.2.

Some of the elasticities were negative. While it is hardly conceivable that total production would decrease were more of any individual

[40]Heady, Earl O. Production functions from a random sample of farms. Journal of Farm Economics, 28: 989-1004. 1946.

factor employed, none of the negative elasticities shown in Table 1.2 were statistically significant at the 5 per cent level of probability. Hence, these elasticities could have arisen with a probability of one in twenty even if the true population elasticities were zero. In every case the sum of the elasticities was less than one, denoting diminishing returns to scale.

Management was not included as an input since there was no objective measure available in the data. The results might well have differed had it been possible to measure the input of this factor. It is the common judgment that better managers are found on larger farms. Accordingly, the true elasticity outcome would differ depending on whether the actual input of management for the farms studied increased at an increasing or at a decreasing rate, as the input of other resources or output of product increased by a given percentage.

Marginal productivities, computed at the geometric mean, and their fiducial limits are presented in Table 1.3 for geographic groups of farms.

Limitations discussed in the study included aggregation of inputs, lack of homogeneity in farms sampled for estimating intra-firm functions, measurement of labor available rather than labor used, measurement of capital inputs, specification of management input, and form of function.

Tintner and Brownlee derived a Cobb-Douglas function for 468 Iowa farms which kept records in 1939.[41] They derived the following mean marginal productivities where inputs and product were measured in dollars: land, .10; labor, .09; buildings, .05; liquid assets, .18; working assets, .16; and cash operating expenses, .28. Tintner derived a similar production function for 609 Iowa farms which kept records in 1942.[42] The magnitude of elasticities is indicated in the following empirical result where product is measured in value terms and inputs are, respectively, land, labor, farm improvements, liquid assets, working assets, and cash operating expense. All inputs but land acres and labor months were measured in dollars.

$$P = aA^{.29} B^{.16} C^{.05} D^{.21} E^{-.01} F^{.16} \tag{1.31}$$

In all three of these exploratory studies, crop and livestock products were aggregated into a single output. Inputs were aggregated largely on the basis of accounting procedures of the time. They were computed from existing data available and the authors recognized that the samples were heterogeneous and likely gave rise to mongrel functions. Also, it was recognized that inputs were not always measured in a logical fashion and thus may have given rise to law or negative productivities. However, it also was recognized that these same

[41] Tintner, G. and Brownlee, O. H. Production functions derived from farm records. Journal of Farm Economics, 26: 566-74. 1944.

[42] Tintner, G. A note on the derivation of production functions from farm record data. Econometrica, 16: 295-304. 1944.

Table 1.3. Marginal Productivities and Fiducial Limits at the 5 Per Cent Level of Probability (per Dollar of Input)

	Land A	Labor B	Equipment C	Livestock and Feed D	Miscellaneous Operating Expense E
All farms					
mean	.0465	.0791	.2013	.8390	.3931
upper limit	.0578	.2329	.2855	1.0772	.4320
lower limit	.0351	-.0742	.1171	.6008	.3543
Northeast dairy area					
mean	.0331	.0973	.1484	.6588	.3783
upper limit	.0518	.3359	.2850	1.0723	.4493
lower limit	.0144	-.1413	.0018	.2452	.3074
Cash grain area					
mean	.0615	.1066	.1809	.4177	.3693
upper limit	.0862	.1460	.3570	.6798	.4662
lower limit	.0368	.0672	.0048	.1556	.2725
Western meat area					
mean	.0382	.0302	.2410	.7130	.4037
upper limit	.0791	.3207	.3930	1.2075	.4738
lower limit	-.0027	.2603	.0890	.2185	.3337
Southern pasture area					
mean	.0187	-.1091	.3133	2.6419	.4025
upper limit	.0485	.1965	.5634	3.8121	.5160
lower limit	-.0111	-.4146	.0631	1.4718	.2989
Eastern meat area					
mean	.0398	.0685	.2147	.5012	.3407
upper limit	.0657	.0891	.4473	.9738	.4159
lower limit	.0140	.0479	-.0178	.0286	.2655

limitations applied to other types of farm management studies using other empirical techniques.

Nicholls used weekly time series data to derive production functions of a midwestern meat packing firm over the period 1938-39 and 1939-40.[43] He used data from 14 departments within the firm and had 101 successive weekly or time series observations for each. Production functions were derived from groups of individual departments, with the variables being hogs processed as output and total man-hours per week (H) and man-hours per man per week (N) the main inputs. As an example, the 1938-39 production function for departments 1-6 was

$$X = -15.30 + 4.73N - .79N^2 + 5.02H - .44H^2 . \tag{1.32}$$

This is a short-run production function where not only plant size but also department capital is fixed in quantity. Nicholls also computed

[43] Nicholls, W. H. Labor productivity functions in meat packing. University of Chicago Press, Chicago. 1948.

some "ultra short-run" functions where only man-hours per man was the variable. He compared results of linear, quadratic, and Cobb-Douglas forms. From equations with three variables, he computed isoquants and estimated combinations of "number of workers" and "hours per week" which minimized costs at specified wage and overtime rates.

Tintner conducted an industry study for American agriculture for the period 1920-41.[44] In contrast to the farm functions based on time series data cited above, his was on industry studies. It was based on time series data and paralleled the nonfarm industry studies of Douglas. The derived function was

$$X_1 = -.187 + 1.698X_2 + .813X_3 + .007X_4 \tag{1.33}$$

where the variables are measured in logarithms with X_1 being the volume of agricultural productions, X_2 the employment of agriculture, X_3 the operating capital of agriculture, and X_4 being time with origin in 1930-31. Tintner obtained an elasticity for labor over twice as great as that for capital. His exponential time trend represents a yearly increase of 1.6 per cent due to innovation and related factors.

Lomax made a similar industry study for English agriculture covering the period 1924-47.[45] His also was a study based on time series data and paralleled that of Douglas and associates for nonfarm sectors. His function was

$$P = \alpha E^{.182} C^{.374} e^{.0103R} \tag{1.34}$$

where P is the predicted index of output, E is the index of labor input, C is the index of capital input, and R is a residual, being a time trend expressed in years. After examining another variable F, the input of bought feeds and fertilizer, he concluded that its effect on production was comparatively unimportant and could be neglected. He obtained an elasticity of capital about double that of labor. His annual increase in production due to trend was 1.03 per cent. Lomax discussed several limitations of the data and estimating procedures.

Recent Production Function Studies

The above review includes only early production function studies for agriculture or those which have contributed methodology for this purpose. The review ends with a period approximating the Second World War. Many more studies have been completed since that time. Because of space limitations, these are not reviewed but are listed in the selected bibliography of recent analyses.

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[44] Tintner, Gerhard. Econometrics. John Wiley and Sons, New York. 1952. Pp. 303-304.

[45] Lomax, K. S. An agricultural production function for the United Kingdom, 1924 to 1947. The Manchester School of Economic and Social Studies, 17: 146-60. 1949.

CHAPTER 2

Economic Applications

ONE REASON for estimating agricultural production functions is to provide basic scientific knowledge. With the large investment made by society for estimating technical relationships in biological and physical sciences and economic structure in farm management science, it is reasonable to expect some of this investment should be devoted to the basic functional relationships as they exist in nature or phenomena. However, most scientific effort is expected to have eventual economic application and productivity. Hence, this chapter is devoted to review and summary of various economic principles and relationships as they relate to choices resting on the coefficients implicit in production function studies. Over time, society is likely to invest but small amounts in production function estimates unless economic criteria can be employed in putting them to use in decisions which benefit farmers and/or consumers.

Agriculture is surrounded by conditions of uncertainty, especially in respect to price and weather. It is virtually impossible to equate price and productivity ratios in a manner to exactly attain the profit maximizing conditions to be outlined later under these conditions of imperfect knowledge. Since this is true, the question may be posed: Is the scientific interest paramount in highly refined production function estimates, since farmers have limited capital and can only approximate certain principles of profit maximization? The answer may be partly positive, for example, in crop fertilization where price and yield from fertilizer inputs cannot be predicted accurately over the crop season ahead. A sufficient number of point estimates on the production function may suffice under these conditions. Selection of one of these discrete points in the lack of complete knowledge of the relevant functional relationship, while the optimum actually lies between it and another point, may result in less profit sacrifice than errors in predicting price and yield levels — and, hence, in the selection of the wrong point. Still, as is pointed out later, more complete knowledge of the production function may come at little cost. Consequently, there may be utility in estimating the continuous function, even though greatest errors in profit maximization may result from price and weather uncertainty. But for problems in livestock rations, such as the proportion of corn and protein supplement to minimize feed costs for 150-200 pound hogs, the answer

is more nearly no. Here decisions are made in respect to inputs and outputs of the near future where price and production can be known with a fair degree of certainty. But even in the case of crop fertilization, the biologically correct relationship is one of a continuous function and the experimental and computational cost, over the long run, is probably no greater in estimating a continuous production function than in estimating a few points on it. Too, knowledge of the appropriate relationships and economic principles can lead to more practical recommendations and inferences. For example, classical recommendations on fertilization were for a single rate of nutrients for a particular soil and climatic situation. No regard was given to prices or the farmer's capital or tenure situation. The same was true for livestock wherein a specific feed mix was recommended regardless of the relative prices of the feeds. The application of principles which recognize price, capital and other market quantities, does not require that information be retailed to farmers in complicated mathematical form.

The practical presentation of findings and recommendations should not be confounded with the scientific estimation and presentation of functions. These are two different activities. But the first is facilitated by accomplishment of the second. Hence, in this chapter we review the various economic quantities and principles which are relevant in application of production function knowledge. Emphasis is on basic principles and relationships. Considerations in practical application are presented later. The method used is to illustrate economic principles and relationships with simple types of algebraic equations. These simple algebraic equations, in contrast to use of greater mathematical abstraction where particular algebraic forms are not specified, is to illustrate the steps involved in deriving from production functions the quantities of importance in economic decisions. The same steps and procedures would be used for other forms of production functions. The functions used are those which adapt themselves to simple manipulation and, therefore, require minimum space in the text and minimum checking by the reader. But before the principles can be illustrated, derivation of relevant quantities from production functions is illustrated. These quantities are illustrated in the immediate following section, for application in the economic principles illustrated in later sections.

QUANTITIES FROM PRODUCTION FUNCTIONS

Production functions, when used for economic analysis and recommendation, provide one of the two sets of information needed for choice and decision-making. The other set of information needed is price data or other quantities which serve as economic criteria. Economic principles and relationships using both types of information are outlined later in this chapter. Since the physical quantities derived from production functions are a necessary part of the data needed to apply these principles, this section explains the quantities which can be derived from production functions.

Production Surfaces

As outlined in a later chapter, physical production ordinarily is a function of many resources. To simplify the discussion which follows, and because additional details are spelled out in later chapters, we suppose a production function where only two resources, X and Z, are variable and a single product, Y, results. The setting might be nitrogen and phosphate used on an acre of land to produce cotton, corn and forage fed to a cow to produce milk or labor and simple capital forms used on a farm of fixed size. With output a function of the two input categories, numerous types of production surfaces may result. Some of these are explained later. For the moment suppose that the surface takes the three-dimensional form of Figure 2.1 where resource magnitudes are measured along the horizontal axes or input plane while output is measured vertically or in product space. For this surface or any other surface, and also in the case of production functions representing more than two variable resources, several sets of interrelated quantities can be derived.

Input-Output Curves

One relationship is that represented by $z_2 l_1$ in Figure 2.1. It represents a vertical slice through the surface, parallel to the X axis. This

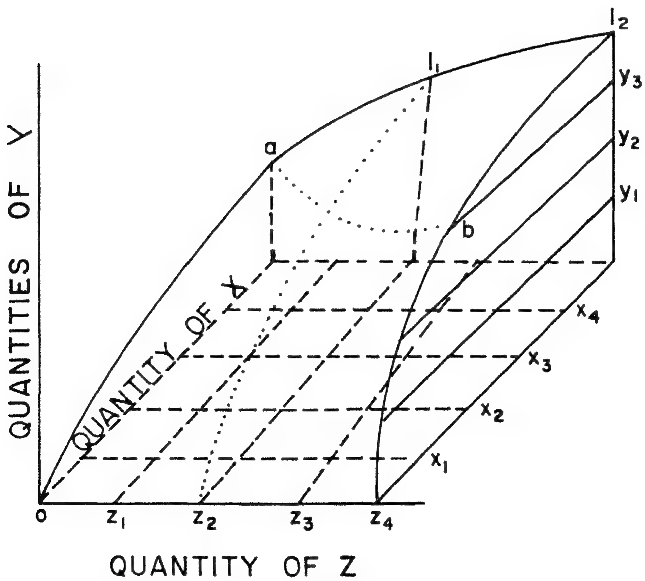

Figure 2.1. Production surface representative of a function with two variable input categories and a single product.

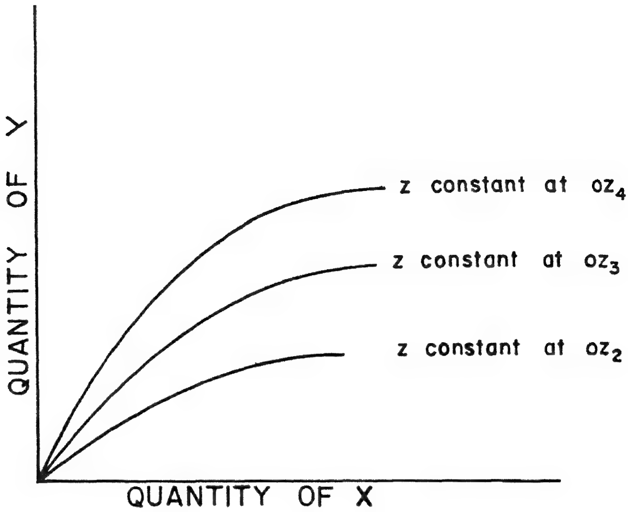

Figure 2.2. Input - output relationship between Y and X with Z constant.

line or curve expresses output in relation to variable input of X when the Z resource is held fixed in the quantity oz_2. Many such vertical slices are possible. For example, similar slices could be made where input of Z is held fixed at oz_1, oz_3, oz_4, etc. In fact, line $z_4 l_2$ is an input-output curve, relating output to the single variable resource X, when Z is fixed at the level oz_4. These input-output curves can be represented in two dimensions as in Figure 2.2. A different input-output curve, showing output for each input of X, exists for each constant level of Z. With vertical slices parallel to the X axis in Figure 2.1, a set of input-output curves similar to Figure 2.2 can be derived, showing the relation between output and quantity of variable resource Z when X is fixed in various magnitudes.

The slopes of the individual input-output curves, such as those in Figure 2.2, indicate the marginal products of the variable input. For the surface in Figure 2.1, marginal products can be derived simultaneously for both resources. The marginal products indicate the amount added to total product by each successive unit of the variable resource. These marginal products (i.e., the slopes *over* the production surface or along the input-output curves) will be used later in applying economic principles in specifying optimum resource use.

Product Isoquants

A second relationship of importance can be represented by horizontal slices through the production surface in Figure 2.1. The height at which we "horizontally slice" the surface represents a specific

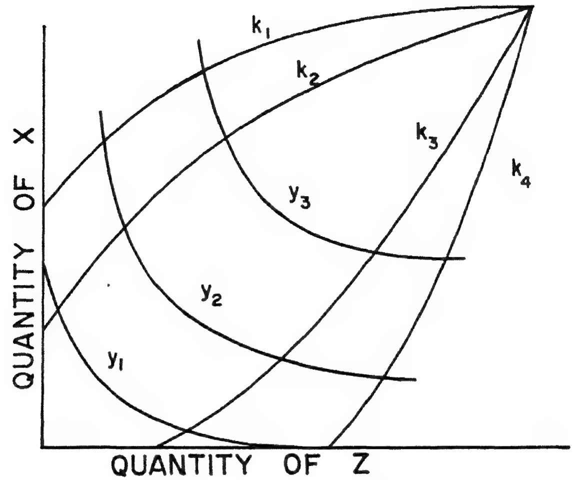

Figure 2.3. Product contours or isoquants and isoclines.

output level. Slices at y_1, y_2, and y_3 in Figure 2.1 would represent three different levels of output. Associated with each vertical slice is a contour *around* the surface at constant height or product level. Hence, it is called an isoquant. Line ab in Figure 2.1 is such a product contour or isoquant. It represents an output level of y_3 and the path it traces over the input or resource plane indicates all of the possible quantities of X and Z which will produce this given output. Many such vertical slices and isoquants can be derived for a particular production surface. They also can be represented in two dimensions. Figure 2.3 provides such a map of product contours or isoquants. The curves y_1, y_2, and y_3 are such isoquants plotted against quantities of the two resources. The production surface is now represented by a contour map with output measured by the specified quantities of y_1, rather than in vertical distance as in Figure 2.1. Graphed over the resource or input plane and with each representing a particular output level, the individual contours indicate all possible combinations of the two inputs or factors which will produce the specified output. The resource X might represent corn while Z represents oil meal. With y_2 representing 100 pounds of pork, this contour would show all possible combinations of corn and oil meal which will produce 100 pounds of gain on a pig. Or, if X is nitrogen, Z is phosphate and y_2 is 30 bushels, the contour would show all possible combinations of the two materials which will produce a 30-bushel product. The resources X and Z might represent labor and capital respectively, while the contours specify level of cotton output on a farm. The other contours, specified as y_1 and y_2, indicate the input combinations and quantities which will produce these other pork, crop, or farm outputs.

The slopes of each isoquant indicate the rate at which one resource

substitutes for or replaces the other, if output is to be maintained at the specified level. When the isoquants are curved, the rate at which one resource substitutes for the other declines, as the given output is produced with more of the former and less of the latter. Indeed, these slopes are crucial in specifying the optimum combinations of labor and capital in farming, the optimum proportions of feeds in livestock production, the optimum proportions of nutrients in crop fertilization, etc. Similarly, the slopes of the input-output curves, in Figure 2.2, the slopes over the surface in Figure 2.1, are important in specifying the optimum quantity of resources to use per farm, the optimum fertilization level and the best marketing weight or milk output of an animal. If the contours are linear, or straight lines with a constant slope, generally only one resource or material should be used in producing the specified output. If the input-output curves are linear, there is no limit to the level of input and output which is profitable, if it is profitable at all. Evidently agriculturists have sometimes supposed linear relationships when, in fact, nonlinear relationships exist for particular phenomena.

The lines in Figure 2.3 indicated by k_1 are isoclines or expansion paths. They are isoclines in the sense that they connect points of equal slope on successively higher isoquants. Hence, they also connect points on the isoquants or product contours which denote equal marginal rates of substitution between factors. For this reason, they also are termed expansion paths: they show the path which the mix of inputs should follow if output is to be expanded. If the isoclines are linear and intersect the origin, the combination of inputs should remain the same as output is expanded to conform to higher product prices or lower resource prices. If the isoclines are curved or if they are linear and do not pass through the origin, the mix of resources should change as output is expanded or contracted to conform with changes in product or factor prices. Two isoclines may serve as ridge lines, denoting zero and infinite rates of substitution between resources. Isoclines k_1 and k_2 in Figure 2.3 are ridge lines: k_2 intersects at slopes of infinite magnitude on the contours while k_2 intersects at slopes of zero magnitude. Hence, k_1 or k_2 indicates that addition of more of X or Z, respectively, will not replace any of the other in producing the output level specified. If the production surface rises to a distinct peak, the isoclines will converge to a single point as in Figure 2.3. Rates of substitution and marginal productivities also are zero at this point. Production surfaces which slope to a plateau do not have a similar convergence point, a limitational input, for isoclines.

Algebraic Derivation

Production functions derived from experimental, sample or other data allow algebraic estimation and arithmetic expression of all the quantities specified above. Hence, we will specify a production function

for two variable factors and illustrate derivation of these quantities and relationships. While it is not taken to represent agricultural situations, suppose the production function is known to be that in equation 2.1 below. The constants a, b, and c would be derived

$$Y = aXZ - bX^2 - cZ^2 \tag{2.1}$$

as regression coefficients in experiments and cross-sectional surveys or from time series data. They would have numerical values. With knowledge of them, we could set X and Z at various levels and estimate a series of Y or output values.

If we plot these values of Y against quantities of X and Z, the production surface can be drawn. Hence, equation 2.1 is the algebraic counterpart of the geometric representation of the production function such as in Figure 2.1.

With knowledge of the production function with two variable resources, such as in equation 2.1, we can specify the algebraic input-output relationship for one resource by holding the other constant at some specified level. Suppose that we fix Z at the level z_2. The products $aZ = az_2$ and $cZ^2 = cz_2$ then will become constants and can be redefined as m_1 and m_2. The resulting single variable production function or input-output relationship is then

$$Y = m_1X - bX^2 - m_2 . \tag{2.2}$$

If we set X at various levels and compute the corresponding quantities of Y, the latter can be plotted to provide a single input-output curve such as the lower curve in Figure 2.2. By setting Z at other levels, additional input-output curves can be derived.

The marginal physical product of a factor is measured as the first derivative of output with respect to input. Hence, for the function in equation 2.2, the marginal product equation for the variable resource X is

$$\frac{\delta Y}{\delta X} = m_1 - 2bX . \tag{2.3}$$

This equation measures the slope on the input-output curve for various magnitudes of X. However, this marginal product equation exists for a specified constant level of Z. We can return to the original production function in equation 2.1, and derive the marginal product equations for both resources. These are the first partial derivatives

$$\frac{\delta Y}{\delta X} = aZ - 2bX \quad \text{and} \quad \frac{\delta Y}{\delta Z} = aX - 2cZ . \tag{2.4}$$

From these equations, representing slopes over the production surface, we can specify the marginal productivity of one factor when the other is fixed at any specific level.

The maximum physical product is attained when the marginal products, or first derivatives, are zero. Hence, in the case of equation 2.3, we could set this equation to equal zero, and solve for the quantity of X which is consistent with a zero marginal product or first derivative. In this case, a value of X equal to $.5m_1 b^{-1}$ drives the first derivative to zero and defines a maximum physical product. The second derivative can be used to define the inflexion point where both increasing and decreasing marginal productivity of the factor hold true.

The isoquant equation can be derived from the production function equation by expressing input of one factor as a function of output level and quantity of the other resource. Using X as the "dependent" resource we can derive an isoquant equation from equation 2.1. Collecting terms and completing the square, the isoquant equation based on equation 2.1 is

$$X = \frac{aZ + \sqrt{(a^2 - 4cb)Z^2 - 4bY}}{2b}. \tag{2.5}$$

This equation is the algebraic counterpart of contours such as those in Figure 2.3. By setting Y at different levels, a number of product contours or isoquants can be derived accordingly. Setting Y to be a specific value, various magnitudes of Z can be plugged into equation 2.5. The corresponding values of X are then computed. If these values of X are plotted against the "plugged in" values of Z, a single contour or isoquant results. Setting Y at different levels and following the same procedure, other isoquants can be derived similarly.

The marginal rates of substitution along a given contour are represented by the derivative of the isoquant equation. Also, they can be measured as the "inverse" ratio of marginal products with a minus sign attached. Hence, the equation of marginal substitution rates, corresponding to isoquant equation 2.5 is

$$\frac{\delta X}{\delta Z} = -\frac{aX - 2cZ}{aZ - 2bX}. \tag{2.6}$$

Having computed isoquant sets from equation 2.5 for plotting of the product contours, these values of X and Z can then be plugged into partial derivative equation 2.6. The quantities so computed represent the slopes of the isoquant and the marginal rates of substitution for the corresponding points on the isoquant.

The isocline equation can be computed as follows: Set the equation of marginal rates of substitution, the derivative of one resource variable with respect to the other, to equal the substitution rate or constant level desired. Following the algebraic example being discussed, we set equation 2.6 to equal -k below. Both sides can be multiplied by -1,

$$\frac{\delta X}{\delta Z} = -k \qquad \text{or} \qquad -\frac{aX - 2cZ}{aZ - 2bX} = -k \tag{2.7}$$

to remove the negative coefficient. Transposing from equation 2.7 and solving for X, we obtain

$$X = \frac{2c + ak}{a + 2bk} Z \; . \tag{2.8}$$

Now selecting a value for k (say that the marginal rate of substitution is one unit of Z for two units of X), we can plug different values of Z into isocline equation 2.8. The computed values of X can be plotted against the "plugged in" values of Z, with the result being an isocline for a given k or substitution rate. Setting k at another level and following the same procedure, another isocline can be defined. In the case of equation 2.8, the isoclines are linear but do not pass through the origin.

Production Elasticities

Another quantity of some importance in production function studies, particularly in analyses relating to product imputation, is the elasticity coefficient. The elasticity of production indicates the change in output relative to the change in input. If output increases by a greater percentage than input, the ratio is greater than 1.0; if output increases by the same rate as input, it is 1.0; for a percentage increase in output less than the percentage increase in input, the ratio is less than unity. These three magnitudes of the elasticity coefficient, if all input categories are increased in the same proportion, indicate respectively increasing, constant and decreasing returns to scale. The elasticity coefficient, E_p, is defined as

$$E_p = \frac{\delta Y}{\delta X} \frac{X}{Y} \tag{2.9}$$

for a single variable resource X. In other words, it is the product of the marginal product and the reciprocal of the average product. Applying this definition for X in the function of equation 2.1, we multiply the derivative of Y with respect to X by the ratio XY^{-1}. Using the derivative in equation 2.3, where Z is fixed at z_2 level, the production elasticity of X is as follows, expressed in terms of X:

$$E_p = \frac{m_1 X - 2bX^2}{m_1 X - bX^2 - m_2} \; . \tag{2.10}$$

Substituting the production function in equation 2.1, with Z fixed at z_2 level, for Y in equation 2.10 and plugging in various values for X, we can compute the elasticity for each value of X. For this function, a different elasticity of X will exist for each magnitude of this factor, and

for each level at which Z is fixed. Similar computations would lead to the production elasticity for Z.

Elasticity of production magnitudes divides the production function into three stages for particular phenomena. However, all three of these stages are not present in all statistical production functions. Stage I of production corresponds to the range of inputs and outputs over which the elasticity of production is greater than 1.0 and, if all input categories have been included, increasing returns to scale prevail. (Generally, stages of production exist only where some factors are fixed.) In this range, the marginal product is greater than the average product. This is an uneconomic range of production where fixed factors are involved since some of the fixed factor can be discarded. Use of more variable resources on a smaller mix of fixed factors then will increase total product and average product. More output can be obtained from a given amount of variable resource. Stage II corresponds to the output and input range where the elasticity coefficient is less than 1.0 but greater than zero. It is within this range in which economical combinations or use of resource is found; within this range, the principles outlined in this chapter can be applied to maximize profits. Stage III corresponds to output ranges where the elasticity coefficient is less than zero. Use of more variable resource depresses total output and average product. Hence, range III never denotes economical production since discarding some variable resource will increase return to both fixed and variable resources.[1]

PROFIT MAXIMIZING QUANTITIES OF INPUTS

Recommendations in agriculture ordinarily are in terms of inputs. The agriculturist suggests quantities of fertilizer, seed or insecticide to apply on a fixed producing unit such as an acre. Or, he recommends daily rations for a dairy cow. Hence, we review principles which emphasize specification of the magnitude of optimum inputs rather than direct specification of optimum outputs. (Obviously, specification of the optimum input implies designation of the optimum output and vice versa.) First, consider the case of a single variable resource where production is related to a fixed plant such as an acre or an animal. Here we suppose that the production function and product and factor prices are known. Profit is maximized for the fixed unit if the marginal value product of the factor is equal to the marginal cost of the factor. We will use a form of production function involving only simple calculations, although the steps and principles illustrated would apply to any function. Assuming the production function in equation 2.11, with Y representing magnitude of output, X magnitude of input, and b

$$Y = a + bX - cX^2 \tag{2.11}$$

[1] For additional details on stages of production, see Heady, Earl O. Economics of agricultural production. Prentice-Hall, Inc., New York. 1952. Ch. 3.

and c positive constants, the total value product, V, is given by equation 2.12 where output in equation 2.11 is multiplied by P_y, the product price.

$$V = P_y Y = aP_y + bP_y X - cP_y X^2 \tag{2.12}$$

The marginal value product is the derivative of equation 2.12 while the marginal cost of the resource is P_x, the price of the resource in a competitive market. Equating the marginal value product of the resource with its marginal cost, we obtain equation 2.13.

$$\frac{dV}{dX} = bP_y - 2cP_y X = P_x \tag{2.13}$$

Transposing terms and dividing by the coefficient of X, we specify the magnitude of input which will maximize profit as that in equation 2.14.

$$X = \frac{bP_y - P_x}{2cP_y} \tag{2.14}$$

As is indicated in later chapters, the coefficients b and c will ordinarily be derived from regression equations estimated from experimental or sample data. With these estimates available and knowing the product and factor prices as P_y and P_x, respectively, these quantities can be substituted into equation 2.14, with the magnitude of X specified accordingly. This value of X indicates the magnitude of input or treatment which will maximize profit from the fixed factor. It might be, for example, 40 pounds of nitrogen on an acre of corn. Substituting this value, $X = 40$, back into the production function in equation 2.11, we could also compute the optimum output or yield level.

The optimum quantity of input is a function of factor and product price, with b and c being constants in the production function. As P_x increases, the optimum magnitude of X declines. As P_y increases, the optimum level of X increases. A decline in factor and product prices will have the opposite effect on the optimum magnitude of X.

If fixed costs are involved in use of the variable factor, profit may be examined better from the standpoint of a profit function. Profit from the use of a factor is the difference between total return from the factor and its total cost. The total cost of using the factor is equal to the sum of fixed costs, K, involved in its use plus the value of the quantity of the variable factor used, or $K + P_y X$. Subtracting this total cost function from the total revenue function, we can define total profit, Π, as in equation 2.15.

$$\Pi = aP_y + bP_y X - cP_y X^2 - (K + P_x X) \tag{2.15}$$

Profit is maximized when the marginal profit from use of the resource is zero. Hence, defining marginal profit as the derivative of the profit function with respect to the variable resource, we obtain equation 2.16.

$$\frac{d\Pi}{dX} = bP_y - P_x - 2cP_y X \tag{2.16}$$

Equating equation 2.16 to zero and solving for X, the optimum input (the profit maximizing quantity of factor), is again that in equation 2.14. The same quantity of input is specified because fixed costs do not affect the magnitude of marginal costs or marginal profits. Since this is true, a more apparent relationship between factor and product prices and production coefficients can be indicated. Profit from use of the factor also is maximized when its marginal product is equated to the price ratio, price of factor divided by price of product or

$$\frac{dY}{dX} = \frac{P_x}{P_y} . \tag{2.17}$$

The marginal product of the factor is the first derivative of the production function in equation 2.11. Setting the derivative of equation 2.11 to equal the factor to product price ratio, we obtain equation 2.18.

$$b - 2cX = \frac{P_x}{P_y} \tag{2.18}$$

Transposing and dividing by 2c, we obtain the quantity of X which maximizes profit. It again is that determined for equation 2.14.[2] But from equation 2.18 it is obvious that to maximize profit the marginal product will need to decrease, through addition of X, as factor price decreases relative to product price. An increase in the magnitude of P_x relative to P_y will call for an increase in magnitude of the marginal product, $b - 2cX$, by a reduction in magnitude of X.

Classically, economists have defined total revenue and total costs as a function of output and have defined optimum production in terms of output level which maximizes profits. While, as indicated earlier, this quantity can be derived from the steps, described by equations 2.11 through 2.18, we illustrate it with the determination now being first in terms of specifying the optimum level of output. Hence, multiplying output, Y, by P_y, the price per unit of output in a competitive market, the total revenue function is

$$R = P_y Y . \tag{2.19}$$

[2] That solution by the approach of equation 2.18 is the same as that by equation 2.15 is easily illustrated. From equation 2.15 we obtain equation 2.16 and set it equal to zero, or

$$bP_y - P_x - 2cP_y X = 0 .$$

Adding P_x to both sides, we have

$$bP_y - 2cP_y X = P_x .$$

Now dividing through by P_y we obtain the quantity already expressed in equation 2.18; indicating that if we follow through from the profit equation in 2.15, we actually equate the marginal product with the factor to product price ratio.

The total cost function is the cost per unit of Y multiplied by the magnitude of Y. The total cost function was represented in equation 2.15 as

$$(2.20) \qquad T = K + P_x X ,$$

a function of X. However, since profit maximization is now to be defined in terms of output, the total cost equation must be expressed as a function of Y. Thus from the production function in equation 2.11, we first express input as a function of output. For the particular algebraic form, this is accomplished by transposing terms with X and X^2 to the left and Y to the right of the equality and completing the square. Solving the quadratic equation for value of X, we obtain equation 2.21.

$$(2.21) \qquad X = (2c)^{-1}[b \pm \sqrt{4c(a - Y) + b^2}]$$

With input defined as this function of output, we substitute the value of X in equation 2.21 into the cost equation in equation 2.20. This quantity is subtracted from total revenue to provide the profit equation in equation 2.22.

$$(2.22) \qquad \Pi = P_y Y - \left[K + \frac{P_x[b \pm \sqrt{4c(a - Y) + b^2}]}{2c}\right]$$

Profit again is at a maximum when marginal profit is zero. Marginal profit now is the first derivative of the total profit equation in equation 2.12, or the first derivative of Π with respect to Y as represented by equation 2.23.

$$(2.23) \qquad \frac{d\Pi}{dY} = P_y - \frac{P_x}{(4ac - 4cY + b^2)^{.5}}$$

Equating equation 2.23 to zero, and solving for Y, we obtain the equilibrium output as that in equation 2.24.

$$(2.24) \qquad Y = a + \frac{b^2}{4c} - \frac{P_x^2}{4cP_y^2}$$

Now if we substitute the value of Y in equation 2.11 for that on the left in equation 2.24 and solve for X, we obtain the equilibrium input already indicated in equation 2.14.

Other Criteria

The principles outlined above suppose that capital is unlimited and profit from the fixed unit can be maximized. However, typical farmers do not have unlimited capital. They must employ other, but related, economic criteria in specifying the optimum quantity of variable factor

to be used per unit of fixed factor. The optimum input defined above is the maximum quantity of the resource which should be used. Most farmers will need to use less, if they maximize profit in their limited capital situation. Similarly, a minimum quantity of input can be defined, with farmers needing to use more if they are to make any profit. In case the production function involves only decreasing marginal returns and fixed costs are not involved, the minimum quantity of input to be used is zero. More than zero quantity should be used only if the price of the product is sufficiently large relative to the price of the factor. However, if fixed costs are involved, the minimum quantity of input which should be used differs from zero even if the production function includes only diminishing returns. The minimum input is defined by equation of total cost of and total revenue (total return) from use of the factor. In a diagram of total costs and total revenue related to use of a variable factor such as Figure 2.4, this minimum quantity of input is ox_1. With fixed costs of oc_1, none of the factor should be used if price of the product is sufficiently low (i.e., if the total revenue curve is lower throughout than the total cost curve for the resource). But with a product price which is sufficiently high relative to the factor price, the minimum quantity of factor which should be used is ox_1. For inputs larger than this, revenue exceeds costs and net profit is generated. Profit from the fixed producing unit can be increased until input reaches the magnitude of ox_2, the level which maximizes profit for the fixed producing unit. At ox_2, the difference between total revenue and total cost, rs, is at a maximum. At this input, marginal cost and marginal revenue from using the resource are equal, as denoted by equality of the slopes of the two curves R and T. The optimum input quantity for a farmer with limited capital and other profitable investment opportunities will fall somewhere between ox_1 and ox_2. Given knowledge of the production function, the minimum input which is profitable can be determined algebraically. With the production function in equation 2.11, the total revenue from resource use is equation 2.12. With the total cost function in equation 2.20, where K is fixed cost from use of the resource and P_x is the price per unit of resource, total revenue and total costs are equated in equation 2.25 below.

$$aP_y + bP_y X - cP_y X^2 = K + P_x X \tag{2.25}$$

Transposing terms, we obtain equation 2.26. Completing the square and solving for the roots of X, we obtain equation 2.27 which provides the X quantity defining the minimum quantity of resource which can be used profitably.

$$cP_y X^2 + (P_x - bP_y)X = aP_y - K \tag{2.26}$$

$$X = \frac{bP_y - P_x \pm \sqrt{4cP_y(aP_y - K) + (P_x - bP_y)^2}}{2cP_y} \tag{2.27}$$

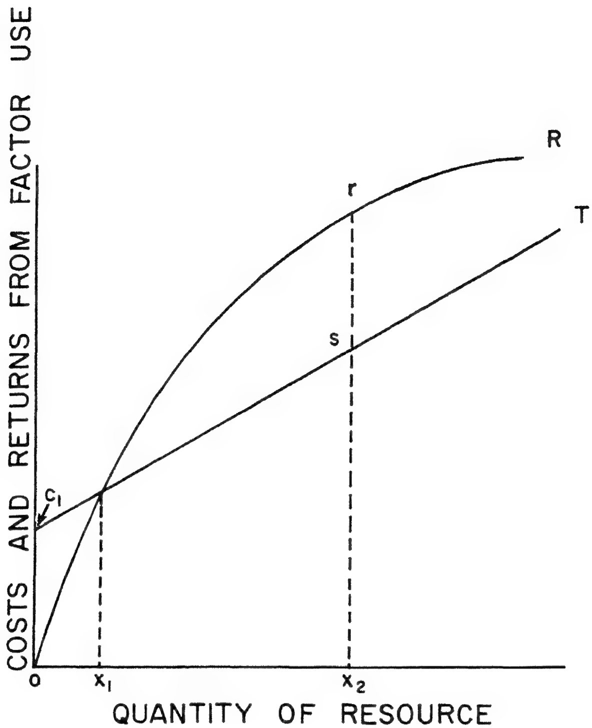

Figure 2.4. Total revenue from and total costs of using a variable resource in relation to minimum and maximum quantities of input.

If the production function has ranges of both increasing and decreasing marginal productivity (i.e., ranges over which the elasticity of production is both greater and smaller than 1.0), the minimum input which should be used is defined at the point where the marginal product is equal to the average product (i.e., the input where the elasticity of production is equal to 1.0). Details indicating why this quantity of resource is minimum for a production function with a range of increasing returns are given elsewhere.[3] Again, the quantity can be derived algebraically for a given production function. Assuming the algebraic form of production function equation in 2.28, the marginal product equation or derivative is equation 2.29. The average product is the total product divided by total input, or the right-hand side of equation 2.28 divided by X.

$$Y = bX + cX^2 - dX^3 \tag{2.28}$$

[3] Heady, Earl O. Economics of agricultural production and resource use. Prentice-Hall, Inc., New York. 1952. Pp. 91-92.

(2.29) $$\frac{dY}{dX} = b + 2cX - 3dX^2$$

Defining the average product thus and equating it to the marginal product in equation 2.29, we obtain equation 2.30. Completing the division

(2.30) $$b + 2cX - 3dX^2 = \frac{bX + cX^2 - dX^3}{X}$$

on the right-hand side of equation 2.30, collecting terms and solving for X, we obtain equation 2.31, indicating the minimum quantity of factor which should be used per unit of fixed resource, if input is to be greater than zero.

(2.31) $$X = .5cd^{-1}$$

Return for Investment in Variable Factor

For many farmers with limited capital, the quantity of variable input which maximizes the rate of return on investment in the variable input is perhaps as relevant as the input which maximizes profit in the sense of equating marginal costs and revenues. For a function with only diminishing marginal returns, the quantity of input which maximizes return per dollar invested in it is zero. However, when there are fixed costs, K, of using the factor and some nonzero quantity of factor is profitable, the input which maximizes return on investment in the variable factor is not zero. The input of variable factor which maximizes rate of return on investment is fixed and variable costs for it can be defined as follows: total monetary return is determined by multiplying output by product price as in equation 2.12 for the production function in equation 2.11. The total revenue function can be divided by the total cost function, to provide an equation defining return per dollar invested in the factor, a rate of return equation. The marginal rate of return per dollar invested is the derivative of the rate of return equation. Setting this derivative, the rate of return in respect to the variable factor, to equal zero and solving for the value of input, we can indicate the quantity of variable factor which maximizes rate of return on investment in the variable resource. The production function in equation 2.11 can be used to illustrate the procedure. The total revenue function is equation 2.12 and we divide it by the total cost function in equation 2.20 to define the rate of return on investment, r, as different inputs are used, as in equation 2.32.

(2.32) $$r = \frac{aP_y + bP_yX - cP_yX^2}{K + P_xX}$$

The marginal rate of return, as a function of input, is the derivative of r with respect to X as shown in equation 2.33. Setting equation 2.33 to

$$\frac{dr}{dX} = \frac{(bK - aP_x)P_y - 2cKP_y X - cP_y P_x X^2}{K^2 + 2P_x KX + P_x^2 X^2} \tag{2.33}$$

equal zero, collecting terms, completing the square, and solving for the roots of the equation, we obtain the value of X in equation 2.34. This is

$$X = \frac{-2cKP_y \pm [4c^2K^2P_y^2 + 4cP_y^2P_x^2(bK - aP_x)]^{.5}}{2cP_y P_x} \tag{2.34}$$

the quantity of input which will maximize the return per dollar invested in the variable factor, considering the magnitude of fixed costs. It increases as product price increases or factor price decreases. It decreases for price movements in the opposite direction.

As a "rule of thumb" criterion, to be used apart from analysis of the whole farm business and in making input recommendations to farmers who are short on capital, the value of input which maximizes return on investment in the variable factor is perhaps more relevant than the input magnitude which defines the minimum input described earlier or the maximum input defined by a marginal product which is equal to the factor to product price ratio. The farmer with limited capital will, of course, maximize profits if he invests in each input category such that the marginal return per dollar invested is equal among all input categories. (We have been considering only a single variable input here. Hence, the problem posed falls more correctly in sections to be outlined later.) Thus specialists working separately (such as specialists in fertilization, hog feeding, or dairy nutrition) on a particular input category, and without attempting to program profit maximizing programs for the entire farm, might best provide information of this type for farmer use under the following procedure: Each specialist should provide the function indicating the marginal "rate of return" per dollar invested in his specialized resource (or resource collection) as different quantities of it are used. The farmer could take these several sets of information and, considering his capital and the numbers of different types of producing units such as acres and animals, decide on how much of each type of variable resource to use with each fixed acre and animal, if profit from the farm is to be maximized. He would allocate investment to each category of variable resource so that the marginal "rate of return" is equal among them. A means of doing this is provided by equations such as equation 2.34. By inserting different values of X in such an equation, marginal returns per dollar invested in the variable factor can be determined as level of X increases. Given this information for numerous input categories (provided by separate agricultural scientists) such as feed, fertilizer, and insecticide, the farmer could estimate rather quickly the amount of each which should be used if he were to allocate his limited funds optimally among them. But if this degree of refinement were to be used, programming procedures might be more appropriate.

Allocation of Limited Capital

An allocation of limited capital among product investment opportunities when continuous production functions are known can be illustrated for a simple situation. Here we might suppose that a single variable resource such as fertilizer is involved, although the basic principle would be the same if other types of resources and producing units also were involved. Suppose that we have three crops, Y_1, Y_2, and Y_3. Each has a production function of the form in equation 2.11. Hence, a subscript is attached to the constants in each equation to indicate the product to which it refers. Similar subscripts indicate the quantity of the resource (e.g., nitrogen fertilizer) used for each product. Funds are available for only a fixed quantity of the resource indicated as $\overline{X}$. What proportion of the total quantity $\overline{X}$ should be allocated to each crop? The marginal physical products of the variable resource are indicated in the first column of equations in system 2.35. By multiplying these equations by P_1, P_2, and P_3, the prices of the three crops, we obtain the three equations for the value of the marginal products (VMP's) which are identical with the marginal value productivities for a competitive industry such as agriculture, in the second column of equations in system 2.35.

$$
\begin{aligned}
&\frac{dY_1}{dX_1} = b_1 - c_1X_1 \qquad && VMP_1 = b_1P_1 - c_1P_1X_1 \\
(2.35) \qquad &\frac{dY_2}{dX_2} = b_2 - c_2X_2 \qquad && VMP_2 = b_2P_2 - c_2P_2X_2 \\
&\frac{dY_3}{dX_3} = b_3 - c_3X_3 \qquad && VMP_3 = b_3P_3 - c_3P_3X_3
\end{aligned}
$$

With a limited quantity, $\overline{X}$, of the variable resource, we wish to allocate a quantity to each crop so that the marginal value productivity of the resource will be equal for the three crops. We do not know the magnitude of the marginal value product which will exist when resources are allocated to each crop so that this quantity is equal for all. Hence, we will denote it as m, and set all three marginal value productivities to equal this value as in the first three equations of system 2.36.

$$
(2.36) \qquad
\begin{aligned}
b_1P_1 - c_1P_1X_1 &= m \\
b_2P_2 - c_2P_2X_2 &= m \\
b_3P_3 - c_3P_3X_3 &= m \\
X_1 + X_2 + X_3 &= \overline{X}
\end{aligned}
$$

But we also have a restriction on resource allocation to be met. It is indicated in the last equation of system 2.36, stating that the sum of the

quantities of the resources allocated to each crop must equal the total available quantity, $\bar{X}$, of the resource. The unknown quantities in this set of four equations are X_1, X_2, X_3, and m. All other quantities are known as parameters of price, resource supply or of the production functions. Hence, system 2.36 is a system of simultaneous equations which can be solved for the unknown values. This may be done as follows: Transpose all terms of known quantities to the right-hand side of the equations and all terms with unknown quantities to the left-hand side, as in system 2.37; we have also introduced each unknown quantity

$$(2.37)\qquad \begin{aligned} -c_1P_1X_1 + 0X_2 + 0X_3 - 1m &= -b_1P_1 \\ 0X_1 - c_2P_2X_2 + 0X_3 - 1m &= -b_2P_2 \\ 0X_1 + 0X_2 - c_3P_3X_3 - 1m &= -b_3P_3 \\ 1X_1 + 1X_2 + 1X_3 + 0m &= \bar{X} \end{aligned}$$

in all equations, by giving it a zero coefficient for the equation in which it did not previously occur. Such a formulation more clearly suggests the algebraic manipulations which follow. The coefficients of the X_i and m form a coefficient matrix for equation 2.38 while the X_i form a vector of unknowns to be solved from the vector of known parameters on the right of the equals sign.[4]

$$(2.38)\qquad \begin{bmatrix} -c_1P_1 & 0 & 0 & -1 \\ 0 & -c_2P_2 & 0 & -1 \\ 0 & 0 & -c_3P_3 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ m \end{bmatrix} = \begin{bmatrix} -b_1P_1 \\ -b_2P_2 \\ -b_3P_3 \\ \bar{X} \end{bmatrix}$$

Denoting by A the matrix of coefficients, by X the vector of unknowns, and by S the vector on the right, we have the matrix product:

$$(2.39)\qquad AX = S$$

Now inverting A, the matrix of known coefficients, to obtain A^{-1}, its inverse, we have

$$(2.40)\qquad X = A^{-1}S$$

from which we form a system of scalar equations expressing the value of X_1, X_2, X_3, and m. Letting c_{ij} represent the elements of this 4 x 4 order inverse matrix, we can compute the values of X_1, X_2, and X_3, the

[4] Readers unacquainted with solving systems of equations are referred to the relatively simple explanation given in Chapter 13 of Heady, Earl O. and Candler, Wilfred V. Linear programming methods. Iowa State University Press, Ames. 1958.

quantities of resource to be allocated to each crop and m, the magnitude of the marginal value product, as follows:

$$
\begin{aligned}
X_1 &= -c_{11}b_1P_1 - c_{12}b_2P_2 - c_{13}b_3P_3 + c_{14}\overline{X} \\
X_2 &= -c_{21}b_1P_1 - c_{22}b_2P_2 - c_{23}b_3P_3 + c_{24}\overline{X} \\
X_3 &= -c_{31}b_1P_1 - c_{32}b_2P_2 - c_{33}b_3P_3 + c_{34}\overline{X} \\
m &= -c_{41}b_1P_1 - c_{42}b_2P_2 - c_{43}b_3P_3 + c_{44}\overline{X}
\end{aligned}
\tag{2.41}
$$

Hence, we have established the quantity of the variable resource to allocate to each crop to provide equal marginal value productivities. With the total quantity, $\overline{X}$, allocated to provide an equal marginal value product, m, for each crop, profit from use of the fixed quantity of the resource is at a maximum. This procedure in resource allocation is appropriate for farmers who have limited funds and cannot apply variable resources to each fixed producing unit until marginal physical products are equal to factor to product price ratios. While the principle has been illustrated with a particular algebraic form of function, the same procedures generally apply to other types of equations, although certain other forms of equations are more difficult to manipulate. This problem could also be approached by the empirical technique outlined later, using Lagrangian multipliers to equate the returns from several resources when expenditures on them are limited by a fixed capital restraint.

Perhaps more often the farmer has a limited fund with which he can purchase different types of physical resources, such as different nutrients or different mixes of nutrients, which he may apply on different enterprises. Then, in contrast to the case of equations 2.35 through 2.41 where a fixed quantity of a given type of physical resource is allocated to different crops, the problem is to allocate the different resources so that the marginal value return is the same per dollar invested in each type of resource for each enterprise. In this case, an equation such as 2.32 should be constructed for each crop. Then setting the partial derivatives of these with respect to the X's equal to v, a quantity expressing a given magnitude of marginal value for each type of resource used for each crop, a system of equations similar to system 2.36 can be formulated. Using a restraint such that the value of the resources used, the sum of the products of prices times the quantities of resources, must equal the fixed expenditure, an equilibrium solution can be derived by the same procedure. It will indicate the quantity of each resource which should be purchased for each crop if profits are to be maximized. In many cases, practical solutions to problems of this type, particularly where many investment opportunities exist, may be more easily obtained by linear programming procedures.[5]

[5] See Heady, Earl O. and Candler, Wilfred V. Linear programming methods. Iowa State University Press, Ames. 1958.

Production Possibilities

An alternative approach also allows specifying optimum allocation of a given quantity of resource such as fertilizer, among competing crops. This is by means of a production possibility curve. For example, suppose that we have the two production functions

(2.42) $$Y_1 = a_1 X_1^{b_1} \quad \text{and} \quad Y_2 = a_2 X_2^{b_2}$$

where Y_1 and Y_2 are the outputs of two crops, X_1 and X_2 are the amounts of a resource such as fertilizer allocated to each and $\overline{X}$ is the total quantity of resource available. From the production functions in equation 2.42, we can derive the requirements equations in equation 2.43 indicating the total amount of resource required to produce a given output

(2.43) $$X_1 = a_1^{-\frac{1}{b_1}} Y_1^{\frac{1}{b_1}} \quad X_2 = a_2^{-\frac{1}{b_2}} Y_2^{\frac{1}{b_2}}$$

of each product. Now the restraint

(2.44) $$X_1 + X_2 = \overline{X}$$

exists, indicating that the sum of resources used on each product must equal $\overline{X}$, the total amount of resource available. Now substituting into equation 2.45 the values of X_1 and X_2 in equation 2.43, we obtain

(2.45) $$a_1^{-\frac{1}{b_1}} Y_1^{\frac{1}{b_1}} + a_2^{-\frac{1}{b_2}} Y_2^{\frac{1}{b_2}} = \overline{X}$$

which also is a resource restraint or requirements equation expressed in terms of the variable outputs Y_1 and Y_2. By subtracting the product containing Y_2 from both sides of equation 2.45 and dividing by the coefficient of Y_1, we derive the production possibility equation 2.46, with

(2.46) $$Y_1 = \left(a_1^{\frac{1}{b_1}} \overline{X} - a_1^{\frac{1}{b_1}} a_2^{-\frac{1}{b_2}} Y_2^{\frac{1}{b_2}}\right)^{b_1}$$

output of Y_1 expressed as a function of Y_2 when quantity of variable resource is fixed at level $\overline{X}$. This equation shows the amount of Y_1, which can be produced when Y_2 is at different levels. From this equation, we can derive an equation of marginal rate of substitution, indicating the amount of Y_1 which must be sacrificed for each unit gain in Y_2. The latter equation is the derivative in equation 2.47. Now setting it to

(2.47) $$\frac{dY_1}{dY_2} = \left(a_1^{\frac{1}{b_1}} \overline{X} - a_1^{\frac{1}{b_1}} a_2^{-\frac{1}{b_2}} Y_2^{\frac{1}{b_2}}\right)^{b_1-1} \left(-b_2^{-1} a_1^{\frac{1}{b_1}} a_2^{-\frac{1}{b_2}} Y_2^{\frac{1-b_2}{b_2}}\right)$$

equal $-P_2 / P_1$, the ratio formed by dividing the price of Y_2 by the price of Y_1, we specify the combination of Y_1 and Y_2 which will maximize profits. Setting 2.47 to equal the price ratio and solving for Y_2, we have the value below.

$$(2.48) \qquad Y_2 = \left(\frac{b_2 a_2^{\frac{b_1}{b_2}} P_2}{b_1 a_1 P_1} \right)^{\frac{b_2}{b_1 - b_2}}$$

Substituting this value into equation 2.46, the value of Y_1 can be derived similarly. From these quantities substituted into equation 2.43, the quantities of resource to be used for either enterprise can be determined as a function of the calculated output of either crop and of the prices of the two crops. But, again, alternative empirical approaches such as those mentioned above and linear programming may be computationally desirable.

TWO OR MORE RESOURCES

Profit maximizing and related quantities of inputs have been specified above from production functions involving a single variable resource. Input quantities which represent an economic optimum can also be derived when two or more resources are variable. But in addition, another problem exists; the proportion of inputs which minimize the cost of any output must be determined. We explain these two steps immediately below. While the illustrations suppose a production function with two variable resources, the steps outlined can be generalized to n variables.

Profit Maximizing Quantities of Inputs

Given a production function where output is a function of two resources, as in equation 2.49, profit is maximized from the fixed producing unit when the marginal products of the two resources, or n resources if more than two are varied, are simultaneously equal to their

$$(2.49) \qquad Y = a + b_1 X_1 - b_2 X_1^2 + b_3 X_2 - b_4 X_2^2 + b_5 X_1 X_2$$

respective prices divided by the prices of the product. This condition is illustrated in equation 2.50 where the marginal products on the left

$$(2.50) \qquad \begin{aligned} \frac{\delta Y}{\delta X_1} &= b_1 - 2b_2 X_1 + b_5 X_2 = \frac{P_1}{P_y} \\ \frac{\delta Y}{\delta X_2} &= b_3 - 2b_4 X_2 + b_5 X_1 = \frac{P_2}{P_y} \end{aligned}$$

(with the first equation for the marginal product of X_1 and the second equation for X_2), are equated to the price ratios. P_1 is the price per unit of the resource represented by X_1, P_2 is the price for resource X_2, and P_y is the price for the product. Solving these two equations simultaneously, because two unknowns are included in the two equations, we have the equilibrium quantities of X_1 and X_2 below. These are the profit maximizing quantities of the two resources which should be used, for

$$
(2.51)\quad
\begin{aligned}
X_1 &= [(2b_4 P_1 + b_5 P_2)P_y^{-1} - (2b_1 b_4 + b_3 b_5)]\ (b_5^2 - 4b_2 b_4)^{-1} \\
X_2 &= [(2b_2 P_2 + b_5 P_1)P_y^{-1} - (2b_2 b_3 + b_1 b_5)]\ (b_5^2 - 4b_2 b_4)^{-1}
\end{aligned}
$$

the particular form of production function under analysis, when capital is not limiting. The procedure has been illustrated with a particular form of production function and two input variables.[6] But the same procedure applies to n input variables. The marginal products, partial derivatives of output with respect to inputs, are obtained and equated to the price ratio. If interaction between inputs is not present in the production function equation, the amount of each factor can be solved directly from its partial derivative equated to the price ratio. If interaction is present, the n partial derivative equations must be solved simultaneously. With more than two variables for the function used we would need to invert the matrix of coefficients to solve simultaneously for the unknowns.

Inputs Under Limited Capital

If funds are limited, the criteria of minimum inputs and the input levels which maximize rate of return on investment, as outlined for a single resource, can be computed for two resources and may be as relevant as the maximum quantities described in equation 2.51. But with two resources an additional problem exists: namely, using a quantity of each such that the marginal returns on investment are equal for the two. These quantities can be computed by procedures similar to those outlined for equations 2.37. The procedure for determining profit

[6] For n variables, we can write a profit equation where Π is profit, P_i is the price and

$$(a)\qquad \Pi = P_y Y - \sum_{i=1}^{n} P_i X_i - K$$

X_i is the amount of the i-th factor while K is fixed costs, if any. Profit is maximized when the n partial derivatives of the nature of (b) are set to equal zero and the values of X_i are

$$(b)\qquad \frac{\delta \Pi}{\delta X_i} = P_y \frac{\delta Y}{\delta X_i} - P_i$$

determined; from single equations if no interaction and simultaneously if interaction is present. Obviously, setting equation (b) to equal zero, we can derive MPP = factor to product price ratio or $\frac{\delta Y}{\delta X_i} = \frac{P_i}{P_y}$ for all n variable resources, the same form of equation as in equation 2.7 of the text.

maximizing input quantities in equation 2.51, with capital considered to be unlimited, results in the same return on the last dollar invested in both X_1 and X_2. Where capital is unlimited, attainment of this condition for fertilization of a crop acre with two nutrients or for marketing weight of an animal fed n kinds of feeds could be attained for every producing unit and every enterprise on the farm under static conditions. But since uncertainty gives rise to capital restrictions, profit maximizing inputs cannot be applied to each animal, acre or enterprise (i.e., to equate profit derivatives to zero). Profit from the farm as a whole is maximized when inputs are applied to all enterprises so that the ratios $\frac{P_y}{P_i}\frac{\delta Y}{\delta X_i}$ are equal for all inputs to a particular animal or acre and for all individual inputs, X_i, to different producing units and enterprises. Under this condition the last dollar invested in each input for each producing unit and enterprise is the same. Differential equations can be used to express this equilibrium condition but such detailed data, even if uncertainty did not exist, are not available for farms. The most practical empirical procedure for approximating this broad equilibrium condition, considering types of data usually available and the lack of precision necessitated by uncertainty, is linear programming where activities considered are fewer than those supposed by a set of n continuous functions.

More typically, a farmer with limited capital may wish to answer such questions as the following: "Given two or more input categories how much of each should I use when I have a fixed amount of capital to invest in them?" In other words, how much of two nutrients should he apply to an acre when he has an amount of capital equal to C. He cannot attain the per acre profit maximizing quantities explained above. But he should use an amount of each nutrient such that return on the last dollar spent on each is equal. Without capital limitations, the profit equation is equation 2.52 where P_i is the price and X_i is the

$$(2.52)\quad \Pi = P_y Y - (P_1 X_1 + P_2 X_2 + \dots P_n X_n) = P_y Y - \sum_{i=1}^{n} P_i X_i$$

quantity of the i-th resource. Of course, in solving this quantity, the production function equation is substituted for Y with the partial derivative for each factor equated to zero to give

$$(2.53)\qquad P_y \frac{\delta Y}{\delta X_i} - P_i = 0 .$$

But when capital is limited to some level C and we wish to equalize the marginal return per dollar invested in each factor, the profit equation must be modified as in equation 2.54 where λ is a Lagrange multiplier,

$$(2.54)\qquad \Pi = P_y Y - \sum_{i=1}^{n} P_i X_i + \lambda\left(\sum_{i=1}^{n} P_i X_i - C\right)$$

C is the fixed capital level and P_i and X_i are as defined above. Again in actual calculations, we would substitute the computed production function equation for Y. The term $\sum_{i=1}^{n} P_i X_i - C$ indicates the restraint that the total amount spent, $\sum_{i=1}^{n} P_i X_i$, on the factors can be no greater than the amount of capital C. When $\sum_{i=1}^{n} P_i X_i$ and C are equal, then the term is zero. To determine the profit maximizing amount of each input (i.e., with return on the last dollar spent on each equal), we take the partial derivatives of profit with respect to each input category and λ and set them to equal zero. Hence for equation 2.54 we would substitute in the production function equations for Y, compute the partial derivatives and set them equal to zero as:

$$
\begin{aligned}
\frac{\delta \Pi}{\delta X_1} &= P_y \frac{\delta Y}{\delta X_1} - P_1 + \lambda P_1 = 0 \\
\frac{\delta \Pi}{\delta X_2} &= P_y \frac{\delta Y}{\delta X_2} - P_2 + \lambda P_2 = 0 \\
&\vdots \\
\frac{\delta \Pi}{\delta X_n} &= P_y \frac{\delta Y}{\delta X_n} - P_n + \lambda P_n = 0 \\
\frac{\delta \Pi}{\delta \lambda} &= \sum_{i=1}^{n} P_i X_i - C = 0 \ .
\end{aligned}
\tag{2.55}
$$

These equations can be solved for the X_i and λ to determine the magnitudes of inputs which maximize profits under the restraint set out above. With a number of resources and interaction terms in each of the differential equations, we would invert the matrix of known coefficients as outlined for system 2.37 and solve the X_i and λ values from the resulting system of scalar equations as in system 2.41. The value of λ can be obtained by solving one of the equations for it, with this value of λ substituted into the other equations. Having solved for these values of the X_i, they can be substituted back into the original production function equations and its marginal product equation to indicate respectively the expected yield and the marginal productivities, both physical and value, for the various nutrients or resources.

The inputs specified are, of course, on an *isocline* defining the minimum cost mix of factors for the particular output. Marginal value productivities of resources, relative to prices of factors, are always equal when the least-cost mix has been specified along an isocline. The method outlined above simply indicates where, along the isocline, input

expansion should stop when capital available for purchase of inputs is limited. This is easily shown by setting the marginal rate of substitution, the partial derivative of one factor with respect to the other, for two factors, X and Z, equal to their inverse price ratio:

$$\frac{\delta X}{\delta Z} = -\frac{P_z}{P_x} . \tag{2.56}$$

Since the marginal rate of substitution is the (minus and inverse) ratio of the marginal products, the same condition is expressed in equation 2.57.

$$-\frac{\delta Y}{\delta Z} \Big/ \frac{\delta Y}{\delta X} = -\frac{P_z}{P_x} . \tag{2.57}$$

From equation 2.57, we can derive equation 2.58.

$$\frac{\delta Y}{\delta Z}\frac{1}{P_z} = \frac{\delta Y}{\delta X}\frac{1}{P_x} . \tag{2.58}$$

The latter states that the ratio of output is equal relative to the value of input for each factor.

m Products and n Factors

The procedure outlined above, or that explained for the problem of equation 2.35, also can be used for m products and n factors. In this case the profit function is equation 2.59

$$\Pi = \sum_{j=1}^{m} P_{yj}Y_j - \sum_{i=1}^{n} P_{ij}X_{ij} + \lambda\left(\sum_{i=1}^{n} P_{ij}X_{ij} - C\right) \tag{2.59}$$

where P_{yj} is the price of the j-th product, Y_j, P_{ij} is the price of the i-th resource, X_{ij}, used for the j-th product, λ is a Lagrange multiplier, and C is the fixed capital amount available for expenditure on all factors used for all products. Profit is at a maximum when the partial derivative for each factor used for each product is equated to zero as in equation 2.60.

$$\frac{\delta \Pi}{\delta X_{ij}} = 0 \tag{2.60}$$

Hence, computing the (m x n) + 1 differential equations for manageable types of functions, if there is one for each of the n factors used on each of the m products plus one for the Lagrange multiplier, we can solve for the X_{ij} values as outlined above; inverting the coefficient matrix and computing the scalar equations. While not all factors need to

be used on all products, the same procedure still applies. But as mentioned previously, linear programming or related approaches may provide sufficient accuracy for decisions on inputs allocated to agricultural products where price and weather uncertainty is large. Sufficient precision, within this imperfect knowledge situation, may be attained by defining a number of discrete points on isoquants and input-output curves as separate activities.

TENANT INPUTS

Optimum input quantities under situations of limited capital have been defined above. Another situation calling for slight modification of principles defining optimum input quantities is that of rented farms. Numerous arrangements exist on the proportion of (a) the price of inputs paid, and (b) the share of the product received by tenants or landlords. We could define optimum allocation under all of the conditions outlined above for an owner. However, we will restrict the example to the case of input quantities to maximize profit from a fixed producing unit. Suppose that the tenant (landlord) receives a share of output equal to s and pays a proportion of input price equal to r. Thus, his share of total output is the production function multiplied by s, or

$$Y = sf(X_1, X_2 \ldots, X_n) \tag{2.61}$$

and the share of marginal product to him is the full marginal physical product of each factor multiplied by s as in equation 2.62

$$s \cdot MPP_i = s \cdot \frac{\delta Y}{\delta X_i} \tag{2.62}$$

for the i-th factor. Hence, his task is to equate this marginal product to the factor to product price ratio multiplied by r for all factors as in equation 2.63.

$$s \cdot \frac{\delta Y}{\delta X_1} = r \cdot \frac{P_1}{P_y},\ s \cdot \frac{\delta Y}{\delta X_2} = r \cdot \frac{P_2}{P_y}, \ldots, s \cdot \frac{\delta Y}{\delta X_n} = r \cdot \frac{P_n}{P_y} \tag{2.63}$$

His optimum magnitude of marginal product is thus

$$MPP_i = \frac{\delta Y}{\delta X_i} = \frac{r}{s}\frac{P_i}{P_y} \tag{2.64}$$

for the i-th factor. Setting the marginal product to equal this quantity, the magnitude of X_i can be solved by the procedures outlined previously. If the tenant pays the entire price or cost of factors and receives only a half share of the product, so that r is one and s is 0.5, the tenant's optimum level of input for the i-th factor is defined by a marginal

product equal to twice that of an owner. Or we can generalize as follows: If $r > s$, the tenant marginal product should be greater than for owner. If $r = s$, the tenant marginal product should equal that of owner. If $r < s$, the tenant marginal product should be smaller than for owner.

Minimizing Cost of Output and the Optimum Resource Mix

Under conditions of capital limitations, the problem of the input mix to be used for a given output is especially relevant. The minimum cost combination of resources is determined by equating the marginal rate of substitution of resources to their inverse price ratio. Hence, for the production function, equation 2.49, we can derive isoquant equation 2.65.

$$(2.65)\quad X_1 = \frac{b_1 + b_5X_2 \pm \sqrt{4b_2(a - Y) + 4b_2b_3X_2 - 4b_2b_4X_2^2 + (b_1 + b_5X_2)^2}}{2b_2}$$

It expresses resource X_1 as a function of resource X_2 for a specified level of output, Y. It shows the amount of X_1 which must be used for a given level of X_2 if output is to be maintained at some fixed level Y. The marginal rate of substitution of X_2 for X_1 is the derivative of equation 2.65 with respect to X_2. With the marginal rate of substitution equation equated to the price ratio, we obtain equation 2.66 where P_1 is

$$(2.66)\qquad \frac{\delta X_1}{\delta X_2} = -\frac{b_3 - 2b_4X_2 + b_5X_1}{b_1 - 2b_2X_1 + b_5X_2} = -\frac{P_2}{P_1}$$

the price per unit of X_1 and P_2 is the price per unit of X_2.[7] Solving for X_1 in terms of X_2 from equation 2.66, we obtain the isocline equation in equation 2.67. It indicates the amount of X_1 which should be used for

$$(2.67)\qquad X_1 = \frac{(b_1P_1^{-1}P_2 - b_3) + (2b_4 + b_5P_1^{-1}P_2)X_2}{(1 + 2b_2P_1^{-1}P_2)}$$

the factor prices P_1 and P_2 if the output represented by Y in equation 2.65 is to be produced at minimum cost and X_2 is to be at the specified value. This same point was illustrated earlier by equating the derivative of the isoquant to a constant substitution rate.

Equation 2.67 defines the amount of X_1 which is optimum, in

[7] The equality in equation 2.66 obviously is that obtained if we divide the marginal product equation of equation 2.50 for X_2 by the marginal product equation for X_1 in equation 2.50. Performing this division, we have

$$(a)\qquad \frac{\delta Y}{\delta X_2}\Big/\frac{\delta Y}{\delta X_1} = \frac{P_2}{P_y}\Big/\frac{P_1}{P_y} = \frac{P_2}{P_y}\cdot\frac{P_y}{P_1}$$

and since the P_y's cancel out on the right of (a), we have, without the minus sign, equation 2.66. Obviously, then, if factor inputs are employed to maximize profits as explained for equation 2.50, the corresponding output also is produced at minimum cost as explained for equation 2.66.

minimizing cost of a given output, when X_2 also is defined for a given output. Hence, we can substitute this value of X_1 from equation 2.67 into the isoquant equation 2.65. The resulting equation then includes only X_2 as an unknown and we can ascertain its value, for the stated Y value, from the known coefficients. Substituting this value of X_2 into the isocline equation 2.67, we solve for X_1. The result indicates the magnitudes of X_1 and X_2 which minimize costs for a given output. If we have already computed a large number of X_1 and X_2 input pairs corresponding to a given Y or output level, and if the corresponding marginal rates of substitution have been computed, we also can pick the substitution rate which is nearest the factor price ratio in question. This method would suffice where only approximate values are required.

COST CURVES

While not so frequently used for these purposes, production functions also can be used to compute cost curves and to provide knowledge of some quantities underlying supply functions. First, we illustrate the relationship between production functions and cost functions when a single resource is variable.

For convenience in computations, we use the production function in equation 2.68

$$(2.68) \qquad Y = aX^bZ^d$$

but suppose that Z is at some fixed level and that aZ^d is therefore a constant equal to c. The single variable production function in equation 2.69 results.

$$(2.69) \qquad Y = cX^b$$

From equation 2.69, we now derive the requirements equation in 2.70,

$$(2.70) \qquad X = c^{-\frac{1}{b}}Y^{\frac{1}{b}}$$

indicating the amount of X required to produce a specified level of output, Y. Now total cost, T, is the sum of fixed costs K plus the product formed by multiplying the per unit price, P_x, of the factor by its quantity as in equation 2.71.

$$(2.71) \qquad T = K + P_xX$$

Substituting the value of X from equation 2.70 for that in equation 2.71, we obtain the short-run total cost function in equation 2.72

$$(2.72) \qquad T = K + c^{-n}Y^nP_x$$

where cost is expressed as a function of output and, for convenience, we define $n = \frac{1}{b}$. Since average cost is equal to total cost divided by total output, we divide equation 2.72 by Y to obtain equation 2.73.

$$A = KY^{-1} + c^{-n}Y^{n-1}P_x \tag{2.73}$$

Here A, average cost per unit of output, is expressed as a function of output level Y. It represents a short-run cost function. A will decline until the magnitude of $c^{-n}Y^{n-1}P_x$, the variable cost per unit, is greater than KY^{-1}, the fixed cost per unit.[8]

From the total cost function we can derive the short-run marginal cost function, M, marginal cost being the derivative of total cost with respect to output. Hence, we obtain

$$M = \frac{dT}{dY} = nc^{-n}Y^{n-1}P_x \ , \tag{2.74}$$

the equation of marginal costs, as the derivative of equation 2.72.[9] Both equation 2.73, the average cost, and in equation 2.74, the marginal cost

[8] An alternative in deriving the short-run cost function is this: From the production function in equation 2.68, express X as a function of Z and Y as in (a).

$$X = a^{-\frac{1}{b}} Z^{-\frac{c}{b}} Y^{\frac{1}{b}} \tag{a}$$

Now total cost is the sum of prices of factors times their quantities or $T = P_x X + P_z Z$ and average cost is $A = (P_x X + P_z Z)Y^{-1}$, or total cost divided by output. Now substituting the value of X from (a) into the total and average cost functions we obtain (b) and (c).

$$T = (a^{-\frac{1}{b}} Z^{-\frac{c}{b}} Y^{\frac{1}{b}})P_x + P_z Z \tag{b}$$

$$A = (a^{-\frac{1}{b}} Z^{-\frac{c}{b}} Y^{\frac{1}{b}-1})P_x + P_z ZY^{-1} \tag{c}$$

If Z is fixed at a particular level while X is variable and we let $a^{-\frac{1}{b}} Z^{-\frac{c}{b}} P_x = m$ and $P_z Z = K$, the average cost equation is (d), with cost per unit expressed solely as a function of

$$A = mY^{\frac{1}{b}-1} + KY^{-1} \tag{d}$$

output Y. This is the same as the average cost equation 2.73 if we suppose that $c^{-n} P_x = m$.

[9] Profit from the fixed factor or plant is maximized when the marginal cost equation 2.74 is equated with marginal revenue. Under the competitive conditions of agriculture, marginal revenue is identical with average revenue or price. Hence, when the equality in (a) holds

$$nc^{-n}Y^{n-1}P_x = P_y \tag{a}$$

true, profit is maximized, just as it is for the condition of equation 2.14 where the marginal product is equated to the factor to product price ratio. As an illustration, we set the marginal product, the derivative, for equation 2.69 equal to the price ratio in (b). Equations (a)

$$bcX^{b-1} = P_x / P_y \tag{b}$$

and (b) then state the same thing, except the former is in terms of level of Y while the latter is in terms of level of X. Substituting the value of Y from equation 2.69 into (a), we obtain (c)

when X is variable, will depend on the level at which Z is fixed, since the value of c, as indicated for 2.69, is $c = aZ^d$. The cost function in equation 2.72 is a short-run cost function for the particular production function in equation 2.69 because Z is at the fixed input level specified earlier. The costs attached to Z at the fixed level represent part of the fixed costs in K for equation 2.71. K also may include other fixed costs, such as those attaching to use of any nonzero quantity of Z. A different short-run cost function, such as that of average cost in equation 2.73, when X is variable will exist for each level at which Z is fixed. Also, as is evidenced by the presence of P_x in the three cost functions above, a different cost curve will exist for each level of price for the variable factor.

Static Supply Basis

The marginal cost function in equation 2.74 would provide the basis for the short-run supply function if capital were not limited, uncertainty of production and price were absent, and the producer otherwise made decisions under perfect knowledge. Profit is at a maximum, for a competitive firm as in agriculture, when the marginal cost is equal to product price. Hence, we set marginal cost from equation 2.74 to equal product price, P_y, in equation 2.75.

(2.75) $$nc^{-n}Y^{n-1}P_x = P_y$$

But since a supply function indicates the amount which will be produced at each product price level, the short-run supply function is more aptly expressed as in equation 2.76.

(2.76) $$Y = [(n^{-1}c^n P_x^{-1})P_y]^{\frac{1}{n-1}}$$

For a given fixed plant, equation 2.76 shows what output, Y, should be produced for each level of product or factor price. Derived from a production function, it would give some notion of the possible algebraic relation between production conditions in agriculture and short-run supply functions. But in practice, production coefficients and prices are variable and uncertain. Too, other restraints and nonprofit goals are in operation. Hence, the actual response function will seldom correspond exactly, although it is related, to the computed supply function

---

(c) $$nc^{-n}(cX^b)^{n-1}P_x = P_y$$

(d) $$\frac{P_x}{P_y} = \frac{bcX^b}{X}$$

which reduces to (d) and is equivalent to (b). We have derived the equality in (b) from that in (a).

corresponding to a given production function. In general, because of uncertainty and discounting, actual response is expected to be less than that postulated from a curve such as in equation 2.76. A different short-run "supply basis" will exist for each level at which Z in equations 2.68 and 2.69 is set because the magnitude of c in equation 2.76 is affected accordingly. The "supply basis" curve will "flatten" and "move to the right" as Z increases.

Static Factor Demand

Similarly, a factor demand function, showing the amount of resource which should be used for each level of factor price, can be derived from the production function. Derived thus it has the same modifications in uncertainty restraints and goals mentioned above. Under perfect knowledge and profit maximization, it would specify the quantity of factor which should be used at each level of factor price. It can be derived by these steps: From the production function we equate the marginal product, or derivative of total product with respect to factor input, with the factor to product price ratio. For the production function in equation 2.69 this is accomplished by equating its first derivative to the price ratio as in equation 2.77.

$$bcX^{b-1} = P_x / P_y \tag{2.77}$$

Dividing both sides of equation 2.77 by bc, taking the (b-1) root, and letting $(b^{-1}c^{-1}P_y^{-1})^{\frac{1}{b-1}} = k$, we have the factor demand function equation 2.78 where X, input level, is a function of factor price, P_x. (Stated

$$X = kP_x^{\frac{1}{b-1}} \tag{2.78}$$

otherwise, factor demand is also a function of product price.) Since, under normal short-run situations diminishing returns hold true, and b or the elasticity coefficient is less than 1, $\frac{1}{b-1}$ will be negative. Hence, if we let $\frac{1}{b-1} = -m$, the particular function takes the form $X = kP_x^{-m}$, obviously indicating that as P_x, factor price, increases the level of X will decrease. The factor demand function for this particular algebraic form of production function will have negative slope, as will hold true for any production function with an elasticity coefficient less than 1.0. But, as is suggested later in Chapter 3, the factor demand curve for this particular function is asymptotic to the P_x or factor price axis. In contrast, the factor demand curve for the production function in equation 2.11 is

$$X = .5c^{-1}b - (.5c^{-1}P_y^{-1})P_x \,, \tag{2.79}$$

a straight line which intersects the factor price or P_x axis. The magnitude of the short-run "demand function" for factor X again will depend on the magnitude at which Z in equation 2.68 is fixed. For a greater fixed level of Z, the derived "demand curve" will be further to the right.

Long-Run Cost Functions

Long-run cost functions also can be derived from production functions, although the algebraic manipulations are more difficult than for short-run cost functions. We can use equation 2.68 to illustrate the steps.[10] First, we express factor requirements for X and Z as in equations 2.80 and 2.81, respectively. With total cost equal to the sum of

$$X = a^{-\frac{1}{b}} Z^{-\frac{d}{b}} Y^{\frac{1}{b}} \tag{2.80}$$

$$Z = a^{-\frac{1}{d}} X^{-\frac{b}{d}} Y^{\frac{1}{d}} \tag{2.81}$$

factor prices times factor quantities as in equation 2.82, we can

$$T = P_x X + P_z Z \tag{2.82}$$

substitute the values of X in equation 2.80 into equation 2.82 to obtain equation 2.83. If there are fixed costs not attached to quantities of X

$$T = P_x (a^{-\frac{1}{b}} Z^{-\frac{d}{b}} Y^{\frac{1}{b}}) + P_z Z \tag{2.83}$$

and Z used, we add K, the sum of such fixed costs, to equation 2.83. Total cost is now expressed as a function of Z and output, Y. For the long-run cost curve, we first need to determine the magnitude of Z which will allow a minimum cost for any specific output, Y. Hence, we can set Y at the desired levels in equation 2.83, then take the derivative of total cost with respect to input Z and set it to equal zero as in equation 2.84.

$$\frac{dT}{dZ} = -db^{-1}a^{-\frac{1}{b}} Z^{-\frac{d}{b}-1} Y^{\frac{1}{b}}P_x + P_z = 0 \tag{2.84}$$

[10] For this function, an alternative method might be preferable: First derive the equation of marginal rate of factor substitution. Set it to equal k, to define an isocline, and solve for X in terms of Z. Substitute this expression in Z into the production function (2.68) and the cost function. Now solve for Z in terms of Y. Substitute this expression in Y into the total cost function just defined. Cost then is expressed alone as a function of output.

With the derivative equated to zero, we can solve for Z, indicating the quantities of this input which minimize cost for the particular output. Setting the derivative equal to zero and solving for Z, we obtain

$$Z = \left(d^{-1}\, ba^{\frac{1}{b}}\, P_x^{-i}\, P_z Y^{-\frac{1}{b}}\right)^{-\frac{b}{d+b}} \tag{2.85}$$

which expresses input, Z, of one factor as a function of the price of the two factors and output. Having obtained this value of Z which minimizes costs for the given output, we can compute the corresponding magnitude of X. It can be computed directly from equation 2.80, or it can be computed from the isocline equation for the production function in equation 2.68 as $X = bkd^{-1}Z$ where we know k, the price ratio of the factors, as well as the magnitude of Z from equation 2.85. Substituting the quantities of X and Z so computed into the long-run cost equation of equation 2.82, we obtain the total cost for the specified output, as a point on the long-run total cost function. If we wish to obtain long-run average cost, A, it can be computed by dividing equation 2.86 by Y or output, to obtain

$$A = a^{-\frac{1}{b}} P_x Z^{-\frac{d}{b}} Y^{\frac{1}{b}-1} + P_z Z Y^{-1}\,. \tag{2.86}$$

Setting the derivative of A, average costs, with respect to Z equal to zero and solving for Z, the similar long-run average cost quantities can be computed by use of equations 2.80 and 2.82.

PRODUCTIVITY COMPARISONS FOR ECONOMIC DEVELOPMENT

The principles outlined above apply to use of production function quantities in estimating allocations and mixes of resources for profit maximization of the farm as an individual firm. Other uses also can be made of production function estimates. Problems in economic development might be tackled partly through data derived from production function studies if these data could be provided in appropriate quantities and algebraic forms. As an example, we may take an underdeveloped country which needs to increase food output. It has limited facilities for producing or importing fertilizer. Of the given quantity of fertilizer available, what quantities should be allocated to different regions or soil types? If there is a single main crop such as rice and its supply is extremely short in most regions so that transportation costs are not important, the problem is approximately that of maximizing total physical production from the given supply of fertilizer. Hence, if fertilizer production functions could be estimated for a sufficient number of soil regions, the general principles outlined above could be used in establishing the quantity of the resource to allocate to each region. For a single type of fertilizer, the procedure might be as follows where

production functions are estimated on a per acre basis: the number of acres in each region would be multiplied times the per acre production function to give the regional production function. The task is to allocate a quantity of fertilizer to each region so that the marginal physical products are equal among the n regions. Given knowledge of the production functions, we could use the method outlined previously in equation 2.36. For the particular problem, it now appears as system 2.87 where $\frac{dY_i}{dX_i}$ is the marginal product, derivative of output in respect to fertilizer input, for the i-th region. This quantity in each region is set to equal m, so that the marginal productivity of the resource is equal in

$$
\begin{aligned}
&\frac{dY_1}{dX_1} = m \\
&\frac{dY_2}{dX_2} = m \\
(2.87) \qquad &\quad \vdots \\
&\frac{dY_n}{dX_n} = m \\
&X_1 + X_2 + \ldots X_n = \overline{X}
\end{aligned}
$$

the n regions. There are n such differential equations. In addition, an equation of resource restraints is added, so that the total of input used in each region, X_i, does not exceed the total available, $\overline{X}$. Hence, there are n + 1 equations and n + 1 unknowns: n unknowns to represent the amount of resource to be allocated to each of the n regions and one unknown to represent the magnitude of marginal product, m, to be attained in all regions. Arranging the systems of equations into the form AX = S where A is a matrix of known coefficients from the production functions, X is a column vector with elements representing the unknowns X_i, and m and S is a vector of known constants from the production function and resource supplies, the procedures of equations 2.40 and system 2.41 could be applied to determine the quantity of fertilizer to allocate to each soil region. An alternative procedure would be one paralleling the method of equation 2.54 and illustrated in equation 2.88 for the present

$$(2.88) \qquad Y_t = \sum_{i=1}^{n} Y_i + \lambda\left(\sum_{i=1}^{n} X_i - \overline{X}\right)$$

problem. We wish to maximize Y_t, total output, from the total resources available. It is the sum of the outputs, Y_i, from the n regions. But we must add the restraint represented by the last term of equation 2.88 where λ is a Lagrange multiplier, $\sum_{i=1}^{n} X_i$ is the sum of the resource used in the n regions and $\overline{X}$ is the total quantity available. Now

substituting the actual regional production functions $Y_i = f_i(X_i)$ into equation 2.88 for the Y_i, we take the partial derivatives of Y_t in respect to all X_i and λ and for functions where this derivative exists, equate them to zero as:

$$
\begin{aligned}
&\frac{\delta Y_t}{\delta X_1} = 0 \\
&\frac{\delta Y_t}{\delta X_2} = 0 \\
(2.89) \qquad &\vdots \\
&\frac{\delta Y_t}{\delta X_n} = 0 \\
&\frac{\delta Y_t}{\delta \lambda} = \sum_{i=1}^{n} X_i - \overline{X} = 0
\end{aligned}
$$

Now, solving as for equation 2.55, we again obtain the quantity of resource to be allocated to each region. The fertilization rates per acre and per farm would then be established accordingly. The resulting quantities are those which would maximize output from the total resource supply.

If two or more resources with predetermined fixed supplies are available, the same general procedures apply. In applying the method of equation 2.87 for two nutrients or input categories, we would now have n differential equations for each nutrient and two equations for resource restrictions, or a total of 2n + 2 equations. We would also have 2n + 2 unknowns, with 2n of these representing the amounts of the two resources to allocate to each region and two representing the unknown marginal products to be equated among regions for each resource. (Since stocks of each are separately predetermined, marginal products of the two factors need not be equal.)

Some underdeveloped countries have food problems somewhat approaching that outlined above. However, the allocative problem is not this simple in others because deficit and surplus regions, transportation costs of factors and products and manufacturing costs of factors must be considered — even where food is short and the goal is to produce a specified quantity of a particular crop. Once the problem is extended even to include only discrete quantities required for consumption at different localities, opportunities for production in different regions, and transportation costs between origin and destination, the problem can become computationally complex. Models of continuous functions similar to those for equations 2.36 and 2.54 can be outlined, to solve for quantities to be produced and consumed and resources to be used in each region if costs are to be minimized. But linear programming or transportation models, using "discrete" quantities derived from production function estimates, may provide ample refinement. These

models would be set up with restrictions requiring that consumption requirements be "exactly met" in each region and that planted acreage be equal to or less than acreage available in each region. Different crop producing activities would be used in each region to represent different mixes or levels of fertilizer which could be used and fertilizer purchasing activities might be added for each region.

Productivity Comparisons

Estimates of firm production functions usually are used for diagnostic purposes, in judging the degree of factor disequilibrium under different degrees of economic and market development. If appropriate samples and algebraic models could be devised, these functions also could be used to estimate equilibrium quantities and proportions of resources. However, practically all such studies have employed Cobb-Douglas forms of functions. As outlined in the next chapter, functions of this algebraic form which assume constant elasticity, generally overestimate equilibrium input quantities under the criterion of profit maximization for a firm. Consequently, firm production functions based on the power-type equation have been used mainly to estimate marginal value productivities for mean inputs of resources based on farm samples. Marginal value productivities computed at the mean then may be used to indicate whether disequilibrium in resource use is great or small. One criterion is the magnitude of the marginal value productivity of a unit of a particular resource as compared to the per unit price of the resource. Since farm production functions typically are derived with value rather than physical output representing the dependent variable, the comparison is made simply as follows: all other resources are set at a magnitude equal to their mean. The marginal value product of the resource being studied, computed as the derivative of output in respect to input, is then derived. For example, suppose we have the production function in equation 2.90 where V is value of output and X_i represents different resources such as labor, land, and capital of different types. With X_i considered to be the variable resource in question, its marginal product at its mean is that in equation 2.91 where

$$V = aX_1^{b_1} X_2^{b_2} X_3^{b_3} X_4^{b_4} \tag{2.90}$$

$$\frac{\delta V}{\delta X_i} = \frac{b_i}{\bar{X}_i} (a\bar{X}_1^{b_1} \bar{X}_2^{b_2} \bar{X}_3^{b_3} \bar{X}_4^{b_4}) = \frac{b_i}{\bar{X}_i} V \tag{2.91}$$

the bar indicates inputs at mean levels. With V computed from equation 2.90 when all factors are fixed at their sample mean, we multiply this quantity times b_i, the coefficient of X_i, and divide by $\bar{X}_i$, to obtain the marginal product of the factor, X_i, when input is at mean level. In turn, these marginal productivities could be compared with the respective factor prices to suggest the degree of disequilibrium in resource use.

Also, from the computed marginal value product, we can derive the quantity, with other inputs at mean level, necessary to cause productivity to equal factor price. For equation 2.91, this is accomplished by setting the derivative or marginal product to equal the factor price P_i, and solving for X_i where the latter is not at mean level. The magnitude of X_i, the input of the particular variable factor to equate marginal value productivity and price when other resource inputs are at mean magnitude, is that of equation 2.92.

$$X_i = b_i V P_i^{-1} \tag{2.92}$$

Of course, if appropriate procedures could be employed in sampling, resource classification, and aggregation and algebraic form of equation, the equilibrium quantities of all resources could be specified. This would be accomplished by procedures outlined previously; namely, by taking derivatives such as equation 2.91 for all factors, equating them to their respective prices and then solving the system of simultaneous equations for the value of the X_i.

Or, if it is supposed that the allocative problem is more nearly one of the distribution of resources within the agricultural economy rather than in relation to the total economy as reflected in factor and product markets, these marginal value productivities might be computed for different samples of farms, representing different sectors in respect to tenure, soil, climate, location, or product specialization. They then might be compared, to determine the degree of resource imbalance among the sectors being compared. If interest were in maximizing the value of product from the collection of resources already existing in agriculture, the differences would suggest the relative degree of resource adjustment required. If the data and empirical method were appropriate, a system of equations could be derived, in which the marginal value product, m_i, of the i-th resource is equal for all regions. We could then solve for the magnitude of the i-th resource to be allocated to each region in a manner paralleling that for equation 2.87. If we simply wish to know by how much a specific mobile resource such as labor for one farm sector should be increased or decreased to cause its marginal productivity to equal that of the same resource in another sector, we could use the following procedure. First, compute the marginal value product of the factor for each region, as in equation 2.91 where we suppose that other factors are fixed in quantity. Then equate these two marginal value products as in equation 2.93 where subscript i

$$\frac{b_{i_1} V_1}{X_{i_1}} = \frac{b_{i_2} V_2}{X_{i_2}} \tag{2.93}$$

refers to the i-th resource and subscripts 1 and 2 refer to two regions. Input in region 2 is held at mean level while input in region 1 is variable. We wish to determine the magnitude of X_{i_1}, if the equality of equation 2.92 is to hold true. The value is that of equation 2.94, derived

(2.94) $$X_{i1} = \frac{b_{i1} V_1 \overline{X}_{i2}}{b_{i2} V_2}$$

from equation 2.93, indicating the input or X_{i1} level if the marginal value productivity of the factor in region 1 is to equal the marginal value productivity of the same factor in region 2, when all other inputs are at mean value and the particular factor also is at mean level in region 2. But this procedure is useful only where the form of function employed, the specification of input categories and sampling variance do not lead to serious errors for estimates of productivities from non-mean quantities of inputs. Farm production functions of the Cobb-Douglas type derived under usual circumstances are not very appropriate for these purposes.

SCALE RETURNS AND PRODUCT IMPUTATION

Farm or firm production functions also may be computed to determine the magnitude of returns to scale. Scale returns are measured by the ratio, per cent increase in output divided by per cent increase in input, under the condition that all factors be increased by the same proportion. For this measure to be applicable, there must not be omission of important input categories or any serious aggregation problems.

For the Cobb-Douglas function, the elasticities of the individual factors are their exponents in the production function or regression equation. These individual elasticities can be summed to determine the scale coefficient, ϵ, indicating the percentage by which output will be increased if all factors are increased by 1 per cent. For a function such as equation 2.90, constant returns to scale hold true if $b_1 + b_2 + b_3 + b_4 = 1.0$. A 1 per cent increase in input of all factors will cause output to increase by 1 per cent. If $\sum_{i=1}^{n} b_i < 1.0$, decreasing returns to scale hold true while for $\sum_{i=1}^{n} > 1.0$, increasing returns to scale hold true. Thus for equation 2.68, suppose that $b = 0.3$ and $d = 0.4$. We increase input of each factor by the proportion R and the function becomes

(2.95) $$Y = a(RX)^{.3}(RZ)^{.4}$$

which by the rule of exponents is

(2.96) $$Y = aR^{.3}X^{.3}R^{.4}Z^{.4} \quad \text{or} \quad Y = aR^{.3+.4}X^{.3}Z^{.4}$$

and output only increases by the proportion of $R^{.7}$, rather than by R as in the case of factors, because the sum of individual elasticities is only 0.7. Had the individual elasticities been $b = 0.2$ and $d = 0.8$, then output would have increased by the proportion R, as input of both factors is

increased by this proportion. As indicated in the next chapter, other forms of functions must be employed to estimate scale returns when elasticity coefficients are not constant over all ranges of inputs. However, the general statements on scale returns and their measurement hold true for all algebraic forms of production functions.

Imputation

Estimation of production functions by Douglas and his associates centered around imputation problems; the division of a total output among the factors used in producing it. Agricultural economists long have divided total outputs in this manner, although somewhat arbitrarily and without knowledge of the underlying production functions. Examples present themselves in land appraisal where net income imputed to land is capitalized into a land value and division of income between landlord and tenant, on the basis of the share of inputs contributed by each. Hence, this section is devoted to a brief summary of the conditions allowing imputation of a product among the factors producing it, so as to just exactly exhaust the product.

Of course, a total product can be allocated to its factors in an infinite number of ways, to just exhaust the total output. But the economists' concern has been in allocating the product in proportion to the marginal products and the quantities of resources. It can be shown that under competitive conditions an allocation of factors rewarded in this manner maximizes output from resources available. Hence, we examine the conditions under which an imputational procedure based on marginal resource productivities can be accomplished. Again, using the production function in equation 2.68, we determine the magnitude of output necessary to reward the two factors according to their marginal productivities. This is determined by computing the marginal products of the two factors from equation 2.68 and multiplying them by the quantities, X and Z, of the two resources, respectively, as in expression 2.97. Multiplying out these products, we obtain the sum in expression 2.98, which can be simplified to $(b + d)aX^bZ^d$. Substituting the Y value

$$(baX^{b-1}Z^d)X + (daX^bZ^{d-1})Z \tag{2.97}$$

$$baX^bZ^d + daX^bZ^d \tag{2.98}$$

from the production function, equation 2.68 in this case, this becomes $(b + d)Y$. Hence, if the sum of b and d is less than 1 for equation 2.68, then $(b + d)Y$ is less than Y, the total product, and the quantity imputed to each factor in expression 2.97 will not exhaust output. If $b + d$ is greater than 1.0, the quantity of expression 2.97 will be greater than Y, and the amount of product is in deficit relative to the amount allocated to the factors. With $b + d = 1$, then the quantity in expression 2.97 is equal to Y and the marginal productivity method of rewarding factors

just exhausts the total product. While a power function has been used to illustrate these conditions, the same principles apply to other forms of functions.

CHOICE UNDER UNCERTAINTY

The general principles discussed thus far assume the parameters of production functions and prices of products to be known with certainty. This condition, of course, is unreal: variance attaches to the estimates of productivity coefficients even for a single year and a given weather condition. Weather itself varies in different years; and prices, especially those of products, are uncertain. Various decision-making principles can be outlined for these situations and range from informal precautionary measures to the more formalized mathematical models of game theory and decision models. No attempt will be made to outline these principles, however, because an entire treatise would be required for this purpose. Explanations of them are given elsewhere.[11] Problems of variance quantities related to estimating productivity coefficients are discussed in later chapters.

Presence of uncertainty does not eliminate the need for use of appropriate and correct economic principle. Physical scientists have long made recommendations as if price and production parameters were known with certainty. Examples are recommendations to fertilize at particular levels or to use specific mixes or rations or feeds for livestock. These recommendations not only suppose that the relevant quantities are known without error, but also that all farmers have the same profit goals, capital availability, or tenure situation. Too, economists have used crude average productivity comparisons drawn from census or farm sample summaries, to suggest resource disequilibrium or allocation needs. The data are applied within the framework of principles as if the quantities were known with certainty. Quite obviously, problems of uncertainty are of no less magnitude when "rule of thumb" principles and discrete or point estimates of input to output ratios are used, as compared to correct economic principle and formal knowledge of production functions. The purpose of this chapter has been to outline and summarize the correct economic principles which can be applied to production function estimates.

As pointed out previously, the cost per unit of knowledge for physical phenomenon is probably no greater, and often less, when complete production functions are estimated, as compared to a similar number of treatments directed toward a few point estimates. Presence of uncertainty perhaps best applies to the degree of refinement needed in

[11] See Heady, Earl O. and Candler, Wilfred V. Linear programming methods. Iowa State University Press, Ames. 1958. Pp. 499-527; Luce, R. D. and Raiffa, H. Games and decisions. John Wiley and Sons, New York. 1957. Pp. 278-308; Dillon, J. L. Theoretical and empirical approaches to program selection within the feeder cattle enterprise. Journal of Farm Economics. 40: 1921-31. 1958.

estimating the production functions and in applying the relevant principles. Because price and productivity coefficients are uncertain, a precise definition of input equilibrium is impossible. Hence, once a reasonably accurate estimate of the production function has been completed, a sufficient number of points may be selected from it, with the appropriate principle applied accordingly. In this case, we may simply compute average marginal products between discrete inputs as $\frac{\Delta Y}{\Delta X_i}$ where ΔY is increase in total output and ΔX_i is the increase in total input, between the two discrete input levels. Input is then increased as long as the marginal product so defined is greater than the price ratio, or

$$\frac{\Delta Y}{\Delta X_i} > \frac{P_x}{P_y}. \tag{2.99}$$

Similarly, the substitution rate between two factors X and Z may be defined as $\frac{\Delta X}{\Delta Z}$ between two input levels of Z. Input of Z then should be substituted for X as long as the substitution ratio is greater than the price ratio, or

$$\frac{\Delta X}{\Delta Z} > \frac{P_z}{P_x}. \tag{2.100}$$

Similar modifications can be made for other principles of choice outlined on previous pages. The only modification is substitution of the marginal ratios (as averages between discrete input levels), $\frac{\Delta Y}{\Delta X_i}$ and $\frac{\Delta X}{\Delta Z}$ for the respective derivatives $\frac{\delta Y}{\delta X_i}$ and $\frac{\delta X}{\delta Z}$. The choice then is between the discrete input levels and ratios relative to which the marginal ratios are defined. If extended to sufficiently complex choice or allocation problems, programming procedures are probably most convenient for specifying input and output mixes and magnitudes. If the relevant quantities can be provided in sufficient detail through experiments and samples designed to provide point estimates, there is no reason why they should not be used as substitutes for continuous functions.

CHAPTER 3

Forms of Production Functions

NUMEROUS ALGEBRAIC equation forms can be used in deriving production functions. Some were reviewed in Chapter 1. No single form can be used to characterize agricultural production under all environmental conditions. The algebraic form of the function and the magnitudes of its coefficients will vary with soil, climate, type and variety of crop or livestock, resources being varied, state of mechanization, magnitude of other inputs in "fixed quantity" for the firm, etc. Hence, a problem in each study is selection of an algebraic form of function which appears or is known to be consistent with the phenomena under investigation. Guides on appropriate algebraic forms may come from previous investigations and the theories of the sciences involved. Selection of any specific type of equation to express production phenomena automatically imposes certain restraints or assumptions in respect to the relationships involved and the optimum resource quantities which will be specified. However, some equations are more flexible than others. To illustrate this fact, algebraic evaluation is made in this chapter of a few general types of production functions. Suggestions also are made of phenomena where these might be empirically appropriate. An infinite number of functional forms are possible in productivity studies but those considered in following sections either (a) have logical implications which cause them to "stand out" from the others or (b) have been widely used in production function studies. Some of the equations examined have not had wide application because their parameters or coefficients are difficult to derive in statistical treatment of data. Some have terms which cannot be transformed, or are not readily transformed into linear regression equations and, hence, can be estimated only by iterative processes. However, a few of these are examined to illustrate the range in technical coefficients and conditions which are allowed by different types of functions.

SINGLE VARIABLE EQUATIONS

Numerous research studies in agriculture revolve around production functions with a single resource or treatment applied at different levels. Different levels of applying nitrogen or insecticides to crops

are examples. Hence, we will use production functions of this nature as a step toward a more general analysis of functional forms. Equations with a single variable defining the input can be used to evaluate certain properties of production functions and their marginal products. These same properties generally apply when there are n variables.

Obviously, however, output is never greater than zero when only one production factor is used. Appropriately, the production function should be represented as

$$Y = f(X_1, X_2, \cdots X_n) \tag{3.1}$$

where Y is the output and the X_i are the inputs. In general, given the existence of the production function, the following quantities can be derived and are of direct importance in economic application:

$$\frac{\delta Y}{\delta X_i} = f'_{x_i}(X_1, X_2, \cdots X_n) \tag{3.2}$$

$$\frac{\delta X_i}{\delta X_j} = -\frac{f'_{x_j}(X_1, X_2, \cdots X_n)}{f'_{x_i}(X_1, X_2, \cdots X_n)} \tag{3.3}$$

$$X_i = f''(Y, X_1, \cdots X_n) \tag{3.4}$$

$$\frac{\delta X_i}{\delta X_j} = -k \tag{3.5}$$

$$\frac{\delta X_i}{\delta X_j} = 0 \tag{3.6}$$

These physical quantities represent the data necessary for economic application of results from production functions. We wish to evaluate the characteristics of these relationships as they are derived from various functional forms. In review, they are as follows: equation 3.2 is the equation of marginal physical products for the i-th resource, equation 3.3 is the equation of marginal substitution rates between the i-th and j-th resources, equation 3.4 is the equation of isoquants, equation 3.5 is the equation of isoclines, and equation 3.6 is the equation of ridgelines. If equation 3.3 is substituted into equations 3.5 and 3.6, it is obvious that any of these quantities for one resource depend on the entire range of resources which can be used in the particular production process. It is with this qualification that we first turn to evaluation of production functions which involve a single variable resource.

It is, of course, possible to hold certain categories of inputs at fixed levels while others are variable. However, certain resources or input categories are exogenous in the sense that they are the "result of outside forces," and are not subject to control by the decision-maker or research worker. Thus where only resources $X_1, X_2, \cdots X_g$ can be

controlled in magnitude, then a random disturbance or effect also is associated with the variables $X_{g+1}, X_{g+2}, \cdots X_n$. Expressing this random disturbance as ϵ, the production function, without control of some factors, is

$$Y = g(X_1, X_2, \cdots X_g) + \epsilon . \tag{3.7}$$

We may use a vertical bar as below to indicate that only one factor, X_i, is variable while others are held fixed at some predetermined level.

$$Y = \phi(X_i | \cdots X_g) + \epsilon \tag{3.8}$$

Similar conventions could be used for two or three variable factors. However, to simplify presentation, and with the above recognition of the existence of variance or random disturbances which attach to predictions, we will present functions in the form $Y = f(X)$ where a single-variable production function is under consideration, or $Y = f(X_1, X_2)$ where two factors are variable in quantity while others are in fixed magnitude or are exogenous. Too, while there are few cases where a single input category is variable, we employ the convention used in some agricultural experiments and suppose that it is possible. It may be possible to add labor in a production process, without varying the quantity of other production factors. However, even a simple capital resource such as nitrogenous fertilizer cannot be varied, without simultaneous variation in labor or other capital inputs. But regardless of this, useful interpretation can be made from a single input equation of the algebraic assumptions implicit in the use of various types of equations for estimating production functions.

Cobb-Douglas

The Cobb-Douglas or power function, in the form generally used, is

$$Y = aX^b \tag{3.9}$$

where X is the variable resource measured, Y is output, a is a constant, and b defines the transformation ratio when X is at different magnitudes. The exponent or b coefficient is the elasticity of production and can be used directly. (The equation is estimated in logarithmic form.) This function allows either constant, increasing, or decreasing marginal productivity. It does not allow an input-output curve embracing all three. With all other inputs held in fixed magnitude, the marginal product is expected to decline. The marginal product equation is

$$\frac{dY}{dX} = baX^{b-1} = \frac{baX^b}{X} \tag{3.10}$$

indicating that if $b = 1$, the marginal product (and also the average product) will be constant at the level a. Where $b > 1$, the magnitude of marginal products will increase as X increases, depending on the magnitude of b. If $b = 2$, for example, the marginal products are ba, 2ba, 3ba, and 4ba when X has the respective values 1, 2, 3, and 4. Where $b < 1$, the magnitude of marginal products will decline as X increases since $X^b < X$.

This function assumes a constant elasticity of production, E_p, over the entire input-output curve, or that

$$(3.11) \qquad \frac{dY_1}{dX_1}\frac{X_1}{Y_1} = \frac{dY_2}{dX_2}\frac{X_2}{Y_2} = \cdots = \frac{dY_n}{dX_n}\frac{X_n}{Y_n}$$

where the subscripts refer to marginal products and total outputs corresponding to various magnitudes of X. This condition of the equation, that successive equal increments of input add the same percentage to total output, can be proved by multiplying the derivative or marginal product equation 3.10 by the inverse of the average product (the definition of production elasticity) as shown below.

$$(3.12) \qquad E_p = (baX^{b-1})\frac{X}{Y} = \frac{baX^b}{X} \cdot \frac{X}{Y}$$

Now substituting the value of Y of equation 3.9 into equation 3.12 we obtain

$$(3.13) \qquad E_p = \frac{bY}{X} \cdot \frac{X}{Y}$$

and since the Y's and X's cancel, we have $E_p = b$, or the elasticity is a constant equal to the exponent of X in equation 3.9.

Given the mathematical properties of the equation, this function cannot be used satisfactorily for data where there are ranges of both increasing and decreasing marginal productivity. Neither can the function be used satisfactorily for data which might have both positive and negative marginal products. Also, since the rate of decline in the marginal product decreases with input magnitude, the power function provides a curve of the nature indicated in Figure 3.1. The curve "flattens out" as input increases and a maximum product is not defined. Unless an economic optimum is defined for small magnitudes of input, the power function especially may overestimate the input of X which equates marginal revenue and marginal cost.

When Y measures total output, the equation assumes the resource to be limitational and that output is zero when input is of zero magnitude. With Y measuring yield added by the variable resource, the factor is not assumed to be limitational.

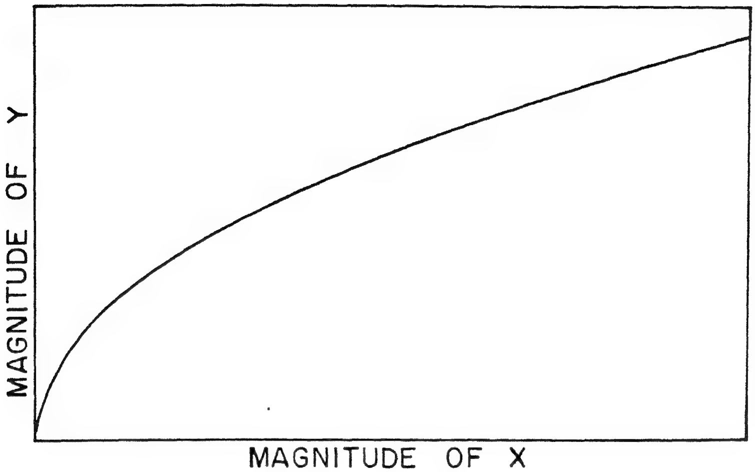

Figure 3.1. Graphic illustration of function computed from equation $\log Y = \log a + b \log X$.

Spillman Function

The exponential-type function suggested by Spillman will be evaluated in the form of equation 3.14

$$Y = M - AR^X \tag{3.14}$$

where Y again measures total output and X measures total input. The other coefficients have this meaning: M is the maximum total output which can be attained by use of the variable resource, A is the total increase in output which can be attained by increasing X, M - A is the level of output defined by fixed resources and a zero input of the variable resource, R is a constant defining the ratio of successive increments to total product. Hence, R also defines the magnitude of the marginal product of input level X_i relative to that of input level X_{i-1}. From equation 3.14 the equation of marginal products is

$$\frac{dY}{dX} = -AR^X \log_e R \tag{3.15}$$

and the marginal product of the i-th input bears the following relationship to the marginal product of the previous input:

$$\frac{dY_i}{dX_i} = R\left(\frac{dY_{i-1}}{dX_{i-1}}\right). \tag{3.16}$$

In equation 3.16, the subscripts on the terms of the derivatives again define levels of input of resource X and the marginal products associated with each. Hence if R = 0.8 and the marginal product for X_{i-1} is 10, the marginal product of X_i is (.8)(10) = 8.

The total product curve of equation 3.14 is asymptotic to M. The marginal product curve is asymptotic to the zero axis, never becoming negative as might be the case of fertilizer used in excess. For this reason, the function is not appropriate for samples drawn from experiments or surveys where input magnitudes are great enough to cause a decline in total product.

The Spillman-type curve in Figure 3.2 illustrates these properties. The increments in product due to increments in inputs are indicated as Δ_i. The product curve approaches the maximum M as X increases and AR^X decreases. If used as a response function, the equation is

$$(3.17) \qquad Y = A(1 - R^X)$$

and the input axis has origin at M - A in Figure 3.2. Similarly, yield then becomes asymptotic to A, rather than M.

Quadratic Forms

The simple quadratic equation in equation 3.18, with a minus before c to denote diminishing marginal returns, does not impose such strict restraints on the production function as does the Cobb-Douglas or

$$(3.18) \qquad Y = a + bX - cX^2$$

Spillman equation. It allows both a declining and negative marginal productivity, but not both increasing and decreasing marginal products. A maximum total product is defined where input magnitude or X is equal to $.5bc^{-1}$. The elasticity is not constant as in the power function, but declines with input magnitude as indicated by the elasticity equation:

$$(3.19) \qquad E_p = \frac{bX - 2cX^2}{a + bX - cX^2}.$$

The marginal products do not bear a fixed ratio to each other as in the case of the Spillman function. However, as indicated by equation 3.20,

$$(3.20) \qquad \frac{dY}{dX} = b - 2cX$$

use of the quadratic equation does assume a particular characteristic in relationship between marginal products; namely, that they decline by a constant absolute amount or $m_i = m_{i-1} - k$ where m_i is the marginal product of the i-th input, m_{i-1} is the marginal product of the i-1 input

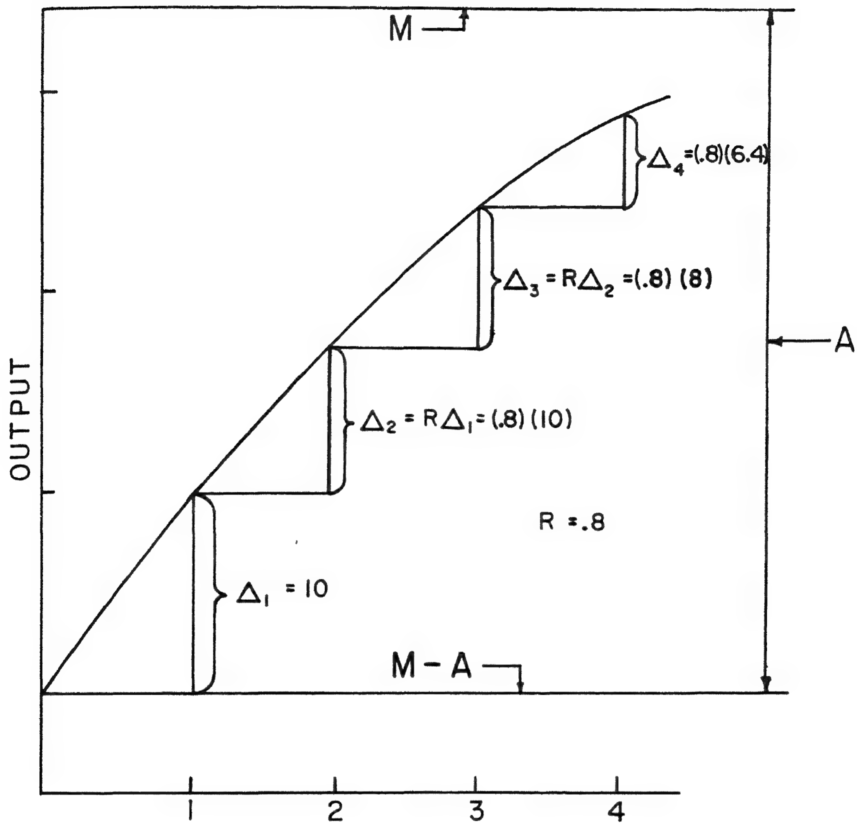

Figure 3.2. Nature of Spillman-type exponential function.

and k (k = 2c) is the constant by which successive marginal products decline. In other words, the marginal product curve is linear (the second derivative being a constant equal to -2c) and the total product forms a "mirror" curve, with the portion on the right of the maximum being the "mirror" of the portion on the left.

The constant a in equation 3.18 represents the product forthcoming from the mix of fixed resources. If Y is used to measure response or output due to the variable resource only, we expect that a is zero. This is also true for other polynomial forms where the exponent or degree of X differs from 1 and 2.

Modifications of the polynomial equation in equation 3.18 may be used to relax the restraint that marginal products decline by a constant amount. For example, using the cube rather than the square for X in equation 3.18 will cause the marginal productivity of the variable resource to decline at an increasing rate (i.e., the second derivative is -6cX).

$$Y = a - bX + cX^{.5}. \tag{3.21}$$

This square root equation 3.21 provides a simple compromise between the power or exponential equations and the quadratic form in equation 3.18. Supposing that all express diminishing marginal productivity, equations 3.9 and 3.14 have marginal products which decline at a diminishing rate, but never allow a declining total product. The equation 3.18 allows a diminishing total product but assumes a constant decline in marginal products. The square root equation 3.21 allows a diminishing total product but also, as equation 3.22 indicates, has marginal products which decline at a diminishing rate. This particular

$$\frac{dY}{dX} = .5cX^{-.5} - b \tag{3.22}$$

relationship may be found under certain biological conditions. The marginal product may be great at low input levels, and decline at slower rates as input level increases; although a sufficiently large dosage may depress total yields. This equation has a maximum total product when X equals $.25c^2b^{-2}$. Its elasticity declines with increases in input and output magnitudes.

The appropriate functional form for estimating production relationships will, of course, depend on the type of phenomena being examined. Ordinarily, those represented by crop response to fertilizer will have diminishing marginal products for all inputs greater than zero. But there are isolated instances where increasing marginal products have been encountered for initial inputs of nutrients. Production functions relating feed input to live weight output of livestock evidently have diminishing marginal products for all input levels. A function allowing some range of increasing marginal products may be needed, however, in estimating output of edible dressed products. In meat production, one does not expect negative marginal product with age of animal and total feed input during the growing and fattening period and may select an equation accordingly. Under many circumstances, except for potash or the single variable under certain soil and climatic conditions, a function allowing negative marginal products would be needed for fertilizer application. However, under normal conditions, one would never expect a single input category such as labor or machine capital to lead to a depressed total output where an entire farm represents the fixed plant for which the production function is being derived.

Resistance Formula

A modified statement of the equation proposed by Balmukand, based on Maskell's "resistance formula," is expressed in equation 3.23.[1]

[1] Balmukand, B. Studies in crop variation. V. The relation between yield and soil nutrients. Journal of Agricultural Science, 18: 602-27. 1928.

(3.23) $$Y^{-1} = a(b + X)^{-1} + c$$

Here a, b, and c are constants and, in the case of a fertilizer function, b might be the amount of nutrient in the soil while X is the amount added. This is an asymptotic function with a maximum yield of c^{-1} and supposes only positive marginal products, as suggested by the marginal product equation 3.24 which has an asymptote of zero. This equation,

(3.24) $$\frac{dY}{dX} = a[a + c(b + X)]^{-2}$$

like that of Spillman and Mitscherlich, assumes that different input factors have effects independent of each other but assumes a marginal relationship such that there are differences among successive responses.

Hyperbolic and Other Functions

Numerous forms of hyperbolic equations have been proposed to describe crop-fertilizer production functions. Most of these are more difficult to fit empirically than quadratic equations, although they possess somewhat similar properties. For example, hyperbolic equation 3.25 has a marginal product curve similar in nature to that of the

(3.25) $$Y = aX(b + X)^{-1} - cX$$

square root equation in 3.21. It has the marginal product equation shown in equation 3.26.

(3.26) $$\frac{dY}{dX} = ab(b + X)^{-2} - c.$$

The marginal product declines at a decreasing rate with a maximum total product at the input level where X equals $\sqrt{abc^{-1}} - b$.

The hyperbolic equation suggested by Thilau in equation 3.27 has the nonlinear marginal product equation 3.28.

(3.27) $$Y = \sqrt{aX + bX^2}$$

(3.28) $$\frac{dY}{dX} = \frac{a + 2bX}{2\sqrt{aX + bX^2}}$$

It has a marginal product which becomes zero at an X value of $-\frac{a}{2b}$.

Increasing Marginal Productivity

A production function embracing both increasing and decreasing marginal productivity is seldom needed for a single variable resource. Where needed, a function allowing this condition is the polynomial

$$Y = a + bX + cX^2 - dX^3 . \tag{3.29}$$

It has increasing marginal products until X is equal to $.3333cd^{-1}$, then diminishing but positive marginal products until

$$X = .3333d^{-1}[c \pm (3db + c^2)^{.5}] \tag{3.30}$$

which is the value of X that maximizes total product. Supposing a to be zero, the elasticity is greater than 1 (stage 1) for $0 < X < .5cd^{-1}$, equal to 1 (constant returns) at $X = .5cd^{-1}$, and less than 1 but greater than zero (stage 2) for $X > .5cd^{-1}$. Marginal products decrease at an increasing rate in the later stage.

Several other types of quadratic equations allow these various stages. Other algebraic forms serve similarly. One proposed by A. N. Halter, *et al.*,[2] is somewhat a hybrid or combination of the power and exponential equations. It is

$$Y = cX^a e^{bX} \tag{3.31}$$

where Y and X measure output and input respectively, a, b, and c are coefficients to be estimated and e is the base of natural logarithms. This function has increasing marginal products up to $X = b^{-1}(-a \pm a^{.5})$, the input defining the point of inflection on the input-output curve. It then has declining but positive marginal products to $X = -ab^{-1}$, the input defining the maximum total product. The elasticity of production is 1.0 at $X = (1 - a)b^{-1}$. Hence, stage 1, with marginal products greater than average products, extends over the input range $0 < X < (1 - a)b^{-1}$ and stage 2, with marginal products less than average products but greater than zero, extends over the range $(1 - a)b^{-1} < X < -ab^{-1}$. The logistic function in equation 3.32 has both increasing and decreasing marginal productivity of the variable resource X. The constants a, b, and c

$$Y = \frac{a}{1 + be^{-cX}} \tag{3.32}$$

are parameters to be estimated from sample observations and e is the base of natural logarithms. Again, this is an asymptotic type curve, with upper asymptote at a, and would not be appropriate for observations characterizing a diminishing total product. The corresponding marginal product equation is equation 3.33.

[2]A note on the transcendental production function. Jour. Farm Econ., 39: 966-74. 1957.

(3.33) $$\frac{dY}{dX} = abce^{-cX}(1 + be^{-cX})^{-2}$$

Marginal product is a maximum where X is equal to c^{-1} log b, the inflection point of the total product curve. It has changing elasticity, with the elasticity coefficient equal to unity at

(3.34) $$X = b^{-1}c^{-1}(b + e^{cX}) .$$

For values of X less than this, production is in stage 1, with an elasticity of production greater than 1; for all values of X greater than this, production is in stage 2 since the elasticity coefficient never becomes negative.

n RESOURCES

Where parameters are to be estimated for more than one variable resource, selection of an equation consistent with known empirical relationships or relevant theories also is involved. As in the case of a single resource, selection of a particular equation implies some assumption about the algebraic nature of the production process. To further illustrate the types of assumptions or restraints imposed by different functions, we evaluate a few of the more commonly used equations in this section. The same algebraic restraints exist for n variables, but we restrict the analysis to two input variables, X_1 and X_2, for purposes of simplicity.

Power Functions

The power function in equation 3.35 has the same mathematical characteristics discussed for equation 3.9, for either resource, when input-output curves are derived for one resource with the other held

(3.35) $$Y = aX_1^{b_1} X_2^{b_2}$$

constant. The assumptions of constant elasticity and marginal products with only a plus or minus sign, regardless of input or output magnitudes, are retained. Again, the regression coefficients, b_1 and b_2 derived with the observations in logarithms, are the production elasticities of the individual resources. Their sum indicates the nature of returns to scale, provided X_1 and X_2 are the only relevant inputs. With the sum $b_1 + b_2 = 1$, a given percentage increase in both inputs will result in an equal percentage increase in output. With elasticity sums being more or less than 1, output will increase by a greater or smaller percentage, respectively, than inputs.

If the sum of elasticities is less than 1, the power function provides

a surface which has no distinct peak. It slopes down to the input axes and tends to form a broad ridge in output space. It never forms a precipice over an input axis, as would be the case where a factor is nonlimitational. Because the isoclines "fan out" in linear fashion from the origin, the point of the wedge representing the surface comes at zero input of both factors, with the wedge broadening along the input axes as the surface rises to greater heights. As mentioned for the one factor input-output line, the surface has only positive slopes when the elasticities are greater than zero (and only a negative slope when they are less than zero).

The isoquants of this production function are asymptotic to the input axes, as suggested in isoquant equation 3.36 where X_1 is expressed as a

$$X_1 = \left(\frac{Y}{aX_2^{b_2}}\right)^{\frac{1}{b_1}} \tag{3.36}$$

function of X_2. Any one isoquant can be derived from another simply by multiplying the input quantities by the appropriate ratio. Because of this property, and as suggested in equation 3.35, each resource serves as a limitational input: No output is forthcoming if either X_1 or X_2 is zero. The marginal rate of substitution is, as indicated in equation 3.37, a linear function of the ratio in which X_1 and X_2 are combined. If

$$\frac{\delta X_1}{\delta X_2} = -\frac{b_2 X_1}{b_1 X_2} \tag{3.37}$$

inputs of X_1 and X_2 are increased in constant proportions, the marginal rate of substitution remains constant at the ratio b_2/b_1, even though level of output changes. This condition is unrealistic for two classes of inputs such as carbohydrate and protein feeds for a growing and fattening animal. At earlier ages, it is known that the marginal rate of substitution of high protein for high carbohydrate feeds is highest when the animal is young, and declines as the animal ages and the growing stage merges into the fattening stage. Hence, the marginal rate of substitution will not remain constant as inputs of two feeds are increased in constant proportion (a specific ration) as the animal gains in weight. Evidence also suggests that the marginal rate of nutrient substitution declines with yield level on a given land area. However, it is entirely possible that rates of factor substitution may remain constant as all inputs of the production process are increased in the same proportion for a firm. While the power function may characterize the production process for certain firm conditions, it is less appropriate for a fixed producing unit represented by an acre, animal, or bird.

The isocline equation, derived from substitution rate equation 3.37 by setting the latter derivative to equal a constant $-k$ (to represent a given marginal rate of substitution or price ratio) and solving for one input quantity in terms of the other, is:

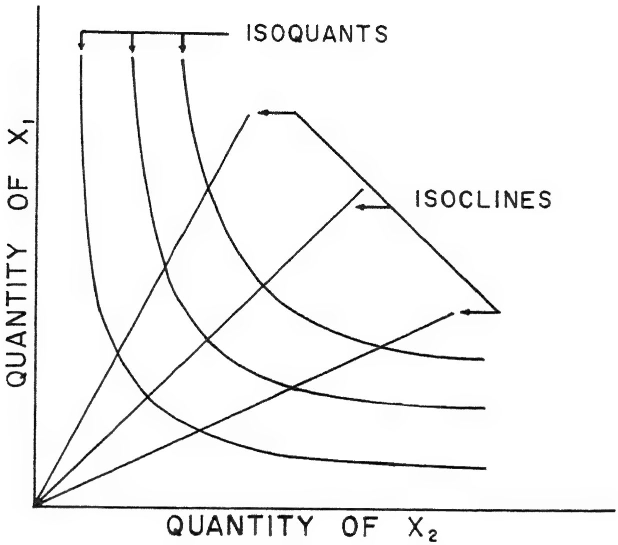

Figure 3.3. Geometric nature of isoquants and isoclines for Cobb-Douglas function.

$$X_1 = b_1 b_2^{-1} k X_2 \quad . \tag{3.38}$$

The quantity of X_1 is thus expressed as a function of X_2. Hence equation 3.38 indicates the quantity of X_1 necessary to provide a rate of substitution of k magnitude when X_2 is given various values. Again, this is a linear equation, indicating that the isoclines are straight lines passing through the origin. A map indicating the geometric nature of isoquants and isoclines "forced" by the power function is included in Figure 3.3. Since the isoclines are straight lines passing through the origin, they also are scale lines, indicating a fixed proportion or mix of the two inputs used at different levels. Because of these characteristics, the power function denotes that the ratio in which the two resources are combined should remain the same regardless of the level of output. The optimum magnitude of input and output changes as the price of the product changes relative to the price of the inputs, but the optimum input ratio does not change if the factor price ratio remains constant. The optimum ratio of factors does change, however, as the factor price ratio changes. Ridgelines for this function do not converge at a factor combination defining maximum output. (No maximum is defined by the function.) In fact, the ridgelines, denoting zero rates of factor substitution, are identical with the input axes in a two-dimension contour map of the surface.

The power function in equation 3.35 can be modified to relax the restraints of constant elasticity over the surface and isoclines that denote

the same ratio of factor inputs regardless of output level. One modification is accomplished in equation 3.39 by adding a constant to each input magnitude.

In logarithms, this function does not reduce to a linear equation. Hence, the added constants may be estimated, with numerous iterations used to determine which magnitudes give smallest deviations from regression. The corresponding equation of marginal substitution rates

$$(3.39) \qquad Y = a(h_1 + X_1)^{b_1}(h_2 + X_2)^{b_2}$$

can be defined as in equation 3.40. When equated to $-k$, equation 3.40

$$(3.40) \qquad \frac{\delta X_1}{\delta X_2} = -\frac{b_2(h_1 + X_1)}{b_1(h_2 + X_2)}$$

allows derivation of the isocline equation in equation 3.41. In this isocline equation, X_1 is still a linear function of X_2, with the slope being

$$(3.41) \qquad X_1 = kb_1b_2^{-1}h_2 - h_1 + kb_1b_2^{-1}X_2$$

$kb_1b_2^{-1}$. However, the isoclines do not pass through the origin but through the X_1 axis if $kb_1b_2^{-1}h_2 - h_1$ is positive and through the X_2 axis if it is negative. Under this modification, the optimum ratio of factors changes as different levels of output are attained.

Since the elasticity coefficients are constants in the Cobb-Douglas production function, the scale returns so indicated simply represent the "average condition" for the sample. If we sum the exponents of the resource variables (regression coefficients with variables measured in logarithms) and the quantity $\sum_{i=1}^{n} b_i$ is less than 1.0, we can only say that, as an average for the sample, decreasing returns to scale hold true provided no relevant inputs are excluded as discussed in Chapter 6. It is possible that increasing returns to scale may hold true for a modest range of inputs, but that the portion of the sample drawn from input ranges characterized by decreasing returns dominates. Or the opposite may hold true, with observations from the range of increasing returns dominating those from the range of decreasing returns.

Spillman Function

In two variables the Mitscherlich-Spillman response function can be evaluated in the form of equation 3.42 where Y is output resulting from the resources being varied, X is the quantity of one factor and Z is

$$(3.42) \qquad Y = A(1 - R_x^X)(1 - R_z^Z)$$

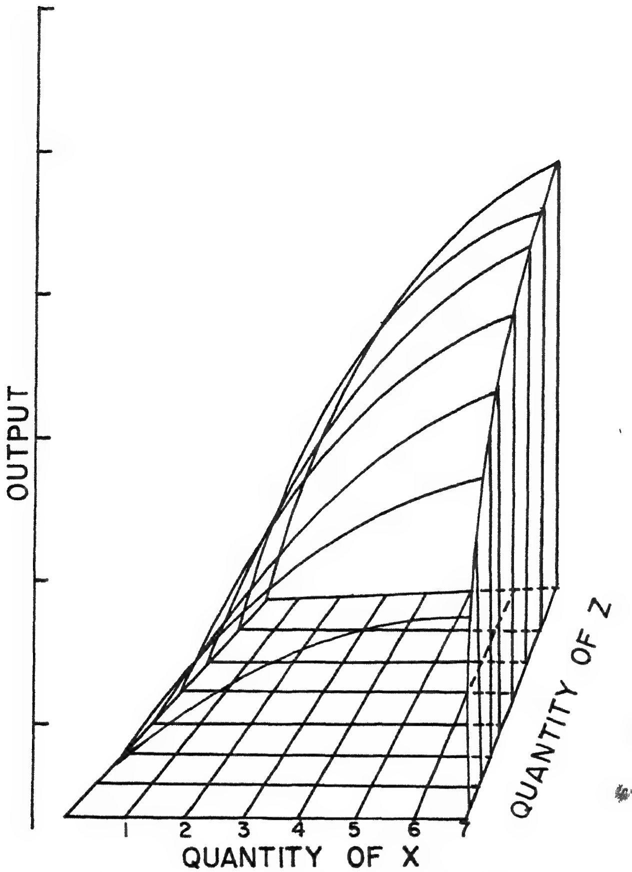

Figure 3.4. Production surface for Spillman or Mitscherlich function.

quantity of another factor. R_x indicates the ratio by which marginal products of X decline while R_z has the same meaning for Z. In this case, A is the maximum response from increasing both factors. The surface has height asymptotic at level A and extends to a long broad "ridge" over the input plane, rather than to a peak at a single point over the input plane. If input of either or both factors is zero, output also is zero. An example of the function is shown in Figure 3.4. The isoquant equation is equation 3.43, indicating that the isoquants are

$$(3.43) \qquad X = \log\left[1 - \frac{Y}{A(1 - R_z^Z)}\right](\log R_x)^{-1}$$

asymptotic to the axes. Hence, one factor can never substitute completely for the other. The isoquants do not, however, maintain a

constant slope (substitution rate) at points of intersection with a scale line extending from the origin. A set of isoquants, the negative sloped curves, illustrating their nature for the Spillman function, is included in Figure 3.5.

The equation of marginal rates of substitution is equation 3.44. Expansion of equation 3.44 indicates, of course, interaction between factors in the sense that the marginal products of the one which is variable depends upon the magnitude in which the other is held fixed. By

$$\frac{\delta X}{\delta Z} = -\frac{(1 - R_x^X)(R_z^Z \log_e R_z)}{(1 - R_z^Z)(R_x^X \log_e R_x)} \tag{3.44}$$

setting equation 3.44 to equal -k, a given price or substitution ratio, the isocline equation becomes equation 3.45 where w has the value

$$X = \frac{\log w}{\log R_x} \tag{3.45}$$

shown in equation 3.46.

$$w = \frac{R_z^Z \log_e R_z}{k(1 - R_z^Z) \log_e R_x + R_z^Z \log_e R_z} \tag{3.46}$$

From equation 3.45, it is evident that if Z is zero, X also is zero indicating that the isoclines pass through the origin. Too, it is evident from this equation that they are not straight lines as in the case of the Cobb-Douglas function. Figure 3.5 indicates that, while the isoclines — the curves with positive slope — are not straight lines, their slope is of "asymptotic nature," and they generally approach linearity out over the input plane. They do not converge because the surface forms a ridge, rather than a peak. Because the isoclines "bow out" from the origin, even if they do approach linearity, they specify that a different input mix should be used for each level of output. (The isoclines are not also scale lines.)

Quadratic Forms

Extension of the quadratic form in equation 3.18 to two resources results in the production surface equation in equation 3.47.

$$Y = a + b_1X_1 + b_2X_2 - b_3X_1^2 - b_4X_2^2 + b_5X_1X_2 \tag{3.47}$$

Diminishing marginal returns exist for either factor alone but there is positive interaction between the two factors.[3] (Negative or zero

[3] The equation might be presented with only positive signs for the b_i's. However, for fixed plants, it is logical that the signs are negative for b_3 and b_4, and either positive or negative for b_5. Hence, this framework is used in the text, although certain general algebraic relationships are the same regardless of the signs used.

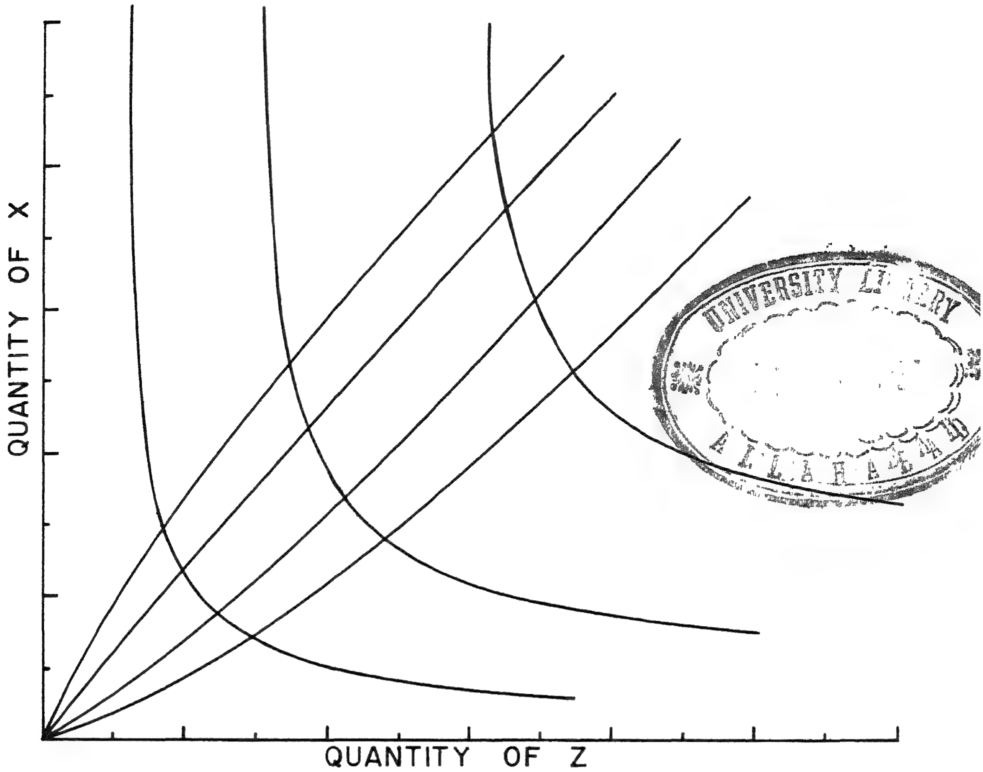

Figure 3.5. Isoquants for Spillman function.

interaction also may exist where diminishing marginal returns hold true for both factors.) The corresponding isoquant equation is equation 3.48. The isoquants are not asymptotic to the input axes, as was true

$$(3.48)\quad X_1 = \frac{b_1 + b_5X_2 \pm [(b_1 + b_5X_2)^2 - 4b_3(Y + b_4X_2^2 - b_2X_2 - a)]^{.5}}{2b_3}$$

for the Cobb-Douglas function. Certain output levels can be attained from input of X_1 alone, with X_2 at zero level, depending on the magnitudes of a, b_1, and b_3. Similarly, certain levels of output can be attained with zero input of X_1 and isoquants are allowed to intersect the input axes of a contour map of the surface. Because of these characteristics, the surface need not slope to zero level at the input axes, but may form a precipice over them. In contrast to the surface formed by the power function, the surface formed by the quadratic equation can have a distinct peak, denoting a maximum output for a single limitational combination of the factors. This limitational combination represents the point in the input plane where the family of isoclines converge

and the output contour, representing the level of maximum output, reduces to a single point.

The equation of substitution rates corresponding to equation 3.48 is

$$\frac{\delta X_1}{\delta X_2} = -\frac{b_2 - 2b_4X_2 + b_5X_1}{b_1 - 2b_3X_1 + b_5X_2}. \tag{3.49}$$

By setting equation 3.49 to equal the constant price ratio or substitution rate, the isocline equation 3.50 is obtained. While the isoclines again

$$X_1 = \frac{kb_1 - b_2}{b_5 + 2kb_3} + \left(\frac{kb_5 + 2b_4}{b_5 + 2kb_3}\right) X_2 \tag{3.50}$$

are linear, they do not impose the same conditions on the production surface and the economic optima in factor combination as do those of the power function. This is true since the isoclines, although linear, are not forced through the origin of the input plane. Only a single isocline, the one representing a substitution rate of k such that

$$\frac{kb_1 - b_2}{b_5 + 2kb_3} = 0 \tag{3.51}$$

does so. In other words, k must be equal to $b_1^{-1}b_2$. If the ratio in equation 3.51 is greater than zero, the isoclines intersect the X_1 axis. If it is less than zero, they intersect the X_2 axis. In general, since they intersect an input axis at quantities greater than zero, they are not also scale lines. As expansion paths, they denote a changing proportion of resources, if the substitution rate is to remain constant at k magnitude as higher levels of output are attained. For this same reason, they indicate that the least-cost path of outputs is attained only if the mix of inputs is changed as the factor to product price ratio changes.

The isocline representing a ridgeline, with a zero marginal rate of substitution of X_2 for X_1, can be derived when the partial derivative shown in equation 3.49 is equated to zero. It has the value

$$X_1 = \left(\frac{2b_4}{b_5}\right) X_2 - \frac{b_2}{b_5} \tag{3.52}$$

and intersects the X_2 axis where the later resource is of input magnitude $X_2 = .5b_2b_4^{-1}$. Similarly, the ridgeline defining a zero rate of substitution of X_1 for X_2 intersects the input axes at $X_1 = .5b_1b_3^{-1}$. The ridgelines generally have a positive slope when the interaction term in equation 3.47 is positive and intersect each other, as is true for all other isoclines, at input values corresponding to the peak of the production surface. If there is no interaction between inputs, isoclines denoting substitution rates greater than zero are positively sloped. However, those isoclines representing zero rates of factor substitution, the ridgelines, form a 90 degree angle at the point of intersection of the

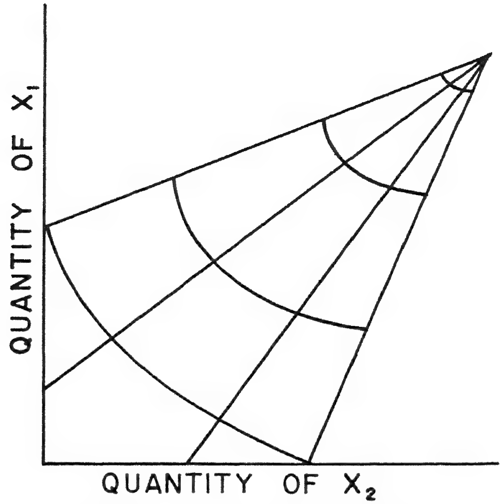

Figure 3.6. Isoquants and isoclines for quadratic function.

isocline family if there is no interaction between inputs. The isocline representing a zero rate of substitution of X_2 for X_1 is parallel to the X_1 axis at the value $X_2 = .5b_2b_4^{-1}$. The one representing a zero rate of substitution of X_1 for X_2 is parallel to the X_2 axis at the value $X_1 = .5b_1b_3^{-1}$. If interaction between resources is negative, isoclines representing nonzero substitution rates can have both positive and negative slopes. The ridgelines have negative slope as defined by the ridgeline equations above when the substitution rates between X_1 and X_2 are zero.

A map of isoquants and isoclines for a quadratic equation with positive interaction between factors is included in Figure 3.6. It provides a geometric illustration of some of the characteristics described above. The isoquants extend only over the range where their slopes are zero and infinite, defining the path of the ridgelines. In this form, with extension of the inputs so that isoclines converge at a "Von Leibig" point, the function would hardly be appropriate for a livestock gain equation where the surface slopes up to a broad plateau, rather than a definite peak.

Square Root

The square root equation in equation 3.53 with positive interaction term allows isoclines which are curved and pass through the origin. In this sense, it is a compromise between such functions as equations 3.35 and 3.47. It does not specify a fixed mix of resources for attaining

$$Y = a - b_1X_1 - b_2X_2 + b_3X_1^{.5} + b_4X_2^{.5} + b_5X_1^{.5}X_2^{.5} \tag{3.53}$$

different output levels, as does equation 3.35, and it does not impose linear isoclines as equation 3.47. The corresponding isoquant equation is that of equation 3.54.

$$(3.54)\quad X_1 = \left[\frac{b_3 + b_5X_2^{.5} \pm \sqrt{4b_1(a - Y - b_2X_2 + b_4X_2^{.5}) + (b_3 + b_5X_2^{.5})^2}}{2b_1}\right]^2$$

It allows product contours nearest the origin in the input plane to intersect the factor axes. The derivative of equation 3.54, the marginal rate of substitution of X_2 for X_1, is provided in equation 3.55.

$$(3.55)\quad \frac{\delta X_1}{\delta X_2} = -\frac{-b_2 + .5b_4X_2^{-.5} + .5b_5X_1^{.5}X_2^{-.5}}{-b_1 + .5b_3X_1^{-.5} + .5b_5X_1^{-.5}X_2^{.5}}$$

The corresponding isocline equation is equation 3.56.

$$(3.56)\quad X_1 = \left[\frac{b_2 - kb_1 - .5h_4X_2^{-.5} \pm \sqrt{2b_5X_2^{-.5}(.5kb_3 + .5kb_5X_2^{.5}) + (b_2 - kb_1 - .5b_4X_2^{-.5})^2}}{b_5X_2^{-.5}}\right]^2$$

From equation 3.56, it is obvious that the isoclines pass through the origin and have positive slope. However, because of the numerous square root terms, the isoclines are not linear and converge to a point in the input plane corresponding to the maximum point on the production surface. For $k = 0$, denoting the zero rates of factor substitution, the ridgelines are curved but intersect the axes at

$$(3.57)\quad X_2 = .25\, b_4^2 b_2^{-2}$$

for a zero rate of substitution of X for X and

$$(3.58)\quad X_1 = .25\, b_3^2 b_1^{-2}$$

for a zero rate of substitution of X_1 for X_2.

A map of isoquants and isoclines for the square root function with positive interaction between the factors is included in Figure 3.7. Again the isoclines converge at an input combination corresponding to the maximum output, the peak of the production surface. As the curved isoclines or expansion paths indicate, the least-cost path to higher outputs denotes a changing mix or ratio of inputs. The input mix should change as product price increases or decreases and output level increases or decreases accordingly.

Other Forms

Other polynomial forms provide a compromise between equations 3.47 and 3.53 in respect to nature of isoclines. One is the function in equation 3.59 where the X_1 are raised to the 1.5 power. The marginal

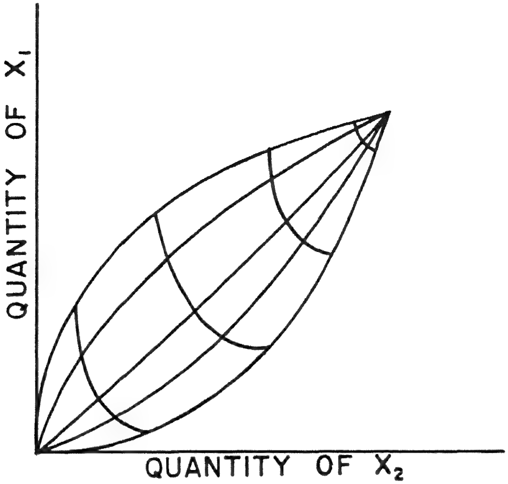

Figure 3.7. Isoquants and isoclines for square root function.

(3.59) $$Y = a + b_1X_1 + b_2X_2 - b_3X_1^{1.5} - b_4X_2^{1.5} + b_5X_1X_2$$

rate of substitution equation, defining slopes of isoquants is equation 3.60.

(3.60) $$\frac{\delta X_1}{\delta X_2} = -\frac{b_2 - 1.5b_4X_2^{.5} + b_5X_1}{b_1 - 1.5b_3X_1^{.5} + b_5X_2}$$

Equating this to -k, the isocline equation in equation 3.61 results.

(3.61) $$X_1 = \left[\frac{-1.5kb_3 \pm \sqrt{4b_5(kb_1 - b_2 + kb_5X_2 + 1.5b_4X_2^{.5}) + (1.5kb_3)^2}}{2b_5}\right]^2$$

Obviously, the value of X_1 can have a nonzero value even when X_2 is zero and the isoclines do not pass through the origin as in equation 3.56. Still they are not linear as in equation 3.50.

A production function combining characteristics of the power and exponential functions is the transcendental function extended to two resources in equation 3.62.

(3.62) $$Y = aX_1^{b_1}e^{c_1X_1}X_2^{b_2}e^{c_2X_2}$$

This function assumes factors are limitational and that Y is zero if either X_1 or X_2 is zero. It has a maximum point at the single peak where one limitational combination of inputs defines the largest output attainable. The isoquant equation in logarithmic form is equation 3.63.

$$\log X_1 + c_1 b_1^{-1} X_1 = b_1^{-1} \log(a^{-1} X_2^{-b_2} e^{-c_2 X_2} Y) \tag{3.63}$$

It has the disadvantage that solutions can only be ascertained by iterative procedures. The corresponding equation of substitution rates is equation 3.64.

$$\frac{\delta X_1}{\delta X_2} = -\frac{(b_2 + c_2 X_2)X_1}{(b_1 + c_1 X_1)X_2} \tag{3.64}$$

Setting equation 3.64 to equal -k, the isocline equation in equation 3.65 is derived. As it indicates, the isoclines pass through the origin, are curved and converge in factor space, defining a maximum physical product, at inputs where the partial derivatives of X_1 and X_2 are zero.

$$X_1 = \frac{kb_1 X_2}{(c_2 - kc_1)X_2 + b_2} \tag{3.65}$$

The function in equation 3.62 allows the surface to display both increasing and diminishing marginal products. Equation 3.47 can be extended to allow this condition by adding a cubed term, for example, for each X_i. The same may be accomplished for equations 3.53 and 3.59 by adding terms of higher order for each resource. The logistic function also allows for increasing and decreasing (but not negative) marginal products.

The "resistance" or Balmukand type production function can be written in the form of equation 3.66 when two input categories are used. A representative surface is included in Figure 3.8.

$$Y^{-1} = a(b + X_1)^{-1} + d(f + X_2)^{-1} + c \tag{3.66}$$

This function does not allow negative marginal products but provides a surface asymptotic to the maximum yield of $\frac{1}{c}$. It has the isoquant equation in equation 3.67.

$$X_1 = \frac{aY(f + X_2)}{(f + X_2) - Yd - Yc(f + X_2)} - b \tag{3.67}$$

The equation of marginal substitution rates is

$$\frac{\delta X_1}{\delta X_2} = -\frac{d(b + X_1)^2}{a(f + X_2)^2}\,. \tag{3.68}$$

Setting it to equal -k, we obtain isocline equation 3.69.

$$X_1 = (\sqrt{kad^{-1}})(f + X_2) - b \tag{3.69}$$

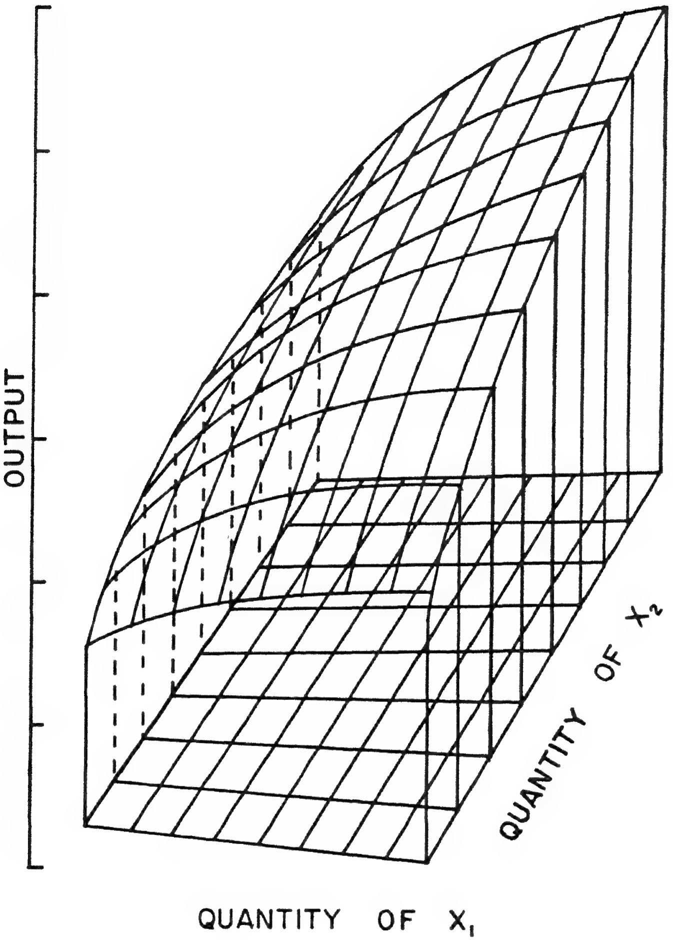

Figure 3.8. Surface representing resistance function.

The isoclines are linear but do not pass through the origin as indicated in Figure 3.9. Hence, the optimum proportion of factors changes as product price increases or decreases relative to factor prices. In other words, different proportions of factors give a constant substitution rate as output is increased to higher levels. The isoclines do not converge to a maximum point.

From the characteristics pointed out above, it becomes apparent that a function of this general nature might well describe a livestock gain function where, with both input categories increasing over the complete factor plane, the function forms a surface which "slopes toward a plateau" or long ridge, rather than to a peak. But equation 3.66 would not be appropriate for a fertilizer response surface where

negative marginal products are encountered and the sample observations include product isoquants which have ranges of both positive slopes (complementary), as well as negative slopes (competition).

Various modifications of the resistance functions are possible. One is shown in equation 3.70

$$\frac{1}{Y} = \frac{a}{X_1} + \frac{b}{X_2} + \frac{c}{X_1 X_2} + d \tag{3.70}$$

where a term has been added to allow factor interaction and the X_1 and X_2 variables are not summed with another constant. A function such as equation 3.70 may apply well where one need not account for that portion of output due to other fixed factors, such as nutrients already in the soil or feed to an animal prior to initiation of the experiment. However, equation 3.70 does not allow negative marginal returns and has an output asymptotic to d^{-1}. Hence, it may be particularly applicable for production functions describing meat production of birds and animals. The isoquant equation corresponding to equation 3.70 is shown in equation 3.71.

$$X_1 = \frac{aX_2 + c}{(Y^{-1} - d)X_2 - b} \tag{3.71}$$

The equation of substitution rates for equation 3.71 is equation 3.72.

$$\frac{\delta X_1}{\delta X_2} = -\frac{(bX_1 + c)X_1}{(aX_2 + c)X_2} \tag{3.72}$$

The isoquants intersect the X_1 axis where X_1 is equal to $-cb^{-1}$. They intersect the X_2 axis where X_2 is equal to $-ca^{-1}$. The isocline equation is equation 3.73.

$$X_1 = \frac{-c \pm \sqrt{c^2 + 4bkX_2(aX_2 + c)}}{2b} \tag{3.73}$$

The isoclines are not linear and do not pass through the origin of the input plane. A particular advantage of equation 3.70 is its computational convenience. It can be estimated directly as a linear regression equation in the form of

$$Y^{-1} = aX_1^{-1} + bX_2^{-1} + cX_1^{-1}X_2^{-1} + d \ . \tag{3.74}$$

APPROPRIATE SURFACE FUNCTIONS

A few of the infinite number of possible production function equations have been examined to illustrate that each supposes some

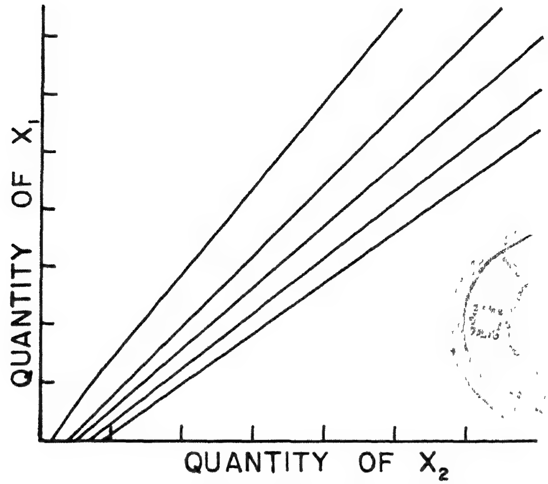

Figure 3.9. Isoclines for a resistance function.

particular property in respect to marginal products, isoquants, marginal rates of substitution, and isoclines. If the production logic, the correct mathematical form, were fully known and data were available for a segment of the surface, the logical function could be fitted to the data. This fitted function then could be used to estimate or predict the whole surface. But if the production logic is totally unknown, the fitted function can only be a "statistical" function, and not a logical function. Hence, it can be used mainly to predict within the range of the observed data. Within the data range, it serves as an approximation to the basic structural phenomena of the process under study. The fitted parameters for the form of the equation may not correspond, variable by variable, to the true structural representation; but taken *in toto* the fitted coefficients have effects that approximate those of the true but unknown parameters, at least within the range of data fitted.

The appropriate form of function to be fitted to a production surface should be selected in terms of the environment and nature of the producing unit for which it is estimated. Biological and production knowledge should provide some basis in selecting forms of functions which are appropriate. A power function with constant elasticities and isoclines which are linear through the origin of the input plane may be useful for certain firm or industry estimates. This might be particularly true where interest revolves only around knowledge of resource productivities at the mean of inputs. However, if the goal is to predict the entire surface and if factor substitution rates change, as well might hold true for labor and capital increased for a fixed firm plant, with larger inputs, other algebraic forms would be more appropriate. Functions estimated from farm samples ordinarily have been of power form because of the smaller number of degrees of freedom involved in

estimating the parameters, and partly because a multiplicative model has seemed logically appropriate. A quadratic or similar form fitted to a farm sample ordinarily proves to have many regression coefficients which are not significant in a probability sense. Usually, only the coefficients for linear terms are significant.

Again, it is quite obvious that if the sample or experiment examined includes a range of increasing marginal products, algebraic forms should be selected accordingly. A modification of the power function to transcendental form as in equation 3.62, or addition of terms of higher power as in equations 3.47 and 3.53, would allow this condition. Similarly, if predictions are to be made for an experiment which indicates a decline in total physical product and hence implies negative marginal productivity, some algebraic forms are not appropriate while others are. A Cobb-Douglas or Spillman function allows only an increasing total product (unless the coefficients are negative, denoting only a decreasing function). If fitted to data of this nature they would generally underestimate the slope of the surface and the magnitudes of marginal products for smaller inputs. If there are reasons for believing that constant elasticity does not prevail, and if knowledge of elasticity and related magnitudes over various parts of the function are important, the power function should not be used. Similarly, the Spillman function should not be used where it is believed that successive marginal products do not bear a constant proportion to each other; nor is the conventional quadratic form, as in equation 3.47, appropriate if a linear marginal product curve seems unrealistic. The same reasoning would apply to use of the "resistance" formula, in finding situations appropriate to the fixed relationships supposed for reciprocals of marginal products.

The surface for meat production from an animal or bird is expected to differ from one for milk production by a dairy cow or fertilizer response for land, where inputs are expanded to the point of maximum physical output from the "fixed" animal or acre. Yet, if the input range does not approach the level defining maximum output, the same or a similar algebraic form might serve nearly as well for meat production as for milk or crop production. Or, a form of function which is appropriate in predicting a fertilizer response surface which rises to a distinct peak might also serve appropriately for meat production where the surface slopes up to a long ridge. The coefficients for the latter, used only to predict over the range of observations, may lead to a surface sloping upward with growing width as is expected to be the case. The production surface appropriate for total gains of meat from a bird or animal ordinarily will not, in relevant growing and fattening periods, have a positively sloped area converging to a peak between ridgelines. Rather, in most cases, the ridgelines will not converge to a point of intersection and the surface will tend to form a long plateau. This plateau widens over the factor plane and along the input axes. These statements apply to total gain or production per bird or animal. On the other hand, if the production period is defined as a day, and output is

measured as daily gain, the resulting surface appears to be defined with converging ridgelines and a definite peak or maximum. The peak of the surface defines the maximum daily gain and ordinarily is possible only with a single limitational ration or resource combination. This surface has certain characteristics paralleling that outlined later for milk.

A fertilizer production function which includes both applied and soil nutrients in the variables, or one relating only to applied nutrients for a pure sand containing no growth elements undoubtedly would have characteristics such as those represented by the isocline family for the square root function, or perhaps for a Spillman function if sample observations do not extend into ranges of negative marginal products. The isoclines, perhaps other than those representing a zero replacement or substitution rate, would necessarily pass through the origin. No yield is possible without some quantity of nutrients required for chemical and physiological processes for growth. To the extent that with non-zero inputs of the various nutrients, the same yield level can be attained with a number of combinations or proportions of nutrients, the isoclines and ridgelines will "spring apart." (Otherwise the surface would reduce to a knife's edge, passing through the origin and indicating that higher yields can be attained only by using more nutrients in the single, limitational ratio.) Then, the isoclines and ridgelines must converge and intersect at inputs corresponding to the maximum height or peak of the yield surface for nutrients which have toxic or yield depressing effects when applied in large quantities. Hence, a production function with characteristics paralleling those of the square root function or the modified power function might be selected for a soil highly deficient in two or more nutrients. But in most cases, except for a pure sand, some quantity of the nutrients ordinarily will be available in the soil. In other words, the basic isocline family, for applied and soil nutrients, might have the nature of those intersecting the origin and point M in Figure 3.10. But, as might be indicated by a soil test, the soil may already contain a quantity of A and B as indicated by point S in Figure 3.10. The soil would still be relatively limited in the two nutrients, however, and the family of isoclines would likely be curved. The square root production function equation allows this characteristic because of the shape of its isoclines.

Suppose, however, that the quantity of nutrients A and B available in the soil is that indicated at Q before addition of fertilizer for the plant. Then a new origin is relevant. This is denoted as Q and the relevant axes are the broken lines. In other words, the available amount of nutrient A is OL and the available amount of nutrient B is ON. A quantitatively accurate soil test would indicate these quantities, denoting that the fertilizer response surface, predicted from fertilizer application, starts from point Q, rather than from point O, in the nutrient plane. In a regression equation including soil-test quantities as observations, coefficients for nutrients already in the soil would be based on quantities ON of B and OL of A. (By including nutrients already in the soil as variables in the regression equation, better knowledge of the production

surface and isocline family should develop.) In cases where the nutrient content of the soil is relatively high, however, and soil-contained nutrients are not employed in the estimates, the production function used for estimating may not provide nonlinear isoclines for the added nutrients. The relevant isoclines to be predicted for the added nutrients are those above and to the right of the dashed axes (with origin Q) in Figure 3.10. Although these portions of the isoclines are curved, the curvature probably would not be great enough to show up in the production function equations. The best-fitting production function may then provide a family of linear isoclines such as those expressed in equation 3.50 for the quadratic form. They intersect at M, the maximum yield per acre. Also, since they are linear and do not intersect the origin, they denote, like those expressed in equations 3.56 and 3.65, that the least-cost nutrient ratio changes as higher yields are attained.

If the response surface estimated has actual origin at S in Figure 3.10 and applied inputs are SC of A and SD of B, a function of the nature in equation 3.70 might be appropriate. It provides isoclines which are nonlinear and which do not converge at either the origin or a maximum yield. While the entire isoclines over the input plane with origin at S are curved, those falling in the observed rectangle SCQD may have so little curvature that it cannot be effectively estimated.

Occasionally, the research worker encounters special properties of production functions which must be considered in his estimates. An example is the milk production surface for dairy cows. While its exact algebraic nature has not yet been established, it is possible to offer some hypotheses about certain of its characteristics. Generally, the limit of a cow's stomach prescribes the maximum amount of an input such as hay which can be fed in a day, month, or year. On the other hand, physiological conditions may specify a minimum amount of hay which she can consume in an extended time period, particularly where the substitute feed is corn or other concentrate. The milk production surface, although not yet fully established, may appear somewhat like Figure 3.11. Diminishing returns exist for transforming any one ration into milk since the surface declines in slope at higher levels of output. Also, diminishing substitution ratios are indicated between feeds since the slope of the milk isoquants changes with feed ratios or rations. Maximum milk production per cow is at level M. Only one ration and level of feeding will result in maximum milk production and is denoted at point G in the feed plane.

Milk is forthcoming if a hay ration alone is fed (i.e., if level of feeding is extended along the forage axis OF). However, the maximum level of milk production, FH, under a pure forage ration is lower than the maximum level GM attainable under the ration following line OG in the feed plane. With the cow fed hay only, OF represents the quantity denoting her stomach capacity. Supposedly, if a cow were fed to the highest milk level possible from a pure hay ration, and if milk level FH were attained, the amount of hay could be reduced along a stomach limit line. By replacing it with successive quantities of grain, milk

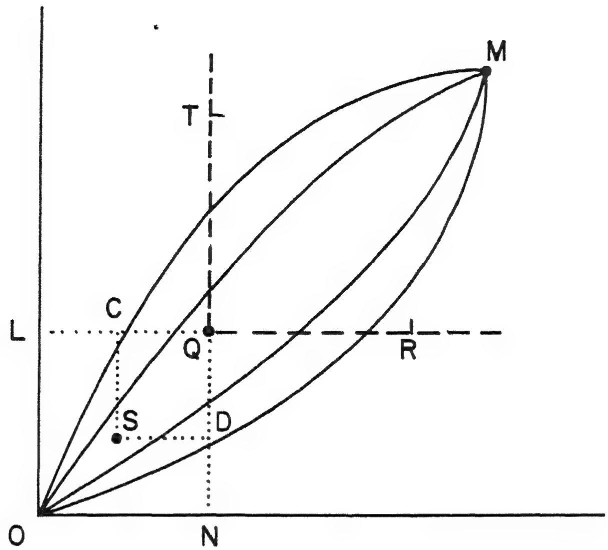

Figure 3.10. Alternative isocline families for production surfaces.

production could be raised to higher levels by following the line on the surface denoted as HM until the maximum milk level of GM is attained. The milk level line denoted as HM parallels the feed line FG in the feed plane. Line FG also represents the limit of the cow's stomach capacity and indicates that, to attain higher milk levels along HM, the amount of forage must be decreased as grain intake and milk production are increased. The extreme grain ration denoted as OG does not follow along the grain axis OC, under the postulate that a physiological minimum of forage must be included in the ration if lactation is to extend over a long period. Hence, the area OFG in the feed plane represents the limits of rations with respect to (1) the grain to hay ratios which may be fed and (2) the maximum possible intake of any particular grain to hay ratio or ration. Over the top of the production or milk surface (i.e., the area OMH in Figure 3.11) are milk contours or isoquants such as a″ through f″, each indicating a particular milk level. If lines corresponding to these milk isoquants are drawn in the relevant feed plane, OFG, they indicate all of the possible rations which allow attainment of a particular milk level.

The two-dimensional isoquant map corresponding to the model of Figure 3.11 is shown in Figure 3.12. Line H′M′ in Figure 3.12 is the stomach limit line and parallels line FG (HM) in Figure 3.11; O′M′ is the physiological limit line corresponding to OM in Figure 3.11. The isoquants a″ through f″ are the counterparts of those shown in Figure 3.11 and, again, indicate all possible rations or feed combinations which allow attainment of the specified milk output. The slope of these

isoquants indicates the grain to hay substitution ratio. No particular function may readily describe this particular production situation. However, if rations extend to the point of isocline convergence, quadratic forms may serve better than exponential forms of functions. However, if sample observations do not approach the convergence point in Figure 3.12, these and other types of functions may serve satisfactorily for a particular experiment.

BASIS FOR SELECTION

The sections above suggest that knowledge of biological, economic, or other environmental factors may exist to provide some guide or basis for selecting a production function. Frequently, however, previous knowledge as a basis for selection may not exist and several algebraic forms may be used initially, with various empirical criteria used for selecting among them. Different research workers emphasize various empirical extremes or approaches in selecting a form to be used. At one extreme, some prefer to start out with an initial hypothesis about the algebraic form of the regression equation. They then use this form and retain all coefficients even if their standard errors are relatively large and even if the term has small numerical importance in predictions. Having selected the particular function on a priori bases, they would retain it unless it appears entirely inconsistent with the numerical observations. At the other extremes, some research workers would prefer selection of a polynomial with sufficient coefficients and powers for the variables and retain them only if the coefficients are significant at the conventional 5 per cent probability level. Or, two other extremes are sometimes followed. In the one case, basic conditions or laws of production are supposed and an algebraic form is selected, with the data fitted to this single form. In the other case, no assumption or hypothesis about the form of the function is formulated and the experimental or sample data are subjected to statistical analysis, leading to selection of an algebraic model which evidently provides the "best fit." Numerous empirical criteria may be used in this selection: one is to select several different algebraic models and fit the observations to each of them. One of "best fit" then may be designated by the magnitude of the coefficient of variation, R^2, if the assumption of normally and independently distributed errors is not violated. With R^2 indicating the proportion of variance in the dependent variable accounted for by a particular type of equation, the larger value might be taken to indicate the form which is most appropriate for estimates. Other related statistics which might be used as empirical criteria include the F ratio and the mean square of deviations from regression (lack of fit). A larger F ratio (the mean square due to regression divided by the mean square of experimental error) or the smaller mean square of deviations from regression (i.e., the lack of fit) is taken to indicate a model most appropriate for the particular set of experimental

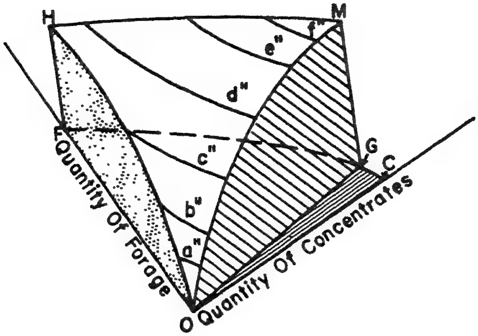

Figure 3.11. Production surface with limits of substitution.

or sample observations. Or, a general polynomial form can be selected, including constants for various powers or degrees of the independent variables and an appropriate number of interaction terms. An analysis of variance may be made for these, with a mean square and F ratio computed for each. Terms can be dropped if the corresponding F ratio is not significant at an acceptable probability level. Or, the constants in the regression equation may be computed directly, with the "t" test (the null hypothesis of regression coefficients at zero level) used to indicate terms which should be dropped because they are not significant at acceptable probability levels. (A procedure sometimes preferred is to select initially a simple polynomial form and add terms one at a time, retaining those which account for a significant incremental portion of variance in output.)

It is not necessary that the research worker confine himself to one or the other of these extremes. In some cases he may have sufficient

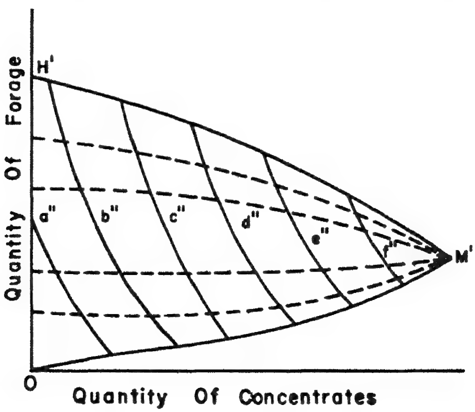

Figure 3.12. Isoquants and isoclines for Figure 3.11.

theoretical and empirical bases for proposing a single initial model and fitting the data to it. This approach has appeal in logic and may be especially appropriate for laboratory experiments in chemistry and physics where control of magnitude for all variables is possible or nearly so. However, for biologic and economic phenomena, it appears unlikely that a single mathematical form of production function is most appropriate for all environmental and resource situations. Too, different individuals may be able to give equally valid reasons for selecting alternative types of functions. Errors in design, measurement, and specification also may cause a function, other than the one initially selected, to be more appropriate. Still, the research worker usually does have some basis for selecting, among the very many algebraic forms possible, one or a few which are appropriate for the phenomenon under consideration. If he feels that the initial hypothesis of functional form should be retained, he may follow this procedure, after applying a simple criterion such as that the lack of fit term is no larger than experimental error for the regression equation. Some of the mathematical properties outlined on previous pages provide guides in selection. For example, the conventional power function would be inappropriate in fitting a surface for a fertilizer experiment where isoclines converge within the range of observations. Yet, while a general type of function may be appropriate for a particular set of observations, the research worker may not have empirical or biological knowledge for specifying a single modification of it. He may then use various empirical tests to determine the particular modification which gives the "best fit." In case previous knowledge and theory are nil, he might be forced to use this approach from the outset, even going so far as to select functions with contrasting algebraic properties and subjecting them to tests of "best fit."

Few experiments and samples exist which do not require exercise of experience and judgment by the analyst. Even if he rejects the somewhat "rigid" approach of using a priori assumptions of the algebraic form, and fitting the data to this single selection, he seldom can let choice rest alone on objective empirical selection of the one "best-fitting" form. In the first place, he is faced with limitations in funds and time. He must reject some types or modifications of equations because it is too costly to compute regression coefficients or estimate the parameters for the entire range of possible functions. Some equations may be rejected simply because their algebraic form makes computations and solution of economic or other optima too difficult. In other cases, the probability level at which a regression coefficient of a polynomial equation will be accepted as nonzero differs among research workers and is a matter of experience and judgment. Some insist that the conventional probability level of .05 must be attained if the coefficient for a particular degree of an independent variable is to be retained in the production function. Others would insist that this pure empirical appeal is too rigid, as perhaps it is. For example, a "t" test for the quadratic equation in equation 3.18 may indicate that the

regression coefficient for the squared term derived from a fertilizer experiment can be considered to differ from zero only at a probability level of .20. If we retain the term in the equation only if its coefficient is significant at a .05 probability level, we assume that the input-output relationship is linear. In general, this assumption is inconsistent with known biological conditions, since most empirical research indicates diminishing marginal productivity when one nutrient is varied and all other resources are fixed in magnitude. Too, it would imply that if fertilizer is profitable at all, the nutrient should be applied in unlimited quantities. The research worker has two alternatives: he can accept the term on the basis of theory and previous experience, and perhaps as a basis for conservatism in recommendations. He can refine the experiment (through improved design, a larger sample, and better measurements) and repeat it under similar environmental conditions. Perhaps both alternatives should be used, with the first serving as a basis for immediate decisions and recommendations and the second used to increase precision of later estimates. Infrequently, the phenomenon being examined does not allow the latter type of refinement. The research worker may lack funds and facilities or the observations may be time series generated from variables of the national economy. Some have suggested the following criterion on decision to include or delete a variable: omit the variable if the standard error of the regression coefficient exceeds the coefficient itself.

Selection among algebraic forms of equations often is no less difficult than decision in respect to the probability level at which variables will be omitted from a polynomial model. There are the logical criteria mentioned previously (such as functions allowing negative marginal products, or converging isoclines for a fertilizer experiment, nonconverging isoclines and an asymptotic maximum for meat production, etc.) which, along with such statistical criteria (as magnitude of R^2 or deviations from regression) do provide guides. Even then, however, the optimum function is not always apparent. For example, the three functions below have been derived from a corn fertilization experiment in Iowa with corn yield, Y, measured in bushels per acre and nitrogen level, N, measured in pounds.[4] Only observations where P_2O_5 is constant at 160 pounds per acre are considered. The magnitude of variance quantities provide little basis for selecting among the three functions. The power function in equation 3.75, while it gives the highest R^2, might be rejected because of the apparent possibility of negative marginal

$$Y = 33.01N^{.2283} \qquad R^2 = .95 \tag{3.75}$$

$$Y = 42.60 + .80N - .0018N^2 \qquad R^2 = .80 \tag{3.76}$$

$$Y = 23.67 + 12.82N^{.5} - 13.37^{N} \qquad R^2 = .91 \tag{3.77}$$

[4] Heady, Earl O., Pesek, John T., and Brown, W. G. Crop response surfaces and economic optima in fertilizer use. Iowa Agr. Exp. Sta. Bul. 424. Ames. March, 1955.

response at large N levels and the equation does not define a maximum. The quadratic (3.76) and square root (3.77) equations are similar in the sense that both allow changing elasticity, diminishing marginal products, and a defined maximum. With their derivatives equated to nitrogen to corn price ratios of .8, .12, .16, and .20, the profit maximizing rates of fertilization are, respectively, 206, 195, 184, and 172 pounds of N for the quadratic and 183, 156, 134, and 117 pounds for the square root equation. (The corresponding rates for the power function are, respectively, 361, 213, 147, and 110 pounds of N.) Generally, the square root function, as compared to the quadratic function, will call for lower input rates with larger price ratios and higher input rates with smaller price ratios because its marginal product curve is not linear (i.e., the second derivative is not constant). Agronomists have not yet established laws of marginal crop products, although some would suggest that they do not "decline in straight line fashion."

In a pragmatic sense, small differences in estimates by alternative equations can be unimportant because price, or even weather underlaying the production function, is so uncertain that exact profit maximizing input combinations can never be selected. Extension specialists, with some rightful basis, would be prone to ask, "Why should extreme effort be invested in refining functions for such inputs as fertilizer because farmers discount future production greatly due to uncertainty and use less than recommended rates (which, in turn, fall short of the level equating marginal revenue and cost)?" Generally, farmers can only guess on future revenues because price and weather are unknown. However, modifications in empirical procedures may be used to lessen differences in optima specified by different functions. Anderson has suggested elimination of observations of zero inputs for variable resources as one method of attaining greater predictional consistency between models.[5] When observations with N = 0 are eliminated for the Iowa data used in the equations above, the following quadratic and square root functions result:

$$Y = 80.77 + 14.22N - 1.0413N^2 \qquad R^2 = .70 \tag{3.78}$$

$$Y = 32.32 - 13.37N + 17.6693N^{.5} \qquad R^2 = .70 \tag{3.79}$$

In terms of magnitude of R^2, the two functions are equally acceptable. The estimated inputs where the marginal products equal the price ratios of .8, .12, .16, and .20 now are, respectively, 212, 181, 150, and 120 for the quadratic and 188, 156, 131, and 113 for the square root equation. Within the range of price ratios historically realized, some greater consistency is attained but there is still considerable variance in recommendations which might be based on the two equations over reasonable price ratios.

[5] Anderson, R. L. Some statistical problems in the analysis of fertilizer response data, in Baum, E. L., *et al.* (eds.), Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames. 1957. P. 192.

Another criterion of general use in deciding between alternative functions is to compare predicted and observed yields over the relevant range of the production surface. Assuming the other criteria mentioned above are adequately satisfied, the best function will be that for which the scattering of positive and negative residuals over the surface is random. Should there be a systematic tendency for predicted yields to exceed observed yields (or vice versa) over parts of the surface, this is an indication that the fitted function does not adequately characterize the observed data. Not unexpectedly, when there are 3 or more input factors involved, the search for systematic tendencies in the residuals becomes messy unless some orderly approach, such as suggested by Valavanis,[6] is used.

Difficulties encountered in selection among continuous models may be eliminated in form free models such as those suggested by Hildreth.[7] In estimating points on the production surface, the investigator is not forced to specify a continuous algebraic form. However, some would argue that if this general procedure is to be used, agronomists might as well use more replicates to obtain better mean yield estimates of the different treatments, and then assume the function to be linear between these points. Again, it is possible that, if sufficient points are estimated, farmers need no more refinement than this. Because of uncertainty and imperfect knowledge, they would not be able to equate the slope of a continuous function with a price ratio even if the algebraic form were known with absolute certainty.

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[6] Valavanis, S. Econometrics. McGraw-Hill, New York. 1959. P. 152.

[7] Hildreth, C. G. Point estimates of ordinates of concave functions. Journal of the American Stat. Assoc., 49: 598-619. 1954.

CHAPTER 4

Data Analysis for Production Function Estimation

PREVIOUS CHAPTERS have shown the usefulness of production functions as decision-making guides. Still, unless empirical estimates are available, knowledge of how production functions may be algebraically manipulated to give meaningful economic information is of no practical use. A skeleton of theory is lifeless without empirical flesh; to be more than playthings in an ivory tower, production functions must be specified in terms of real-world situations. In deriving these empirical specifications, statistical procedures are generally used complemented by the biological, physical, and economic logic relevant to the particular production process being examined.

An introductory outline of the more important statistical techniques and problems pertinent to production function estimation is presented in the current chapter. The aim is not to present a treatise on techniques of estimation. Rather, it is to give the non-statistician working on production functions knowledge of the relevant concepts. He can then make better use of the statistical services and references available to him. Also, they provide the statistical background for the empirical work discussed in later chapters.

There are two phases in the fitting of production functions: (a) the collection of data and (b) its analysis. Data may be collected from either experimental or nonexperimental sources. The source used will generally depend on the particular production process being examined, the stage of the investigation, and the research resources available. However, once a choice has been made between experimental and nonexperimental data, the techniques and problems of analysis largely dictate the particular methods of data collection to be used. On these grounds, the techniques and problems of analyzing data are considered in this chapter; procedures and problems of data collection are discussed in Chapter 5. Difficulties involved in the "economic specification" of production functions are discussed in Chapter 6. Their appreciation requires some understanding of the statistical procedures relevant to the collection and analysis of data. It must be emphasized that the statistical and economic aspects of specification are very closely interrelated; indeed, so closely that the line of demarcation between them is somewhat arbitrary. They are considered separately for expository convenience.

SINGLE EQUATION APPROACH

Two basic approaches to production function estimation are possible. On the one hand, the production function may be regarded as but one of a number of interdependent relationships constituting a model of the economic situation being studied. Such is the *simultaneous equations approach*. Alternatively, the *single equation approach* may be adopted: the production function being considered as a single independent relationship, uninfluenced by other relationships relevant to the economic milieu enveloping the production process. Which of these approaches is strictly appropriate is a question of fact. It can only be answered in terms of the logic — economic, physical, or biological — underlying the production process being examined. Such problems of specification are taken up in Chapter 6. From a computational viewpoint, however, a single equation model is often regarded as an alternative to a more strictly correct but computationally irksome simultaneous equations model. The estimational procedure for the two approaches, while overlapping, are not identical. Since the single equation approach has generally been used in deriving empirical production functions, the statistical procedures and problems pertinent to it will be outlined first. Essentially, this implies a discussion of the technique of least squares multiple regression.[1]

Multiple Regression Analysis with Linear Parameters

Suppose the production relationship between an output Y and inputs $X_1, X_2, \cdots, X_k$ has, on a priori grounds of economic, biological, or physical logic, been postulated as the following single equation model:

$$(4.1) \qquad Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \cdots + \beta_k X_k + \epsilon .$$

Regarding equation 4.1 as a *regression law*, Y is termed the *dependent* variable. The explanatory factors $X_1, X_2, \cdots, X_k$ are the *independent* variables in the sense that the values that they take arise independently of equation 4.1. ϵ is the *error* due to the fact that the postulated independent variables do not completely explain Y; some input factors of minor importance have not been taken into account. The *parameters* $\beta_0, \beta_1, \cdots, \beta_k$ are the *population regression coefficients*. From a mathematical viewpoint, given the correctness of the single equation model, equation 4.1 may be described as a continuous unilateral causal relationship. In such terms Y is the effect variable and $X_1, X_2, \cdots, X_k$ the causal factors. It is desired to specify equation 4.1 empirically, i.e., to determine the values of the parameters. If these values are

[1] Full discussion is to be found in: Williams, E. J. Regression analysis. John Wiley and Sons, Inc., New York. 1959. See also Bradley, R. A. Fitting response surfaces. Industrial Quality Control, 15: 16-24. 1958.

known, the economic implications of the production function may be ascertained and applied in the real world. Thus interest is in the causal relation as a whole, especially the β's, and not just in the values of Y that may be estimated by using the equation. The causal mechanism is of economic relevance in itself — as was explained in Chapter 2. Provided that some basic assumptions hold true, the fact that equation 4.1 is linear in the parameters makes estimation of these parameters by multiple regression rather straightforward. All that is required is a sample of $n \geq k + 1$ sets of data showing the value of Y for various levels of each of the independent variables. Suppose such data are available. They might be listed as follows:

$$\begin{array}{ccccc} Y_1 & X_{11} & X_{21} & \cdots\cdots & X_{k1} \\ Y_2 & X_{12} & X_{22} & \cdots\cdots & X_{k2} \\ \vdots & & & & \vdots \\ Y_n & X_{1n} & X_{2n} & \cdots\cdots & X_{kn} \end{array}$$

where Y_j ($j = 1, 2, \cdots, n$) is the level of Y attained when the i-th of the k inputs is at the level X_{ij}. The estimate of equation 4.1 to be derived from the given data will be of the form:

$$\hat{Y} = b_0 + b_1X_1 + b_2X_2 + \cdots + b_kX_k \tag{4.2}$$

where b_i, termed a *sample regression coefficient,* is an estimate of β_i, and $\hat{Y}$ estimates Y. Since equation 4.2 is but an estimate of equation 4.1, the relation between Y and $\hat{Y}$ is of the form:

$$Y_j = \hat{Y}_j + e_j. \tag{4.3}$$

In other words, it is not to be expected that equation 4.2 will exactly predict the level of output, Y_j, forthcoming from a given set of input quantities, $X_{1j}, X_{2j}, \cdots, X_{kj}$. There will generally be some discrepancy or residual between the observed and predicted value of Y. This residual is the term e of equation 4.3. It is an estimate of the theoretical error, ϵ, of equation 4.1. For convenience in algebraic manipulation, equation 4.2 is often written as:

$$\hat{Y} = b_0X_0 + b_1X_1 + \cdots + b_kX_k \tag{4.4}$$

where X_0 is a dummy variable always equal to one.

Estimation of the regression coefficients

From the given data it is desired to calculate suitable values of the coefficients $b_0, b_1, \cdots, b_k$ of equation 4.4. Such estimates of the β's are regarded as suitable if, over all possible samples of data, they are

correct estimates on average and have the smallest possible variation. In statistical terms, it is desired that the estimates be *unbiased* and have minimum *variance.* By unbiased it is meant, metaphorically, that the estimates do not lie consistently to the right or left of the target. A brief explanation of the concept of variance is perhaps necessary. Consider a variable x. The expected, mean, or average value of x in a population of x's is written E(x). The expected value of $[x - E(x)]^2$, i.e., the expected value of the square of the deviations of the x values from their mean, is the variance of x. It is an indication of the variation that exists among the values of x in the population. Usually this population variance is denoted by σ_x^2. Thus

$$\sigma_x^2 = E[x - E(x)]^2. \tag{4.5}$$

The variance found among a sample of x values drawn from the population will generally not equal the population variance. However, for practical purposes, this sample variance provides an estimate of the population variance. The sample variance is usually denoted by s^2. The covariance of two variables x and y is a measure of the extent to which they vary together. For a population of x and y values, it is denoted by σ_{xy} where

$$\sigma_{xy} = E[x - E(x)]\,[y - E(y)]. \tag{4.6}$$

A statistic related to variance is the *standard deviation* of a variable. It is the positive square root of the variance of the variable. The concepts of variance and covariance are important in the assumptions that must be fulfilled if *least squares* estimates of the regression coefficients are to be unbiased and of minimum variance. The assumptions are:

(a) the expected value of the error ϵ is zero

(b) the covariance between the error associated with one value of Y and that associated with any other value of Y is zero

(c) the variance of the error associated with one value of Y is the same as the variance of the error associated with any other value of Y

(d) the covariance between the error and each of the independent variables is zero[2]

(e) the observations of the independent variables are measured without error.

---

[2] Assumptions (a) to (d) might be expressed more succinctly as follows: (a) $E(\epsilon) = 0$. (b) $\sigma_{\epsilon_j\epsilon_p} = 0$ for $j \neq p$ and $j, p = 1, 2, \cdots, n$. (c) $\sigma_{\epsilon_j}^2 = \sigma^2$ for $j = 1, 2, \cdots, n$. (d) $\sigma_{\epsilon x_i} = 0$ for $i = 0, 1, \cdots, k$.

Consideration will later be given to the procedures that might be adopted if the above assumptions are not fulfilled.

Least squares principle

Using least squares, the aim is to minimize the sum of the squared deviations (SSD) between the observed Y values, $Y_1, Y_2, \cdots, Y_n$, and the corresponding estimated values of Y, $\hat{Y}_1, \hat{Y}_2, \cdots, \hat{Y}_n$, calculated from the data by using equation 4.2. Thus it is desired to minimize

$$(4.7) \qquad SSD = \sum_{j=1}^{n} (Y_j - \hat{Y}_j)^2$$

which, by virtue of equation 4.4, may be written as

$$(4.8) \qquad SSD = \sum_{j=1}^{n} (Y_j - b_0X_{0j} - b_1X_{1j} - \cdots - b_kX_{kj})^2 .$$

Notice that this application of the least squares principle is centered upon Y, the dependent variable. Such an orientation implies that the derived estimates of the parameters will not be the same as the estimates obtained if, say, not Y but one of the X's had been made the dependent variable. Thus the estimated regression equation, equation 4.2, is an irreversible relationship in terms of causal influences. However, the direction of inference may be reversed. Hence, given values of $b_0, b_1, \cdots, b_k$ and a set of levels for $Y, X_1, \cdots, X_{k-1}$, equation 4.2 could be used to "predict" the level at which X_k was set in producing the given level of Y.

Normal equations

To minimize the sum of the squared deviations, the partial derivatives of equation 4.8 with respect to $b_0, b_1, \cdots, b_k$ must be equated to zero. By doing this, the following set of (k + 1) equations are obtained; terms involving Y having been moved to the right-hand side. These are the normal equations expressed in terms of the raw data.

$$(4.9) \qquad \begin{array}{l} b_0\Sigma X_{0j}X_{0j} + b_1\Sigma X_{0j}X_{1j} + \cdots + b_k\Sigma X_{0j}X_{kj} = \Sigma X_{0j}Y_j \\ b_0\Sigma X_{1j}X_{0j} + b_1\Sigma X_{1j}X_{1j} + \cdots + b_k\Sigma X_{1j}X_{kj} = \Sigma X_{1j}Y_j \\ \vdots \\ b_0\Sigma X_{kj}X_{0j} + b_1\Sigma X_{kj}X_{1j} + \cdots + b_k\Sigma X_{kj}X_{kj} = \Sigma X_{kj}Y_j \end{array}$$

where Σ implies summation over $j = 1, 2, \cdots, n$. Since there are $(k + 1)$ equations and $(k + 1)$ unknowns, $b_0, b_1, \cdots, b_k$, the system of normal equations 4.9 may usually be solved for these unknowns. However, for computational purposes and as a theoretical simplification, it is more convenient to express the normal equations in other ways. One

such is in terms of the deviations of the observations on each variable from their average value. Consider the first equation of system 4.9. Since $X_{0j} = 1$, it may be written as in equation 4.10.

$$b_0 n + b_1 \Sigma X_{1j} + \cdots + b_k \Sigma X_{kj} = \Sigma Y_j \tag{4.10}$$

Dividing through by n and writing $\overline{X}_i$ and $\overline{Y}$ for the mean of X_i and Y, respectively, equation 4.10 becomes

$$b_0 + b_1\overline{X}_1 + \cdots + b_k\overline{X}_k = \overline{Y} \tag{4.11}$$

and thus

$$b_0 = \overline{Y} - b_1\overline{X}_1 - \cdots - b_k\overline{X}_k\,. \tag{4.12}$$

Hence it is really only necessary to find $b_1, \cdots, b_k$. Substituting equation 4.12 into equation 4.8 and rearranging terms, we have

$$\text{SSD} = \sum_{j=1}^{n} [(Y_j - \overline{Y}) - b_1(X_{1j} - \overline{X}_1) - \cdots - b_k(X_{kj} - \overline{X}_k)]^2 \tag{4.13}$$

$$= \sum_{j=1}^{n} (y_j - b_1 x_{1j} - \cdots - b_k x_{kj})^2 \tag{4.14}$$

where $y_j = (Y_j - \overline{Y})$ and $x_{ij} = (X_{ij} - \overline{X}_i)$. The system of k normal equations obtained by taking the partial derivatives of equation 4.14 with respect to $b_1, \cdots, b_k$ is then

$$\begin{aligned} b_1 \Sigma x_{1j}^2 &+ b_2 \Sigma x_{1j} x_{2j} + \cdots + b_k \Sigma x_{1j} x_{kj} = \Sigma x_{1j} y_j \\ b_1 \Sigma x_{2j} x_{1j} &+ b_2 \Sigma x_{2j}^2 + \cdots + b_k \Sigma x_{2j} x_{kj} = \Sigma x_{2j} y_j \\ &\vdots \\ b_1 \Sigma x_{kj} x_{1j} &+ b_2 \Sigma x_{kj} x_{2j} + \cdots + b_k \Sigma x_{kj}^2 = \Sigma x_{kj} y_j\,. \end{aligned} \tag{4.15}$$

Each term $\sum_{j=1}^{n} x_{ij} x_{hj}$ or $\sum_{j=1}^{n} x_{ij} y_j$, $(i, h = 1, 2, \cdots, k)$ in equation 4.15 is a *corrected sum of cross products*. Of course, if i = h, a term in X_i is a *corrected sum of squares;* these are found on the main diagonal of the left side of the normal equations. Each cross product term might be spelled out as exemplified in equations 4.16 and 4.17.

$$\Sigma x_{1j} y_j = x_{11} y_1 + x_{12} y_2 + \ldots + x_{1n} y_n \tag{4.16}$$

$$= (X_{11} - \overline{X}_1)(Y_1 - \overline{Y}) + (X_{12} - \overline{X}_1)(Y_2 - \overline{Y}) + \ldots + (X_{1n} - \overline{X}_1)(Y_n - \overline{Y}) \tag{4.17}$$

Moreover it can be shown, by rearranging equations such as equation 4.17, that

$$\Sigma x_{ij}x_{hj} = \Sigma X_{ij}X_{hj} - \frac{\Sigma X_{ij}\Sigma X_{hj}}{n} \tag{4.18}$$

$$\Sigma x_{ij}y_j = \Sigma X_{ij}Y_j - \frac{\Sigma X_{ij}\Sigma Y_j}{n}. \tag{4.19}$$

These relationships provide an easy method of calculating the corrected sums of squares and cross products. So far as solving equation 4.15 to obtain the b_i values is concerned, the easiest way of visualizing the problem — and of understanding the methods used — is as a matrix equation.[3] If we let

$$A = \begin{bmatrix} \Sigma x_{1j}^2 & \Sigma x_{1j}x_{2j} & \cdots\cdots & \Sigma x_{1j}x_{kj} \\ \Sigma x_{2j}x_{1j} & \Sigma x_{2j}^2 & \cdots\cdots & \Sigma x_{2j}x_{kj} \\ \vdots & & & \\ \Sigma x_{kj}x_{1j} & \Sigma x_{kj}x_{2j} & \cdots\cdots & \Sigma x_{kj}^2 \end{bmatrix} \tag{4.20}$$

$$G' = [\, \Sigma x_{1j}y_j \quad \Sigma x_{2j}y_j \quad \cdots\cdots \quad \Sigma x_{kj}y_j \,] \tag{4.21}$$

$$B' = [\, b_1 \quad b_2 \quad \cdots \quad b_k \,] \tag{4.22}$$

then the set of normal equations, 4.15, may be written as

$$AB = G. \tag{4.23}$$

The matrix B is obtained as

$$B = CG \tag{4.24}$$

following the usual notation of denoting A^{-1} by C. It is noteworthy that A, and hence C also, is a symmetric matrix — a fact which simplifies the computational effort involved in solving the normal equations. Usually, when formulated in terms of deviations of the variables from their means, the normal equations are solved by the forward solution of the abbreviated Doolittle method.[4]

Frequently, the normal equations are expressed in terms of the *standard partial regression coefficients*, b'_1, b'_2, $\cdots$, b'_k, and the

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[3] For an extremely simple exposition of the matrix concepts involved see Heady, E. O. and Candler, W. V. Linear programming methods. Iowa State University Press, Ames. 1958. Pp. 378-407.

[4] An easily followed example of the abbreviated Doolittle method is to be found in Anderson, R. L. and Bancroft, T. A. Statistical theory in research. McGraw-Hill Book Company, Inc., New York. 1952. Pp. 197-99.

coefficients of correlation, r_{ij} $(i, j = X_1, X_2, \cdots, X_k, Y)$, between pairs of variables.[5] In such form, the normal equations are:

$$
\begin{aligned}
b'_1 r_{11} + b'_2 r_{12} + \cdots + b'_k r_{1k} &= r_{1y} \\
b'_1 r_{21} + b'_2 r_{22} + \cdots + b'_k r_{2k} &= r_{2y} \\
\vdots \qquad\qquad\qquad\qquad & \quad \vdots \\
b'_1 r_{k1} + b'_2 r_{k2} + \cdots + b'_k r_{kk} &= r_{ky}
\end{aligned}
\tag{4.25}
$$

where, for $i = 1, 2, \cdots, k$,

$$
b'_i = b_i \sqrt{\frac{\sum_{j=1}^{n} x_{ij}^2}{\sum_{j=1}^{n} y_j^2}}
\tag{4.26}
$$

and the correlation between X_i and Y is given by

$$
r_{iy} = \frac{\sum_{j=1}^{n} x_{ij} y_j}{\sqrt{\sum_{j=1}^{n} x_{ij}^2 \sum_{j=1}^{n} y_j^2}}.
\tag{4.27}
$$

The correlation between X_i and X_h $(i, h = 1, 2, \cdots, k)$ is calculated as in equation 4.27, replacing y_j by x_{hj}. These correlation coefficients, which by construction lie between plus and minus one, indicate the degree of association between the relevant variables. They have no causal implications. If negative, the correlation coefficient indicates that as one of the pair of variables increases, the other tends to decrease. If the correlation coefficient is positive, both variables tend to move in the same direction. The closer the coefficient is to plus or minus one, the greater is the degree of association between the two variables. Thus r_{ii}, the correlation of a variable with itself, is necessarily unity. Hence, the main diagonal of equation 4.25 really consists only of b'_i values. If the correlation coefficient between a pair of independent variables is, roughly speaking, greater than $|0.8|$, the problem of multicollinearity may arise. It is discussed later in this chapter.

As shown by equation 4.26, the standard partial regression coefficients are related to the sample regression coefficients. In fact, they are the sample partial regression coefficients standardized in terms of

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[5] For an illustrative example of an efficient, widely used calculation procedure based on the b', r_{ij} formulation of the normal equations, see Ostle, B. Statistics in research. Iowa State University Press, Ames. 1956. Pp. 207-14.

the standard deviations of Y and X_i, as equation 4.26 indicates. Being standardized, the various b' values are comparable. They enable comparisons to be made of the relative influence on Y of the various independent variables. Such comparisons cannot be made via the sample regression coefficients because, in general, the independent variables will not be measured in comparable units. To illustrate, from equation 4.2, it can be seen that

$$\frac{\delta \hat{Y}}{\delta X_i} = b_i, \qquad i = 1, 2, \cdots, k. \tag{4.28}$$

Hence, if X_i changes by one unit, all the other independent variables remaining constant, $\hat{Y}$ will change on average by b_i units; alternatively, Y is estimated to change by b_i units. Now suppose X_1 is measured in tons and X_2 in acres. Clearly, b_1 and b_2 tell nothing of the relative effects of X_1 and X_2 on Y. However, a comparison of b'_1 with b'_2 indicates the number of standard deviations by which $\hat{Y}$ would change if X_1 or X_2, considered separately, were each changed by one standard deviation. In such fashion, the relative influence of the independent variables may be gauged.

Variances and covariances of the sample regression coefficients

The matrix C of equation 4.24 plays an important role in estimating the variances and covariances of the sample regression coefficients. Denoting the general element of C by c_{ij} $(i, j = 1, 2, \cdots, k)$, the variance of b_i is given by

$$\mathrm{Var}(b_i) = c_{ii}\sigma^2 \tag{4.29}$$

where σ^2 is the true (but unknown) variance of the error, ϵ. Of course, c_{ii} is a diagonal element of C. As an estimate of σ^2, s^2 is used where

$$s^2 = \frac{\sum_{j=1}^{n} (Y_j - \hat{Y}_j)^2}{n - k - 1}. \tag{4.30}$$

The covariance between pairs of regression coefficients is given by

$$\mathrm{Covar}(b_i b_j) = c_{ij}\sigma^2, \qquad i \neq j \quad \text{and} \quad i, j = 1, 2, \cdots, k. \tag{4.31}$$

The variance of the sum or difference of two of the sample regression coefficients may be calculated as follows:

$$\mathrm{Var}(b_i \pm b_j) = \mathrm{Var}(b_i) + \mathrm{Var}(b_j) \pm 2\,\mathrm{Covar}(b_i b_j). \tag{4.32}$$

Likewise, using the relationship given in equation 4.12, the variance of b_0 may be estimated as in equation 4.33:

$$(4.33)\quad Var(b_0) = Var(\bar{Y}) + \sum_{i=1}^{k} \bar{X}_i^2 Var(b_i) + 2 \sum_{i=1}^{j-1} \sum_{j=1}^{k} \bar{X}_i \bar{X}_j Covar(b_i b_j)$$

where $Var(\bar{Y})$ equals s^2/n.

Significance tests of the sample regression coefficients

Having solved the normal equations to ascertain the b values, it is desirable to ascertain the reliability of these estimates. This may be done by calculating confidence limits for the estimates or by testing to see if the estimates are significantly different from some selected value. For such procedures to be accurate, it is necessary that the distribution of the error term ϵ be known so that a test statistic may be devised. In general, the derivation of such a test is extremely complicated. Because of this, it is desirable that the errors be normally distributed; in which case the "t" statistic, for which tables are available, may be used. Assuming that the errors are normally distributed with a mean of zero and variance σ^2, the value of t to test whether the sample regression coefficient b_i is significantly different from zero at some probability level α is given by

$$(4.34)\qquad t = \frac{b_i}{\sqrt{Var(b_i)}}.$$

If this t is larger than the tabled level of t_α with (n - k - 1) degrees of freedom, then b_i is significantly different from zero at the α level of probability. The *confidence limits* on b_i are given by

$$(4.35)\qquad L = b_i \pm t_\alpha s \sqrt{c_{ii}}$$

where t_α is read from the "t" table with (n - k - 1) degrees of freedom.[5a] There is then a probability of $(1 - \alpha)$ that the value of β_i actually lies within these limits. The value of t to test the significance of the difference between two of the sample regression coefficients is given by

$$(4.36)\qquad t = \frac{b_i - b_j}{\sqrt{Var(b_i - b_j)}}$$

where t_α is obtained from the "t" table with 2(n - k - 1) degrees of freedom.

[5a] Methods of deriving confidence limits for such estimates as the location of the maxima, etc., of a fitted function are given in Williams, E. J. Regression analysis. John Wiley and Sons, Inc., New York. 1959. Pp. 90-116.

Variances and confidence intervals for predictions

Using the estimated regression equation 4.2, estimates of Y may be made. Such estimates will be valid so long as they do not involve levels of the independent variables outside the range found in the data used to derive equation 4.2. No estimates of the production parameters are available outside of this range. That this is so is explicitly shown in equation 4.24; the matrices C and G relate only to the range of data examined. If extrapolations are made beyond this range, they must be interpreted extremely cautiously.

Two types of estimate of Y may be made. Consider the case when it is desired to estimate the *average* Y that might occur from many trials with a given set of values of the independent variables. In such circumstances, the variance of $\hat{Y}$ is given by

$$\text{(4.37)} \qquad \operatorname{Var}(\hat{Y}) = \left(\frac{1}{n} + \sum_{i=1}^{k} c_{ii}x_i + 2\sum_{i=1}^{j-1}\sum_{j=1}^{k} c_{ij}x_ix_j\right)s^2 .$$

It is noteworthy that the variance of $\hat{Y}$ depends on the actual level at which the independent levels are set. The larger the deviation between the level of X_i studied and its mean, $\bar{X}_i$, the greater will be the variance of $\hat{Y}$. These comments also apply for the other type of prediction of Y: the estimate of a *particular* Y from a given set of values of the independent variables. The variance of this prediction of a specific Y value, denoted $\operatorname{Var}(\hat{Y}^*)$, as might be expected, is larger than for the first type of prediction, as is shown by comparison of equations 4.37 and 4.38.

$$\text{(4.38)} \qquad \operatorname{Var}(\hat{Y}^*) = \left(1 + \frac{1}{n} + \sum_{i=1}^{k} c_{ii}x_i + 2\sum_{i=1}^{j-i}\sum_{j=1}^{k} c_{ij}x_ix_j\right)s^2$$

The fact that $\operatorname{Var}(\hat{Y})$ depends upon the level of X_i ($i = 1, \cdots, k$), coupled with the aim of minimizing $\operatorname{Var}(\hat{Y})$, has some important implications for experimental design relevant to production function estimation, as is indicated in Chapter 5.

Confidence intervals for $\hat{Y}$ and $\hat{Y}^*$ are given by $\hat{Y} \pm t_\alpha\sqrt{\operatorname{Var}(\hat{Y})}$ and $\hat{Y}^* \pm t_\alpha\sqrt{\operatorname{Var}(\hat{Y}^*)}$, respectively, where t_α has $(n - k - 1)$ degrees of freedom.

Coefficient of multiple determination

The correlation between the n observed values of Y and the corresponding $\hat{Y}$ values is shown by the *coefficient of multiple correlation.* It is denoted by R. The *coefficient of multiple determination,* R^2, indicates the percentage of the variation in the n observed Y values that is explained by the fitted regression equation 4.2. Thus it is a measure of

the goodness of fit of the estimated regression equation. As such, it is of use in choosing between alternative single equation models of a production process. Of course, this is only true in the absence of strong a priori evidence as to what exact form the model should take. It can be shown that

$$R^2 = \frac{\sum_{j=1}^{n} y_j^2 - \sum_{j=1}^{n} (Y_i - \hat{Y}_i)^2}{\sum_{j=1}^{n} y_j^2} \tag{4.39}$$

$$= \frac{\sum_{i=1}^{k} b_i \sum_{j=i}^{n} x_{ij} y_j}{\sum_{j=1}^{n} y_j^2}. \tag{4.40}$$

When the number of parameters to be estimated is large or, as often happens in production function estimation, the number of sets of observation is small, the above calculations tend to overestimate R^2. To take account of this, $\bar{R}^2$, the *adjusted coefficient of multiple determination,* may be used.[6] It is calculated as

$$\bar{R}^2 = 1 - (1 - R^2)\left(\frac{n - 1}{n - k}\right) . \tag{4.41}$$

In equation 4.39, the denominator is the corrected sum of squares of the n values of Y. The numerator is the amount of the corrected sum of squares of Y accounted for or explained by the fitted regression equation. As is indicated in equation 4.39, this sum of squares attributable to regression (SSR) equals the total corrected sum of squares minus the sum of squares of the deviations of $\hat{Y}$ from Y. If equation 4.2 fitted the data perfectly, $Y_i - \hat{Y}_i$ would be zero and R^2 would be one.

Analysis of variance for multiple regression

The relationships between the sums of squares can be shown conveniently in an analysis of variance table, as illustrated in Table 4.1.

The mean-square values are obtained by dividing the sums of squares by their associated degrees of freedom. The value of the "F" statistic given by

$$F = \frac{\text{Regression mean square}}{\text{Error mean square}} = \frac{SSR/k}{s^2} \tag{4.42}$$

---

[6] See Ezekiel, M. and Fox, K. A. Methods of correlation and regression analysis. John Wiley and Sons, Inc., New York. 1959. P. 300.

where F has k and (n - k - 1) degrees of freedom, provides an over-all test of the significance of the regression. Equivalently, it provides a test of the null hypothesis that $\beta_1 = \beta_2 = \cdots = \beta_k = 0$. Thus if F from equation 4.42 is larger than the tabled value of F at the desired probability level, the null hypothesis is probably not true.

Curvilinear regression with linear parameters

So far in this chapter, concern has been with the estimation of the production function in equation 4.1 which was linear in both its variables and parameters. Least squares multiple regression may also be applied directly to estimate the parameters of production functions nonlinear in the variables; the only proviso being that the parameters enter linearly. If they do not, some indirect or "round about" estimation procedure has to be used. Assuming that the parameters enter linearly, all that is necessary is that some functional transformation of the variables be available which will transform the production function model into a regression law linear in the transformed variables. To date, two types of transformations have been used extensively in empirical work: a logarithmic transformation for multiplicative Cobb-Douglas type production models; and a simple "relabeling" of the variables in polynomial type functions.

Consider the ordinary Cobb-Douglas production function for an output P from inputs $Z_1, Z_2, \cdots, Z_k$. It is

$$(4.43) \qquad P = \alpha Z_1^{\beta_1} Z_2^{\beta_2} \cdots Z_k^{\beta_k} q$$

where q is a proportional error due to the exclusion of some relevant inputs. By expressing each of the variables, and also α, in terms of logarithms (either natural or to the base 10), a linear function is obtained. Thus equation 4.43 becomes

$$(4.44) \qquad \log P = \log \alpha + \beta_1 \log Z_1 + \cdots + \beta_k \log Z_k + \log q.$$

If log P is denoted by Y, log α by β_0, log Z_i by X_i and log q by ϵ, equation 4.44 is identical to equation 4.1. To wit:

$$(4.1) \qquad Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_k X_k + \epsilon.$$

Thus, while equation 4.43 represents a curved surface in the (k + 1) dimensional space with axes P, $Z_1, \cdots, Z_k$, equation 4.44 represents a flat surface or plane in the (k + 1) dimensional space with axes log P, log $Z_1, \cdots$, log Z_k.

As an example of a production function postulated as a polynomial, consider the following function of second degree in two explanatory inputs:

Table 4.1. Analysis of Variance for Multiple Regression

Source of Variation in Y	Degrees of Freedom	Sum of Squares	Mean Square
Regression	k	SSR	SSR/k
Deviations from regression	n - k - 1	$\sum_{j=1}^{n} (Y_j - \hat{Y}_j)^2$	s^2
Total	n - 1	$\sum_{j=1}^{n} y_j^2$	

(4.45) $$Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_{11} Z_1^2 + \beta_{22} Z_2^2 + \beta_{12} Z_1 Z_2 + \epsilon.$$

By a simple relabeling of the terms involving Z, it becomes

(4.46) $$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \epsilon$$

where, for instance, $\beta_4 X_4$ corresponds to the *quadratic effect* $\beta_{22} Z_2^2$ and $\beta_5 X_5$ to the *interaction effect* $\beta_{12} Z_1 Z_2$. While equation 4.45 denotes a curved surface in the three dimensional space with axes Y, Z_1, and Z_2, equation 4.46 specifies a plane in the six-dimensional space with axes Y, $X_1, X_2, \cdots, X_5$.

When transformations of the above type have been carried out, care must be taken in interpreting the estimated regression coefficients. Thus if the derived estimate of equation 4.46 is

(4.47) $$\hat{Y} = b_0 + b_1 X_1 + \cdots + b_5 X_5$$

(4.48) $$= b_0 + b_1 Z_1 + \cdots + b_5 Z_1 Z_2$$

then the partial derivative of $\hat{Y}$ with respect to X_1 is meaningless. X_1 cannot vary with $X_2, \cdots, X_5$ held constant; because of equation 4.45, when X_1 varies, X_3 and X_5 must also vary. However, we can derive partial derivatives from equation 4.48 as shown in equation 4.49.

(4.49) $$\frac{\delta \hat{Y}}{\delta Z_1} = b_1 + 2b_3 Z_1 + b_5 Z_2$$

Orthogonal polynomial transformations

First, a word of explanation is necessary. Two polynomials, say ξ_1 and ξ_2, where

(4.50) $$\xi_1 = a_0 + a_1 X + a_2 X^2 + \cdots + a_k X^k$$

(4.51) $$\xi_2 = b_0 + b_1 X + b_2 X^2 + \cdots + b_k X^k$$

are orthogonal if

$$(4.52) \qquad \sum_{i=0}^{k} a_i b_i = 0.$$

Equivalently, ξ_1 and ξ_2 are uncorrelated. The latter fact makes the orthogonal transformation of polynomial functions useful for regression analysis, at least under certain circumstances. These circumstances are that (a) the levels of each explanatory variable be equally spaced; and (b) there are an equal number of observations for each set of levels of the explanatory variables. Obviously, these restrictions will only be met if specialized techniques of data collection are used. Such techniques will usually only be possible under experimental conditions. If the restrictions are not met, the use of orthogonal polynomials is still possible — but not conveniently enough to be worthwhile.

Consider the k degree polynomial production function in a single explanatory variable.

$$(4.53) \qquad Y = \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots + \beta_k X^k$$

Further suppose that n observations of different X values and corresponding y values are available. The orthogonal polynomial transformation of equation 4.53 consists of a function of the form

$$(4.54) \qquad Y = \alpha_0 + \alpha_1 \xi_1 + \alpha_2 \xi_2 + \cdots + \alpha_k \xi_k$$

where α_0 equals $\bar{Y}$ and the other α's are known functions of the β's of equation 4.53. The ξ's are orthogonal polynomials in $(X - \bar{X})$. In particular, ξ_i corresponding to X^i of equation 4.53 will be a polynomial of degree i in $(X - \bar{X})$. Moreover, there will be a set of n ξ_i values, ξ_{ij}, $(j = 1, 2, \cdots, n)$, corresponding to the observed values X_j, $(j = 1, 2, \cdots, n)$, of X.[7] For equally spaced levels of the explanatory variables or input factors, tables of ξ_i values are available.[8] They do not have to be computed by the researcher.

Since any pair of the ξ's are uncorrelated, their sums of cross products are zero. Thus the normal equations for equation 4.54, corresponding to equation 4.15, are:

---

[7] Simple numerical examples of the use of an orthogonal polynomial transformation are given by Ostle, B. Statistics in reserarch. Iowa State University Press, Ames. 1956. Pp. 142-46; and Davies, O. L. (ed.), Statistical methods in research and production. Oliver and Boyd, London. 1957. Pp. 245-49.

[8] Such tables, together with illustrative examples of the use of orthogonal polynomials, are to be found in Pearson, E. S. and Hartley, H. O. Biometrika tables for statisticians. Cambridge University Press. 1958. Vol. 1, Pp. 91-95. Also Anderson, R. L. and Houseman, E. E. Tables of orthogonal polynomial values extended to n = 104. Iowa State University Research Bul. 297. Ames. 1942.

$$
(4.55)\quad
\begin{array}{lllll}
a_1 \Sigma \xi_{1j}^2 & & & & = \Sigma \xi_{1j} y_j \\
& a_2 \Sigma \xi_{2j}^2 & & & = \Sigma \xi_{2j} y_j \\
& & \cdots\cdots & & \vdots \\
& & & \cdots\cdots & \\
& & & a_k \Sigma \xi_{kj}^2 & = \Sigma \xi_{kj} y_j
\end{array}
$$

where the a's are estimates of the α's of equation 4.54. Since all the cross product terms have vanished, each equation of system 4.55 may be solved independently. Thus

$$
(4.56)\qquad a_i = \frac{\sum_{j=1}^{n} \xi_{ij} y_j}{\sum_{j=1}^{n} \xi_{ij}^2} \qquad \text{for } i = 1, 2, \cdots, k.
$$

Also, again because the ξ's are uncorrelated and lead to independent estimates of the α's, the estimate of α_i of equation 4.54 remains unaltered when variables are added to or taken out of equation 4.53. For instance, if a term in X^{k+1} were added to equation 4.53, direct estimation procedures would involve recalculating all of the estimates of the β's. By using the indirect orthogonal polynomial approach of adding ξ_{k+1} to equation 4.54, the estimate of α_{k+1} can be calculated without having to recompute the estimates of $\alpha_0, \alpha_1, \cdots, \alpha_k$. It is then a simple matter to derive the new set of sample regression coefficients estimating $\beta_0, \beta_1, \cdots, \beta_{k+1}$.

The orthogonal polynomial transformation can be applied to polynomials in more than one explanatory variable. All that is necessary is a simple extension of the procedures outlined above. For example, suppose it was desired to estimate the following second degree polynomial in three variables.

$$
(4.57)\qquad
\begin{aligned}
Y = \beta_0 &+ \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \\
&+ \beta_{11} X_1^2 + \beta_{22} X_2^2 + \beta_{33} X_3^2 \\
&+ \beta_{12} X_1 X_2 + \beta_{13} X_1 X_3 + \beta_{23} X_2 X_3
\end{aligned}
$$

The required transformation would be of the form:

$$
(4.58)\qquad
\begin{aligned}
Y = \alpha_0 &+ \alpha_1 \xi_1 + \alpha_2 \eta_1 + \alpha_3 \omega_1 \\
&+ \alpha_{11} \xi_2 + \alpha_{22} \eta_2 + \alpha_{33} \omega_2 \\
&+ \alpha_{12} (\xi_1 \eta_1) + \alpha_{13} (\xi_1 \omega_1) + \alpha_{23} (\eta_1 \omega_1)
\end{aligned}
$$

where α_0 equals $\bar{Y}$ and the other α's are known functions of the β's and the ξ's, η's and ω's are pairs of orthogonal polynomials in $(X_1 - \bar{X}_1)$, $(X_2 - \bar{X}_2)$ and $(X_3 - \bar{X}_3)$, respectively.

As shown by equation system 4.55, the sample regression coefficients in an orthogonal polynomial regression are independent. This fact makes it possible to use analysis of variance directly to test the significance of the contribution to the sum of squares due to regression of each term of the transformed equation; and hence of individual terms of the original equation. For instance, analysis of variance relevant to the estimate of system 4.58 may be carried out as shown in Table 4.2 in which the a's are estimates of system 4.58. It should be contrasted with the analysis of variance possible for ordinary multiple regression as shown in Table 4.1.

As shown in Table 4.2, the mean square for regression can be partitioned into a component for each effect when orthogonal polynomials are used. The significance of each of these individual effects may be tested by an F test. Thus, from Table 4.2, the significance of the linear interaction between X_1 and X_3 may be ascertained from equation 4.59, s^2 being the mean square of the deviations from regression.

$$F = \frac{a_{13}^2 \Sigma (\xi_1 \omega_1)_j^2}{s^2} \tag{4.59}$$

If the value of F from equation 4.59 is larger than the tabled F value with one and (n - 10) degrees of freedom at the desired probability level, the X_1X_3 interaction is significant. Similar tests apply to the other effects. It should be mentioned that, reading down the analysis of variance table, nonsignificant lower order effects may precede significant higher order effects. In the estimation of production functions, therefore, the significance or nonsignificance of lower order effects provides no guide as to the possible role of higher order effects and interactions.

Multiple Regression Analysis with Nonlinear Parameters

So far, least squares multiple regression has only been considered as it applies to functions linear in the parameters. Of course, such a function may be a transformation of a production function that is nonlinear in the parameters to be estimated. For instance, a logarithmic transformation of a Cobb-Douglas function leads to linearity in the parameters (as well as in the transformed variables). Nothing has been said of the estimation of production functions nonlinear in the parameters for which no suitable transformation to linear form is available. The commonest group of such production relations are those known to biological researchers as "growth functions." Their common feature

Table 4.2. Analysis of Variance for a Second Degree Orthogonal Polynomial Multiple Regression in Three Explanatory Variables

Source of Variation in Y	Degrees of Freedom	Mean Square
Regression	9	SSR/9
Linear effect of X_1	1	$a_1^2 \ \Sigma \ \xi_{1j}^2$
Linear effect of X_2	1	$a_2^2 \ \Sigma \ \eta_{1j}^2$
Linear effect of X_3	1	$a_3^2 \ \Sigma \ \omega_{1j}^2$
Quadratic effect of X_1	1	$a_{11}^2 \ \Sigma \ \xi_{2j}^2$
Quadratic effect of X_2	1	$a_{22}^2 \ \Sigma \ \eta_{2j}^2$
Quadratic effect of X_3	1	$a_{33}^2 \ \Sigma \ \omega_{2j}^2$
Linear X_1 by linear X_2	1	$a_{12}^2 \ \Sigma \ (\xi_1 \eta_1)_j^2$
Linear X_1 by linear X_3	1	$a_{13}^2 \ \Sigma \ (\xi_1 \omega_1)_j^2$
Linear X_2 by linear X_3	1	$a_{23}^2 \ \Sigma \ (\eta_1 \omega_1)_j^2$
Deviations from regression	n - 10	$\Sigma \ (Y_j - \hat{Y}_j)^2/(n - 10)$
Total	n - 1	$\Sigma \ y_j^2$

is that they are exponential relationships of one type or another.[9] Growth functions that have been used empirically are the Spillman-Mitscherlich functions of the form

$$Y = \lambda(1 - \delta e^{-\beta x}) \ , \tag{4.60}$$

the Gompertz function

$$\log_e Y = \lambda(1 - \delta e^{-\beta x}) \ , \tag{4.61}$$

and the logistic growth function

$$Y = \frac{\lambda}{1 + \delta e^{-\beta x}} \ . \tag{4.62}$$

Production function researchers will, in general, not be interested in these functions with only one explanatory variable. To be useful, production functions should include a number of explanatory variables. Extension of the above functional forms to include k explanatory inputs leads to production functions of the following types.

[9] An interesting account of the derivation of growth functions is to be found in Hald, A. Statistical theory with engineering applications. John Wiley and Sons, Inc., New York. 1952. Pp. 658-62.

(4.60.i) $$Y = \lambda(1 - \delta_1 e^{-\beta_1 X_1})(1 - \delta_2 e^{-\beta_2 X_2}) \cdots (1 - \delta_k e^{-\beta_k X_k})$$

(4.60.ii) $$Y = \lambda(1 - \delta_1 e^{-\beta_1 X_1} - \delta_2 e^{-\beta_2 X_2} - \cdots - \delta_k e^{-\beta_k X_k})$$

(4.62.i) $$Y = \frac{\lambda}{1 + \delta_1 e^{-\beta_1 X_1} + \delta_2 e^{-\beta_2 X_2} + \cdots + \delta_k e^{-\beta_k X_k}}$$

(4.62.ii) $$Y = \frac{\lambda}{1 + \delta e^{-\beta_1 X_1 - \beta_2 X_2 - \cdots - \beta_k X_k}}$$

For functions such as these, no simple transformations to a form linear in the parameters, λ, δ, β_1, $\cdots$, β_k, are available. Indeed, iterative or successive approximation procedures for estimating such functions are largely unexplored. An exception is an iterative procedure for estimating Spillman-Mitscherlich functions such as equation 4.60.i. that has been developed by Ibach.[10] His approach utilizes a set of standard values that markedly reduce computational procedures. A number of possible approaches to the estimation of logistic functions involving a single explanatory variable have been summarized by Nair.[11] In general, experience to date indicates that cumbersome iterative procedures, as have to be used for most exponential functions of the above types, are not desirable. Especially is this so when account is taken of the fact that research resources are not limitless. An additional disadvantage is that the estimates obtained by such iterative procedures are not readily amenable to statistical significance tests. Since any production surface can be fitted reasonably well by an easily calculated polynomial-type function, there seems little justification for persevering with functions requiring complex iterative procedures.

Functions of the following algebraic form,

(4.63) $$Y = \frac{\alpha + \beta X}{1 + \delta X},$$

have also sometimes been suggested for production functions. To fit equation 4.63, which is nonlinear in both its variables and parameters, the method of internal least squares regression developed by Hartley may be used.[12] However, computational procedures become extremely complicated when such algebraic functions are extended to more than one explanatory variable, as in "resistance functions" like equation 4.64.

---

[10] Ibach, D. B. Use of standard exponential yield curves. ARS. 43-69. USDA, Washington, D. C. 1958. Also, A graphic method of interpreting response to fertilizer. Handbook No. 93. USDA, Washington, D. C. 1956.

[11] Nair, K. R. The fitting of growth curves, in Kempthorne, O. *et al.* (eds.), Statistics and mathematics in biology. Iowa State University Press, Ames. 1954. Pp. 119-32.

[12] See Hartley, H. O. The estimation of non-linear parameters by "internal least squares." Biometrika, 35: 32-45. 1948.

$$(4.64) \quad \frac{1}{Y} = \alpha_0 + \frac{\alpha_1}{\delta_1 + \beta_1 X_1} + \frac{\alpha_2}{\delta_2 + \beta_2 X_2} + \cdots + \frac{\alpha_k}{\delta_k + \beta_k X_k}$$

As for the exponential type functions, the computational disadvantages of equation 4.64 outweigh any advantages resistance functions may have as logical expressions of the production process under consideration. For this reason, it will be assumed in the following discussion that if a single equation approach is taken, the regression function being estimated is linear in the parameters. The restriction on the algebraic form of the production function is that it be capable of being transformed to a form linear in the parameters.

Statistical Problems in Least Squares Estimation of Production Functions

Early in this chapter, the assumptions of the least squares approach were listed. If these assumptions hold true, least squares estimates of the production function parameters will be unbiased and of minimum variance. In general, the assumptions will not be satisfied exactly. Problems of estimation and interpretation then arise. Some of these difficulties are discussed here, insofar as they commonly arise in production function research. It should be emphasized, as a rider to the following discussion, that statistical techniques, no matter how "high powered," cannot be used to derive satisfactory estimates from bad data.

Errors of observation in the independent variables

In the prior discussion of multiple regression, it was assumed that the data contained no errors. The only error allowed was an error in the equation due to the omission of some input factors. Of course, for a model such as equation 4.1 to be satisfactory, any excluded inputs should only be of minor importance. In point of fact, the assumption of no errors whatsoever in the measurement of the input variables is extremely strong. Errors of observation, of some magnitude, will always be present. Such errors may arise for a variety of reasons, including the obvious one that a continuous variable can rarely be measured with absolute exactness. Other causes of observational errors are the human element involved—for instance, mistakes may occur in the collection and transcription of data—and, most importantly, the fact that in many situations a satisfactory conceptual basis of measurement is not available. The latter, of course, is the essence of the *problem of aggregation.* It arises whenever the input or output under consideration is not homogeneous, either within or between observations. Insofar as aggregation is a problem of economic specification, it will be discussed in Chapter 6. However, some explanation of the problem is necessary

here. A typical example of the aggregation problem occurs in the measurement of land, labor and capital, say, as inputs for the derivation of a farm production function from cross-sectional data. Variations in quality and mode of use will occur in all these variables from one observation to another. Consider the land input. Not only may there be variations in soil type within the one observation and between the cross-sectional observations, there may even be variations in quality within the same soil type. To measure the land inputs simply in terms of acres would ignore these differences in quality. What is needed, in such circumstances, is some basic unit in which the observations may be measured so that they are strictly comparable. To attempt to formulate such a basic unit involves conceptual difficulties of a high order, as the reader will quickly realize should he try to solve the problem. Similar difficulties arise with regard to any input or output category that is not strictly homogeneous both within and between observations. If the observed values of the variables are not comparable because of a lack of homogeneity, use of the raw observations means that observational errors are necessarily present in the data. If some method of adjusting the data to take account of heterogeneity is used, the adjusted data will contain observational errors insofar as the adjustment process or aggregation procedure is imperfect. Moreover, in general, no aggregation procedure will be perfect because of the conceptual difficulties involved. On the other hand, if the relevant variables are homogeneous, the degree to which errors of observation occur is simply a question of the mechanics of data collection. Provided sufficient care is exercised, mechanical errors of observation can be reduced to negligible proportions. On these grounds, in fitting empirical production functions, the most important source of errors in the observations is the lack of homogeneity.

Given the fact that observational errors will generally be present in the data, the researcher may take a number of approaches. If the errors are thought to be negligible — averaging no more than 1 or 2 per cent, say, and with no gross errors present — then the ordinary least squares multiple regression approach may be used. If the errors cannot be regarded as negligible, the alternative approach of weighted regression may be used. However, before discussing this technique, mention should be made of a procedure sometimes used to determine whether or not the observational errors may be regarded as negligible. It is known as confluence analysis.

Confluence analysis

Briefly, the steps involved in confluence analysis are as follows. Suppose equation 4.65 is the production model being examined, X_1 being the output, X_2 and X_3 the inputs.

$$X_1 = \beta_1 + \beta_2 X_2 + \beta_3 X_3 + \epsilon \tag{4.65}$$

The sample regression coefficient of X_1 on X_2 is b_2 of equation 4.66. It is obtained by excluding X_3 from the analysis and minimizing the sum of squares in the direction of X_1. The latter procedure implies the assumption that the only errors of observation are in X_1.

$$X_1 = b_1 + b_2X_2 \tag{4.66}$$

If the sum of squares is minimized with respect to X_2, again excluding X_3, an equation such as 4.67 results.

$$X_2 = c_0 + c_1X_1 \tag{4.67}$$

If equation 4.67 is put in the same form as equation 4.66, as shown in equation 4.68

$$X_1 = -c_0\,c_1^{-1} + c_1^{-1}\,X_2 \tag{4.68}$$

then c_1^{-1} may be regarded as the sample regression coefficient of X_1 on X_2 when all the errors of observation are in X_2.

The procedures typified by equations 4.66, 4.67, and 4.68 are then carried through in similar fashion for the set X_1, X_2, and X_3; the sum of squares being minimized in the direction of each variable in turn. The next step is to convert each of these "estimates" of b_2 to the standardized partial regression coefficient form. The procedure for doing this is given by equation 4.26. The standardized coefficients of X_1 on X_2, obtained with X_3 excluded, are then compared with those obtained with X_3 included. For convenience, this comparison is usually made graphically by way of *bunch maps*. Consider first the situation when the inclusion of X_3 leads to the standardized coefficients becoming more similar, i.e., a bunching together. It may then be concluded that the effect of any observational errors in X_3 is negligible. Also, that X_3 is a "useful" variable to include in the model. On the other hand, the inclusion of X_3 may lead to an explosion of the standardized coefficients. The researcher then has an indication that if any errors are present in X_3, they will probably induce nonsensical estimates of the parameters of equation 4.65. In such circumstances, it would be wise to exclude X_3 from the model — unless the logic underlying the production process strongly supported its inclusion.

For guidance as to whether the inclusion of X_2 would be useful or detrimental, the same analysis as above must be carried through; the roles of X_3 and X_2 being interchanged. In similar vein, a model containing k inputs can be checked.[13] However, as k increases, the

[13] Detailed accounts and illustrations of the method are to be found in Frisch, R. Statistical confluence analysis by means of complete regression systems. Publication No. 5. Oslo University Institute of Economics. 1934; and Stone, R. The measurement of consumers' expenditure and behaviour in the United Kingdom, 1920-1938. Cambridge University Press. 1954. Vol. I, pp. 300-1 and 342 ff. For an application of confluence analysis in the fitting of production functions, see Antill, A. G. Towards a production function for dairy farms. The Farm Economist. 8: 1-11. 1955. Also, Valavanis, S. Econometrics. McGraw-Hill, New York. 1959. P. 146.

number of combinations of variables that must be examined increases rapidly. The necessary computations quickly become immense. The method also has other disadvantages. One, perhaps the most important, is that confluence analysis is a subjective technique, there being no objective manner of deciding whether the standardized partial regression coefficients are sufficiently similar to imply negligible errors. Nor whether an explosion really implies nonnegligible errors. Also, it assumes that no relevant variables have been omitted from the model. Obviously, in most production function research, this latter assumption will not be true.

Weighted regression

With errors of observation present, each of the observed values X_{ij} ($i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, n$) may be considered to be made up of a systematic component M_{ij} which is the true quantity of the variable, and an error of observation η_{ij}. Thus

$$X_{ij} = M_{ij} + \eta_{ij} \quad . \tag{4.69}$$

Under the assumptions that (a) there are no relevant variables omitted from the single equation model being used, i.e., no errors in the equation, and (b) the errors η_{ij} are noncorrelated, random, and normally distributed the method of weighted regression may be used. The essential feature of the weighted regression technique is that the variances and covariances of the observational errors are used as weights in estimating the regression coefficients. By the nature of the problem, the errors and their variance-covariance matrices are generally unknown. These variances and covariances have to be estimated in some fashion — either from past experience, by guessing, or by an iterative procedure.

A detailed exposition of the technique of weighted regression, and some examples of its application, has been given by Tintner. A variation of the method, in a production function context, is illustrated in some work of Antill.[14]

There are a number of disadvantages to using weighted regression. The computations involved are cumbersome. It may be difficult to obtain satisfactory estimates of the variances and covariances of the errors. The assumptions that the observational errors are noncorrelated, random and normally distributed may be just as false as the assumption of no observational errors. Indeed, when the observational errors arise because of an imperfect aggregation procedure, it is rather unlikely that these errors are normally distributed and arise at random. In addition, the assumption that there are no errors in the equation, i.e., that no relevant variables have been left out, will not be

[14] Tintner, G. Econometrics. John Wiley and Sons, New York. 1952. Pp. 121-43; Antill, G. A. Towards a production function for dairy farms. The Farm Economist, 8:1-11. 1955.

true in general. Against the latter criticism, it must be noted that no method has so far been developed by which account may be taken of both errors in the variables and errors in the equation. Still, given the complexity of the weighted regression approach and the assumptions that it makes, it may be that the use of weighted regression raises more problems than it solves. The researcher who does not have negligible errors of observation in his data might be best advised to seek better data rather than try to develop reliable estimates out of unreliable data.

Autocorrelation

By autocorrelation is meant correlation between successive items in a time series of observations. Since time series data are frequently used in production function estimation, the difficulties raised by autocorrelation are pertinent to this discussion. Two facets of the problem may be distinguished: autocorrelation in the observations on one or more of the variables, say $Y, X_1, X_2, \cdots, X_k$ of equation 4.1, and autocorrelation of the errors ϵ_{ij} ($i = 1, 2, \cdots, k$; $j = 1, 2, \ldots, n$) of equation 4.1.[15]

Autocorrelation in the variables

If autocorrelation is present in the observations of one or more of the variables, the estimates of the regression coefficients obtained via least squares are still unbiased and of minimum variance. However, if the variable X_i is autocorrelated, the variance of its regression coefficient b_i will be affected so that equation 4.29 does not hold true. In consequence, there is no need to worry about autocorrelation among the variables unless it is desired to calculate confidence limits and significance tests for the regression coefficients. Such calculations are generally necessary in production function research. Hence any time series observations to be used in estimating a production function should be tested for autocorrelation unless the observations are obviously independent. Some explanation of "obviously independent" is required. Suppose 40 readings of hog weight, each reading taken from a different hog at a different point in time, are to be used in estimation. There would be no reason to expect these 40 readings to be autocorrelated. On the other hand, it is very likely that 40 readings of hog weight at different points in time, all made on the same hog, would be autocorrelated. The reason, of course, is that the readings are not independent — they all relate to the same hog.

The simplest test for autocorrelation is the *von Neumann ratio test.* The first step is to calculate the ratio σ^2/s^2, where, for the n observations on the variable X_i

[15] See Anderson, R. L. The problem of autocorrelation in regression analysis. Journal of the American Statistical Association, 49: 113-29. 1954. Also, Beach, E. F. Economic models: an exposition. John Wiley and Sons, Inc., New York. 1957. Pp. 176-80.

$$\sigma^2 = \frac{\sum_{j=1}^{n-1} (X_{ij+1} - X_{ij})^2}{n - 1} \tag{4.70}$$

and

$$s^2 = \frac{\sum_{j=1}^{n} (X_{ij} - \bar{X}_i)^2}{n} \tag{4.71}$$

The calculated von Neumann ratio is then checked against tabled values of the ratio, as presented by Hart, to see if the series is significantly autocorrelated.[16]

If autocorrelation is present in the observations of X_i, the variance of b_i may be calculated by the following modification of equation 4.29 due to Wold.[17]

$$\mathrm{Var}(b_i) = c_{ii}s^2f^2 \tag{4.72}$$

where f^2 is estimated by the first few terms of

$$f^2 = 1 + 2r_{i1}\rho_1 + 2r_{i2}\rho_2 + \cdots + 2r_{in}\rho_n, \qquad i = 1, 2, \cdots, k\ . \tag{4.73}$$

The r_{ij} and ρ_j ($j = 1, 2, \cdots, n$) values of equation 4.73 are the noncircular autocorrelation coefficients for X_i and the errors, ϵ of equation 4.1 estimated by e of equation 4.3, respectively.[18] Approximate tests of significance and confidence limits may then be calculated as in equation 4.34 by using the estimate of the regression coefficient's variance obtained from equation 4.72. Little reliability can be attached to these procedures if the number of observations, n, is less than 30. Moreover, equation 4.72 assumes that the regression is strictly linear in the variables. If the regression is curvilinear, that is, nonlinear in the untransformed variables, the procedure is only approximate. Insofar as the majority of production functions are nonlinear in the original variables, this is unsatisfactory.

An alternative approach may be taken to the problem of autocorrelation if the autocorrelated series is replicated. The procedure is based on the fact that the effect of autocorrelation is to, in a sense, reduce the number of effective observations to which the regression function is fitted. In other words, the number of degrees of freedom used in entering tables for tests of significance relevant to an autocorrelated variable

[16] Hart, B. I. Significance levels for the ratio of the mean square successive difference to the variance. Annals of Mathematical Statistics. 13: 445-47. 1942.

[17] Wold, H. Demand analysis. John Wiley and Sons, New York. 1953. Pp. 44 and 209-12.

[18] The rather complicated formulae by which these autocorrelation coefficients may be estimated, together with a detailed discussion of autocorrelation, are to be found in Tintner, G. Econometrics. John Wiley and Sons, New York. 1952. Pp. 240-55, esp. 243. See also Ezekiel, M. and Fox, K. A. Methods of correlation and regression analysis. John Wiley and Sons, Inc., New York. 1959. Pp. 325-47.

is smaller than it would be if the observations of the variable were not autocorrelated. Procedures are available for approximating the effective number of degrees of freedom in autocorrelated series.[19] However, the necessity of calculating the autocorrelation coefficient and approximating the effective number of observations may be avoided. How? By basing the tests of significance on the minimum number of effective observations to which the series could be reduced by autocorrelation. To give an example, 20 observations of weight taken on one hog could, through strong autocorrelation, be equivalent to a minimum of one effective observation. However, a series of 100 hog weight observations made up of 20 observations on each of five hogs could, at worst, be reduced to five effective readings since the experimental units (hogs) are independent. If significance tests of the regression coefficients calculated in the ordinary fashion of equation 4.34 but with degrees of freedom based on the minimum number of effective observations are significant, the null hypothesis that the regression coefficient is zero may be rejected. Should the test be nonsignificant, the null hypothesis cannot be accepted without further testing. The test must then be conducted on the basis of the actual number of observations, disregarding the presence of autocorrelation. If the test is still not significant at an acceptable probability level with the greater number of degrees of freedom, the null hypothesis that the regression coefficient is not significantly different from zero may be accepted. The effective number of observations need be estimated only if the tests are nonsignificant on the basis of the minimum effective number of observations, but significant on the basis of the actual number of observations taken.

Autocorrelation in the errors

If the errors ϵ_j, $(j = 1, 2, \cdots, n)$, relevant to equation 4.1 are autocorrelated, the least squares assumption that $E(\epsilon_i \epsilon_j) = 0$, $i \neq j$, does not hold true. Under such circumstances, ordinary least squares estimates of the regression coefficients will not be unbiased, nor will they have minimum variance. However, if the variances and covariances of the errors are known, a modified least squares procedure leading to suitable estimates may be used.[20] Insofar as the error variances and covariances usually have to be estimated from the data, or guessed, this adaption of least squares may not, in practice, lead to reliable estimates of the production parameters. In general, the errors might be expected to be autocorrelated when (a) the algebraic form of the estimated production function is not very satisfactory in terms of fitting the data, (b) there are errors of observation in the data, and (c) important explanatory variables have been omitted from the analysis. In estimating production functions from nonexperimental time series data, all

[19] See Orcutt, G. H. and James, S. F. Testing the significance of correlations between time series. Biometrika, 35: 397-413. 1949.

[20] See Tintner, G. Econometrics. John Wiley and Sons, New York. 1952. Pp. 279-283.

three of these causes of autocorrelation may occur. However, there is no reason why these sources of error autocorrelation should not be minimized under experimental conditions, especially if care is taken in designing the experiment and collecting the data. Insofar as most agricultural production function estimation involving time series uses experimental data, the problem of autocorrelation of the residuals is not too frequent. A number of tests for testing if the errors are autocorrelated are possible. Perhaps most useful is that developed by Durbin and Watson.[21]

It should be pointed out that the lack of independence — known as autocorrelation in time series data — between observations of a given variable, and between the errors in the regression model, may also occur between observations adjacent in space. To guard against this is one of the reasons why randomization procedures are adopted in designing experiments. Should spatial observations be "autocorrelated" — which may often occur with nonexperimental data — little account can be taken of the lack of independence since, while time series are generally equally spaced in time, spatial data are usually irregularly spaced in terms of distance between observations.

Multicollinearity

By multicollinearity is meant the situation when there is more than one relationship among the variables being considered. For purposes of exposition, two types of multicollinearity may be distinguished. The first relates to the problem of whether the single equation model is valid or whether the production function should be estimated as one of a number of simultaneous relations. Since this question is largely one of economic specification, it is deferred until Chapter 6. The other type of multicollinearity relates to the single equation model when two or more of the independent variables are so highly correlated that there is, for all intents, one or more linear relations between some or all of the independent variables. Such cases of multicollinearity are statistical in emphasis and will be considered here. They are closely related to the problem of errors of observation in the independent variables.

As a simple example consider the problem of estimating the production function

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \epsilon \tag{4.74}$$

when X_1 and X_2 are perfectly correlated so that

$$r_{X_1X_2} = 1 = r_{X_2X_1} \tag{4.75}$$

[21] Durbin, J. and Watson, G. S. Testing for serial correlation in least squares regression. Biometrika, 37: 409-28. 1950.

The normal equations, expressed in terms of the partial correlation coefficients, as in equation 4.25, are then as follows:

$$(4.76)\qquad \begin{aligned} b'_1 r_{11} + b'_2 r_{12} &= r_{1y} \\ b'_1 r_{21} + b'_2 r_{22} &= r_{2y} \end{aligned}$$

Since all the partial correlation coefficients are equal to one, the two normal equations are identical. In other words, there is only one equation from which to estimate the two unknowns, b'_1 and b'_2. Obviously, the values of b'_1 and b'_2 — and hence of b_1 and b_2 — are indeterminate. Now suppose that X_1 and X_2 are, in truth, perfectly correlated but because of observational errors in the data, the partial correlation between the observed values of X_1 and X_2 happens to be 0.98. The normal equations would be:

$$(4.77)\qquad \begin{aligned} b'_1 + 0.98b'_2 &= r_{1y} \\ 0.98b'_1 + b'_2 &= r_{2y}\ . \end{aligned}$$

Since these two normal equations are not identical, they could be solved for b'_1 and b'_2. However, these values of b'_1 and b'_2 would have no relevance in terms of the production process in equation 4.74. Being merely the consequence of errors in the data, they are meaningless. Referring to equation 4.69, it can be said that the sample regression coefficients computed from equation 4.77 are determined completely by the unsystematic errors of observation η_{ij}. The estimated coefficients do not reflect a relationship between the systematic portions M_{ij} of the observations on the variables.

A geometric interpretation of multicollinearity is also possible. Suppose Y, X_1, and X_2 were measured without error, X_1 and X_2 being perfectly correlated. Plotting the observed values of Y against the observed values of X_1 and X_2 would give a straight line in the three-dimensional space with axes Y, X_1, and X_2. Since an infinite number of planes can satisfy the condition of passing through one straight line, regression analysis using such data would lead to indeterminate results. However, since the observations will always be subject to some error, plotting the observed Y values would not give a straight line. The result is an apparently determinate regression plane.

If, instead of there being one or more exact linear relationships among the independent variables, there is a high degree of correlation between some of the independent variables, the collinearity will be only approximate. Such a situation results in the estimates of the regression coefficients having large variances because the errors in the observations tend to play a dominant role in determining the values of the parameters. Moreover, the estimated values of the sample regression coefficients tend to be relatively unstable due to their sensitivity to observational errors in the data.

A number of approaches to multicollinearity are possible. Since the

correlation coefficients between the independent variables serve as indicators of the possible presence of linear or near linear relations among the independent variables, it is wise to calculate and examine all these coefficients for the sample data. If any of these coefficients are close to plus or minus one, say greater than $|0.8|$, the regression analysis should be carried through with one of the highly correlated variables omitted. Which variable(s) to omit and which to retain should be decided on the basis of the logic — physical, biological, or economic — relevant to the production process being examined. If the R^2 is satisfactory and the logic of the production situation does not dictate that the excluded variable(s) must be included, the new regression estimates may serve satisfactorily. The method of confluence analysis, discussed above with regard to errors in the observations, may be of some use as a guide to deciding whether nuisance relations are present among the independent variables. Also, bunch maps may help in deciding whether a particular variable should be included in the analysis or whether its inclusion might lead to meaningless estimates of the regression coefficients. Frisch has outlined the use of confluence analysis as a guide to the problem of multicollinearity.[22] An example of the use of confluence analysis in production function estimation has been given by Antill.[23]

An alternative procedure is to attempt to test statistically whether there are any other linear relations among the variables apart from that postulated by the production model.[24] These test procedures are rather closely related to the technique of weighted regression and may conveniently be used when weighted regression is used. Such complementarity exists because weighted regression is aimed at overcoming the effect of errors in the observations while the effects of multicollinearity, of the type discussed here, are also closely related to the presence of errors in the variables. The uninitiated reader must be warned though that such testing procedures, like weighted regression analysis, involve formidable computations and a number of assumptions that may not be well founded. For instance, it is assumed that the η's of equation 4.69 are known. So far as possible, therefore, it is very worthwhile to try and overcome the possibility of multicollinearity in the sample data — where it may be present by fortuitous circumstances — by the use of appropriate sampling methods. Such methods are discussed in Chapter 5.

Should the researcher have a priori knowledge of the value of any of the structural parameters, the normal equations may be made determinate by utilizing these known values. Thus, referring to equation 4.76, if b_1 were known a priori, there would be no problem of indeterminancy. The value of b'_2 could be ascertained by inserting the known value of b'_1 in equation 4.76 and solving for b'_2.

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[22] Frisch, R. Statistical confluence analysis by means of complete regression systems. Publication No. 5. Oslo University Institute of Economics. 1934.

[23] Antill, A. G. Towards a production function for dairy farms. The Farm Economist, 8: 1-11. 1955.

[24] Such a test is illustrated in Tintner, G. Econometrics. John Wiley and Sons, New York. 1952. Pp. 127-38.

Unequal error variances

One of the assumptions under which the least squares principle gives unbiased estimates with minimum variance is that the ϵ's, arising as in equation 4.1, have the same variance. In short, that $E(\epsilon_j^2) = \sigma^2$ for $j = 1, 2, \cdots, n$. In production function estimation, this assumption may be doubted if there is thought to be a relation between any of the independent variables studied and the omitted variables accounted for by the error term, ϵ. Consider the following hypothetical example. Suppose the crop response function to a fertilizer mixture is being studied. It may be that the effect of the omitted variable sunlight, measured exclusively by ϵ say, depends on some crucial enzyme whose production in turn depends on the quantity of fertilizer applied. At low levels of fertilizer, the effect of sunlight may be inhibited so that ϵ is small and relatively constant, having a small variance. At high levels of fertilizer, because the effect of sunlight is not inhibited but varies in response to other features of the crop, ϵ may be large and have a large variance. The assumption of constant error variance would, therefore, be violated.

When it is thought that the error variances may not be constant, the data may be partitioned into a number of sets to obtain an estimate of the error variance in each range of the data. Comparison of these variance estimates then indicates whether the assumption of constant error variance is violated. If it is, the variance for the j-th range of the data may be weighted by a factor, w_j, satisfying equation 4.78.

$$w_j s_j^2 = \sigma^2 \tag{4.78}$$

An arbitrary value is selected for σ^2, say $\sigma^2 = 1$, and equation 4.78 solved to determine w_j; s_j^2 having been determined for each range of the data by way of equation 4.30. Using these w_j values as weights on the cross product sums in the normal equations, unbiased estimates of the regression coefficients with minimum variance may be derived. An exposition of the method and some illustrative examples are given by Anderson and Bancroft.[25] Unlike the procedures necessary when many of the other assumptions of least squares are violated, the above technique for handling unequal error variances is quite straightforward. All that is necessary is that a sufficient range of data and number of observations be available to calculate the s_j^2 and w_j values.

SIMULTANEOUS EQUATIONS APPROACH

So far in this chapter, production function estimation has been considered within the framework of a single equation model. If the

[25] Anderson, R. L. and Bancroft, T. A. Statistical theory in research. McGraw-Hill Book Company, Inc., New York. 1952. Pp. 182-86.

production relationship is in fact a unilateral causal relation with output dependent upon a number of predetermined input variables, the single equation model is logically appropriate. Under such circumstances, the least squares multiple regression procedure provides the best estimates of the production function parameters. However, as mentioned at the beginning of the current chapter, the production relationship may be but one of a number of simultaneously determined relationships involving output and inputs, and other variables also, not as dependent and independent variables but as mutually determined variables.[26] If the economic, biological, or physical logic relevant to the production process dictates that such is the case, then, ideally, the parameters of the production relationship should be estimated in terms of the complete set of simultaneous equations in which the production relation is embedded. Leaving questions of model formulation till Chapter 6, the estimation procedures relevant to the simultaneous approach are discussed briefly below. No attempt is made to elaborate the mechanics of the estimational procedures. With a knowledge of the statistical and mathematical concepts elaborated in the prior sections of this chapter, the researcher should be able to follow the simpler expositions of the simultaneous equations techniques to be found in the references noted. Also, many of the problems that arise in dealing with a system of equations are similar to those that occur with single equation models.[27] Some such problems are the choice of variables, selection of the algebraic form of the functions, the validity of the assumptions made and the interpretation of results.

Types of Variables

In a model consisting of a system of simultaneous equations, two types of variables may be distinguished. *Endogenous variables* are those whose current values are determined simultaneously by the system. Contrasting with these are the *predetermined variables;* their current values may be taken as given. Predetermined variables may be of two types: *exogenous* if their values are determined outside the system of equations, and *lagged endogenous* variables. Which variables are classified as endogenous and which as exogenous depends upon the logic of the situation being examined. To be complete, a model with G endogenous variables should consist of G equations.

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[26] For examples of simultaneous equation models oriented to production function estimation, see the following: Marschak, J. and Andrews, W. H. Random simultaneous equations and the theory of production. Econometrica, 12: 143-63. 1944; French, B. L. Simultaneous economic relationships and derivation of the production function, in Heady, E. O. *et al.* (eds.), Resource productivity, returns to scale, and farm size. Iowa State University Press, Ames. 1956. Pp. 97-105. Also, Jarrett, F. G. Estimation of resource productivities as illustrated by a survey of the Lower Murray Valley. Australian Journal of Statistics, 1: 3-11. 1959.

[27] See Foote, R. J. A comparison of single and simultaneous equation techniques. Journal of Farm Economics. 37: 975-90. Also, Valavanis, S. Econometrics. McGraw-Hill, New York. 1959.

Assumptions

The assumptions made in using simultaneous equation techniques of estimation are rather similar to the assumptions of least squares. There are assumed to be no errors of observation in the variables; only errors in the equations are permitted. Like the ϵ of equation 4.1, these errors are supposed to arise because some relevant variables are omitted. The expected value of the error in each equation should be zero. Moreover, the error in a given equation should be independent of the predetermined variables in the equation and should not be autocorrelated. Obviously, this list of assumptions is just as formidable as those for least squares estimation. If tests of significance and confidence intervals are to be derived for the estimates of the parameters, the additional necessary assumptions are generally stronger than those made in the least squares case.

Identification

In order to estimate the parameters of a production relation embedded in a system of simultaneous equations, it is necessary that the production equation be either just-identified or over-identified. For a system of G equations with G endogenous variables, the following rules of thumb can be used to decide whether a given equation is identified or not. Let D be the number of predetermined variables appearing in the system but not in the equation at hand; and H the number of endogenous variables that appear in the equation being studied. The given equation is *just-identified* if $D = H - 1$; *over-identified* if $D > H - 1$; and *under-identified* if $D < H - 1$. The parameters of an under-identified equation cannot be estimated satisfactorily. If an equation is just-identified or over-identified, a number of statistical techniques are available for estimating the parameters.

Estimation of Just-Identified Equations

A model consisting of G interdependent equations in G endogenous variables may, by algebraic manipulation, be written so that each endogenous variable appears as a function only of the predetermined variables in the system. In such equations, known as *reduced form equations,* the coefficients of the predetermined variables are combinations of the coefficients in the original equations of the model. Each reduced form equation may be considered as a regression law, the endogenous variable being taken as dependent upon the predetermined or independent variables in the reduced form equation. Hence the coefficients of a reduced form equation may be estimated by least squares multiple regression. If the production relation is just-identified, estimates of its parameters can be derived from the least squares estimates of the

parameters in the reduced form equations.[28] It should be noted that, in themselves, the estimates of the parameters of the reduced form equations mean nothing. What is wanted are estimates of the parameters of the original equations of the model. These parameters depict the structure — economic, biological or physical — of the situation being studied. Hence they are known as structural coefficients. As emphasized previously, it is knowledge of the structure of the process that is pertinent to the economic analysis of a production relationship.

Estimation of Over-Identified Equations

A number of approaches are possible if the production relation is over-identified. From the point of view of obtaining the most satisfactory estimates of the structural coefficients, the *full information maximum likelihood* method is best. This technique involves solving all of the equations in the system simultaneously. As might be expected, the procedure involves an extremely large number of complicated computations. For this reason, it has seldom been used. Instead, the *single equation limited information maximum likelihood* procedure has generally been followed. Using this algorithm, each equation of interest in the model is estimated separately; utilizing, however, the information that is available about the role of the predetermined variables in the other structural equations. Statistically, the limited information approach does not give estimates of the structural coefficients that are as reliable as those obtained by the full information approach. But it is very much simpler in terms of computational procedures; and gives, relevant to the amount of information that it uses, estimates that are statistically satisfactory.[29]

Mention should also be made of a newer single equation technique, known as the Theil-Basmann method, for estimating the structural coefficients in a simultaneous equations model. The procedure is rather similar to the single equation limited information approach but is much simpler computationally.[30]

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[28] The procedures involved have been explained and illustrated by Tintner, G. Econometrics. John Wiley and Sons, New York. 1952. Pp. 166-72. See also: Koopmans, T. C. (ed.), Statistical inference in dynamic planning. Cowles Commission for Research in Economics, Monograph No. 10. John Wiley and Sons, New York. Pp. 153-237; and Hood, W. C. and Koopmans, T. C. Studies in econometric method. Cowles Commission for Research in Economics, Monograph No. 14. John Wiley and Sons, New York. 1953. Pp. 112-99.

[29] For a simple, complete explanation of limited information estimation procedures, see Friedman, J. and Foote, R. J. Computational methods for handling systems of simultaneous equations with applications to agriculture. Agriculture Handbook No. 94. USDA, Washington, D. C. 1955.

[30] An explanation of the Theil-Basmann approach is to be found in Wallace, T. D. and Judge, G. G. Econometric analysis of the beef and pork sectors of the economy. Tech. Bul. T-75. Oklahoma State University, Stillwater. 1958. See also Basmann, R. L. The computation of generalized classical estimates of coefficients in a structural equation. Econometrica. 27: 72-81. 1959.

Recursive Systems

A particular type of simultaneous equations model is that known as a *recursive causal chain* system.[31] Each equation in such a system specifies a unilateral causal relationship between a dependent variable and a number of independent variables, as in equation 4.1. The equations of the model must be read in sequence, the dependent variable of an earlier equation entering the system again as an independent variable in a later equation. Because it contains only a single dependent variable, each equation in such a system may be estimated by least squares. As for the general simultaneous equations approach, whether or not a recursive causal chain model should be used depends on the underlying logic of the particular production process being examined.

CHOICE BETWEEN ESTIMATION PROCEDURES

If research resources were free goods, the choice between estimating production parameters from a single equation or a simultaneous equations model could be decided in terms of the logic underlying the production process being examined; provided, of course, that this basic logic is known. But research personnel, funds, equipment and time are not free goods. Some account has to be taken of their availability. As previously mentioned, least squares estimation is generally least expensive and simplest to carry out. Moreover, it is the logically appropriate technique when a single equation model is correct or when a recursive causal chain model applies. Also, least squares may be used indirectly via the reduced form equations if the production equation in a simultaneous model is just-identified. If the production relation in a simultaneous equations model is over-identified, least squares estimates — at least theoretically — are not as good as those derived by simultaneous equation techniques. The commonest such techniques are the full information maximum likelihood, limited information single equation maximum likelihood and the Theil-Basmann limited information procedures. For most practical purposes, the full information procedure must be dismissed as too expensive and complicated. The limited information procedures are also far more expensive than if least squares is applied to the production relation. The practical question that arises is whether the additional reliability of the estimates derived by single equation limited information procedures is worth their additional expense. No general answer can be given to this question. It is noteworthy, though, that in the United States, where the relevant research resources are far less restricted than in any other country, the majority of agricultural production function estimates have been based on least squares. Still, that was before the advent of electronic computers.

[31] For exhaustive references and major recent developments, see Wold, H. Ends and means, in Grenander, U. (ed.), Probability and statistics. Almgrist and Wicksell, Stockholm. 1960. Pp. 355-434.

CHAPTER 5

Data Collection for Production Function Estimation

IN OUTLINING the statistical procedures and problems relevant to the fitting of empirical production functions, it was assumed that data were available. Other than noting that the techniques of data collection are in part dictated by the estimation procedures to be used, no mention was made of the possible methods of data collection nor of the difficulties inherent in data collection. Still, it must be strongly emphasized that the collection of data — whether it be by experiment, survey, or transcription — is an integral part of research aimed at the empirical specification of production functions. The fitted function can only be correct to the extent that the data behind it reliably represent the production process. If the data are wrong, the estimated production parameters must also be wrong. Likewise, if the data are incomplete, only incomplete implications can be drawn from the fitted function. In short: shabby data can only lead to shabby results.

The present chapter considers only the more general techniques and problems of data collection. Those procedures and difficulties that are pertinent only to a particular product or particular circumstances will be considered later in Chapter 7 and the empirical chapters relevant to particular production processes.

The primary problem of data collection is to decide which variables should be observed. Only then does the question arise of how best these variables might be studied and in what form the observations should be made. So far as determining which variables are pertinent, the researcher must have a fair degree of familiarity with the production process under study. He must, at least, be sufficiently familiar with the situation to sketch a model indicating, perhaps tentatively, the variables involved. Such a tentative model, for the single equation case, might be written as in equation 5.1 with output merely specified as some unknown function of inputs $X_1, X_2, \ldots, X_k$.

$$Y = f(X_1, X_2, \ldots, X_k) \tag{5.1}$$

So far as possible, any a priori logic relevant to the process should be utilized; irrespective of whether it be based on theoretical considerations or previous empirical study. Such logic is essential in deciding whether a single equation or a simultaneous equations model is

appropriate; and, if the latter, which variables are endogenous and which predetermined. Moreover, as discussed in Chapter 3, familiarity with the production process and its underlying logic provides a basis for delimiting the possible algebraic forms the production function might take. If a simultaneous equations model appears relevant and computationally feasible, it may be necessary to collect data on one or more variables that enter the system of equations but do not enter the production relationship itself. The sketching of a model enables such variables to be easily pinpointed. Also, from an examination of the model and the practical circumstances surrounding the production process, it may be decided that satisfactory observations cannot be made of some of the variables. Such a judgment might be necessary if some variables have no scale of measurement or if the surmounting of mechanical difficulties of observation would be too expensive.

For all the above reasons, model construction and practical familiarity with the production process are primary requirements for meaningful data collection. They are essential. Oftentimes, too, the researcher must take account of institutional restrictions on the type and range of data he may collect. In this category, perhaps the most important class of restrictions are those that arise in co-operative research.

ALTERNATIVE APPROACHES TO DATA COLLECTION

There are two broad approaches to data collection. The researcher may sometimes have a choice between the two. More generally, the availability of research funds and the circumstances of the production process will be such that it is feasible or advisable to utilize only one of these approaches to data collection. The first approach is to use experimental data. Contrasting with this is the use of real-world or nonexperimental data. For both experimental and nonexperimental material it is pertinent to make a distinction between time series and cross-sectional observations.

Time Series and Cross-Sectional Data

As the words imply, time series data consist of a series of observations on a particular unit — such as an animal or a farm — made at different points in time. Typically, these sequential observations will be equidistant in time. Cross-sectional data, on the other hand, consist of observations made on a number of units, one reading per relevant variable being taken from each unit. Frequently a given set of data will consist of both cross-sectional and time series observations. For example, most livestock production functions are based on such data; observations being made relative to each of a group of animals at a number of points in time. As discussed in Chapter 6, both time series and cross-sectional data may lead to problems of economic specification.

In addition, time series data may introduce estimational difficulties if it is autocorrelated.

Occasionally, a choice may be possible between time series and cross-sectional data. More often, only one or the other type of observation will be feasible in terms of the circumstances surrounding a given production process. For example, whole-farm production functions generally have to be fitted from cross-sectional data because time series observations of sufficient length are unavailable; or, if available, may relate to not one but a number of production functions because of changes in technology over the period. Thus time series data over the period 1900-1955, for a United States farm producing corn, might reflect at least four basic technological patterns or production functions, as follows: 1900-1917 horsepower; 1918-1932 tractor power; 1933-1949 tractor power and hybrid corn; 1950-1955 tractor power, hybrid corn, weedicides, and insecticides. Only a mongrel production function could be fitted to observations covering such a range of technologies, unless the data were adjusted in some fashion to allow for shifts in technology. A suggestion of Solov as to how this might be done is discussed in Chapter 7.[1]

Just as for whole-farm functions, crop response functions relative to fertilizer, rainfall, etc., generally have to be estimated from cross-sectional data. Satisfactory time series observations on yield from a given plot are difficult to obtain. Chiefly due to the effects of variations in unmeasurable micro and macro climatic factors, year to year observations are usually not comparable. However, time series observations are generally used in deriving animal production functions. With animals, the observations are usually made at short intervals of a few days or weeks over a period of months. Over such a short period, sufficient control can often be exercised to reduce to negligible proportions variations caused by extraneous factors. The drawback, of course, is that observations on the same animal will generally be autocorrelated. Still, such autocorrelation may lead to fewer difficulties than intrinsic variations between animals might cause if only cross-sectional observations were utilized.

Experimental and Nonexperimental Data

Experimental data, in contrast with nonexperimental data, is characterized by the fact that it is largely generated under the researcher's control. In large measure, the experimenter decides which variables are to be held constant and which allowed to vary. He also has the opportunity of setting the levels at which the controlled variables will be allowed to operate, and of deciding which particular combinations of levels he will examine. Moreover, by virtue of his controlling influence,

[1] Solow, R. M. Technical change and the aggregate production function. Review of Economics and Statistics, 39: 312-20. 1957. Also Aukrust, O. Investment and economic growth. Prod. Meas. Rev., 16: 35-53. 1959.

the experimenter can generally ensure that there are no significant errors of observation in the input factor data. This feature of experimentation is most important in production function research since it fulfills one of the basic assumptions of least squares multiple regression. Still, it must be emphasized that, in practice, the generation of experimental data can never be controlled absolutely by the researcher.[2] Extraneous influences will always be present. Especially is this true with respect to experiments aimed at the estimation of agricultural production functions. Whether the function to be fitted relates to a single technology, such as fertilizer use, or an aggregation of technologies, as found on a farm, there will always be some factors that cannot be controlled by the experimenter. Unlike experimental data, nonexperimental data is generated independently of the researcher; the only control he may exercise is to use some purposive method of data collection *ex post*. Such control, being *ex post*, cannot be nearly so effective as the *ex ante* control that is possible over experimental data. For instance, real-world data generally being collected "after the event," errors of observation are to be expected in the explanatory variables. This state of affairs is particularly apt to be true with survey data where the researcher has to rely on the respondent's recall of his past actions.

In terms of the control exercised by the researcher, the distinction between experimental and nonexperimental data, might be formalized as follows. Suppose it were desired to estimate the production function in equation 5.2, X_1 to X_k being the totality of input factors relevant to output Y.

$$Y = f(X_1, X_2, \ldots, X_k) \tag{5.2}$$

Under nonexperimental conditions it might be possible to gather observations on the first g input factors. The estimated production function would be equation 5.3,

$$Y = f(X_1, X_2, \ldots, X_g) + \epsilon \tag{5.3}$$

the error ϵ being due to the omission of the input factors X_{g+1} to X_k, assuming no errors of observation on X_1 to X_g. Among the factors X_{g+1} to X_k, some will be fixed and some variable. Thus, if the researcher knew which of the factors X_{g+1} to X_k were fixed, equation 5.3 might be written more explicitly in the manner of equation 5.4.

$$Y = f(X_1, X_2, \ldots, X_g \,/\, X_{g+1}, \ldots, X_h \,/\!/\, X_{h+1}, \ldots, X_k) + \epsilon \tag{5.4}$$

In this equation the factors X_1 to X_g are represented as variable and observed, X_{g+1} to X_h as fixed at some known or unknown levels and

[2] For an interesting discussion of the distinction between experimental and nonexperimental data, and their relative roles in physical and social science, see Wold, H. Causal inference from observational data. Journal of the Royal Statistical Society, Series A. 119:28-50.

X_{h+1} to X_k, occurring after the double bar, as variable and unobserved. So far as its real-world implications are concerned, the empirical production function corresponding to equation 5.4 is more or less unreliable depending on the importance of the factors X_{h+1} to X_k. Also, to the extent that factors X_{g+1} to X_h are fixed at atypical levels, the usefulness of the fitted function is lessened. Consider now the situation when experimental observations are used. Assuming that normal experimental procedures aimed at curbing the effects of "extraneous" variations are used, the estimated production function should tend to be of the form of equation 5.5 where, relative to equation 5.4, $\bar{g}$ is greater than or equal to g and $\bar{h}$ is greater than or equal to h. In other words,

$$(5.5)\quad Y = f(X_1, X_2, \ldots, X_{\bar{g}} \,/\, X_{\bar{g}+1}, \ldots, X_{\bar{h}} \,//\, X_{\bar{h}+1}, \ldots, X_k) + \epsilon$$

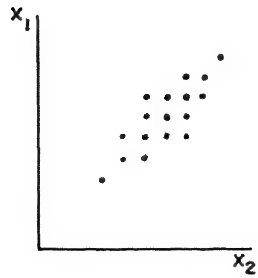
by using experimental procedures it is generally possible to increase the number of variable factors observed and to increase the number of additional factors held fixed. The net effect tends to be a reduction in the number of unobserved variable factors. To the extent that $\bar{g}$ is larger than g, and $\bar{h}$ larger than h, production functions fitted from experimental data are more reliable and tell more of the workings of the production process than do functions based on nonexperimental data. In so far as $\bar{g}$ and $\bar{h}$ may always be made as large as g and h, respectively, production functions based on experimental data can always be as reliable and useful as those derived from real-world observations. The above statements do not imply that data collection should always be based on experimentation. For many production processes, experimental procedures are either mechanically infeasible or not worthwhile in terms of their costs and benefits relative to nonexperimental data collection.

Relative to the use of real-world observations, the other major advantage of experimentation is that it allows the individual input factors that are controlled to be varied at the researcher's discretion. Consider a production process involving two inputs and a single output. The output response might be represented by a production surface in three dimensional space. By tradition and other institutional circumstances, it may be that in the real-world the two inputs are always used in the same proportions and over but a small range of values. From a bird's-eye view of the production surface, real-world observations of input quantities would always lie on a straight line across a small part of the surface. Such a circumstance is illustrated in Figure 5.1.a. Since an infinite number of planes can be fitted to pass through a straight line, the researcher using such real-world data would be unable to fit a plane that might be identified as the production surface.[3]

[3] Such a criticism has been leveled at the pioneering production function research of Douglas. See Mendershausen, H. On the significance of Professor Douglas' production function. Econometrica, 6: 143-53. 1938.

A

B



C

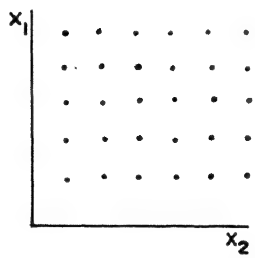


Figure 5.1. Bird's-eye view of observed points on production surface for $Y = f(X_1, X_2)$.

Or, as discussed in Chapter 4, the multicollinearity present among the input data may lead to the fitting of a completely irrelevant production surface, determined not by the structure of the production process but by errors of observation in the input data. Moreover, even if the real-world input combinations are not strictly proportional, it is very likely that they will tend to conglomerate in certain sections of the surface, assuming a random selection of observations. Figure 5.1.b illustrates such a situation. While a determinate surface may be estimated for the observed portion of the surface, no information is available relative to the major part of the surface. In this respect the implications of figures 5.1.a and 5.1.b are similar. Sometimes, of course, real-world observations may be scattered over all parts of the surface that are thought relevant by the researcher. Still, if the scatter of observations

is markedly uneven, there will be great differences in the reliability with which the surface is estimated over its different regions. Such a lack of reliability reduces the usefulness of the empirical estimates.

The advantage of an experimental approach to data collection is illustrated by the contrast between the scattering of observations in Figure 5.1.c and that in figures 5.1.a and 5.1.b. As shown in Figure 5.1.c, the researcher using experimental procedures may select the input combinations studied so as to give an even scattering of observations over the section of the surface that he wishes to study. Of course, any distribution of the observations deemed desirable could be arranged at the discretion of the experimenter by virtue of his ex ante control over the generation of the experimental data.

While the above discussion has been in terms of only two input factors, it applies equally well to the estimation of production functions involving more than two inputs. Over-all, therefore, experimental data collection has two highly desirable features relative to the collection of data from real-world sources. First, under experimental conditions it is generally possible to control a greater number of the relevant variables. Secondly, the experimenter may purposively arrange the observations on each variable input so as to cover all sections of the production surface that are of interest. In addition, he may select the levels at which the input factors are allowed to operate so as to simplify computational procedures in fitting the production function. For instance, estimation of a single equation production function model is greatly simplified if orthogonal polynomial regression can be used. As indicated in Chapter 4, this technique requires that the observations on the independent variables be equally spaced. Only under extremely special circumstances would real-world observations meet this restriction. Conversely, it is a simple matter to arrange an experiment so that the factor levels are equally spaced.

It has been argued that production function research based on experimental data has a major disadvantage relative to the use of real-world observations. The criticism is that experimental conditions are not typical of the uncertainty-ridden environment with its multitudinous range of factors in which the farmer has to operate.[4] In terms of equation 5.5, the farmer may be able to control factors X_1 to $X_{\bar{g}}$ but will have little or no control over $X_{\bar{g}+1}$ to $X_{\bar{h}}$. Consequently, from the function estimated experimentally, the farmer will have no idea of the variation that may occur in Y due to real-world variations in any of the factors $X_{\bar{g}+1}$ to $X_{\bar{h}}$. He only knows the values Y may take for certain fixed values of $X_{\bar{g}+1}$ to $X_{\bar{h}}$; assuming the factors $X_{\bar{h}+1}$ to X_k to be of negligible importance. Such a criticism of the experimental approach is not fatuous. Still, strictly speaking, it is not a criticism of experimental data collection per se but of the inadequacy of research funds. Given

[4] Johnson, G. L. Planning agronomic-economic research in view of results to date, in Baum, E. L. *et al.* (eds.), Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames. 1957. Pp. 217-25.

sufficient research resources, it would be possible to estimate the production function 5.6 in which, unlike equation 5.5, all the controllable inputs are variable.

$$(5.6) \qquad Y = f(X_1, X_2, \ldots, X_{\bar{h}} \,//\, X_{\bar{h}+1}, \ldots, X_k) + \epsilon$$

Data for fitting such a function need not necessarily be collected from a single large experiment. That would be rather infeasible. It might be obtained from a drawing together of the analyses and results of experiments conducted and reported by many different authorities. In this regard, a striking example of such an amalgamation process formed the basis for Britain's World War II fertilizer policy. Despite the fact that experiments on crop-fertilizer relations had been carried on for many years in Britain, at war's start there was no clear basis for devising a policy aimed at optimal allocation of the scarce fertilizer supplies. By sifting and amalgamating meaningfully all the experimental information available, Crowther and Yates produced a clear and consistent picture of fertilizer responses that served as the cornerstone of Britain's wartime fertilizer policy.[5]

Alternative to such an amalgamation process, a co-ordinated program of sequential experimentation might be carried out. Initially, interest may lie in determining the general shape of the production surface and in deciding which inputs are most important. At later stages stress might be laid increasingly on more applied analysis oriented more and more to a controlled simulation of the farmer's operating conditions. An extension of the experimental program in this manner appears preferable to using real-world data in an attempt to introduce "extraneous" variations of the type typically encountered by farmers. Real-world data does not incorporate these variations in any systematic fashion. Under experimental conditions these effects can, in large measure, be introduced systematically; so allowing meaningful interpretation of their real-world implications.

Choice between Experimental and Nonexperimental Data

The two modes of data collection — experimental and nonexperimental — are not entirely competitive. Within certain ranges they are complementary. The extent of such complementarity depends on the particular production process being studied, the region of interest on the surface, and the estimation procedures to be used. In general, some real-world information is necessary in order to plan experimental procedures. For the major part, however, the collection of real-world and experimental data are competitive activities in terms of available research resources and the benefits to be derived therefrom. The

[5]Crowther, E. M. and Yates, F. Fertilizer policy in war-time. The fertilizer requirements of arable crops. Empire Journal of Experimental Agriculture, 9: 77-97. 1941.

researcher has to make a choice between them. Ideally, this choice should be made in terms of the marginal costs and benefits associated with each type of data collection. In actuality, the decision cannot be made in such terms since research benefits are never known *ex ante.* However, the researcher does at least know that with a sufficiently complete experiment he can always generate data as suitable, if not better, than any real-world data that might be collected. The choice, therefore, hinges primarily on relative costs. It is usually readily apparent which procedure would be most expensive in terms of collecting a specified quantity of data. Indeed, within the range of research resources usually available, one or the other method of data collection will often be infeasible although not impossible. Thus it would be possible, but surely infeasible, to set up a farm or a region as an experimental unit from which production function data could be generated at the researcher's discretion. Conversely, biological production functions often have to be fitted from experimental data because satisfactory nonexperimental data could not be collected without great expense. In cases where the choice is not so clear cut, account has to be taken of the research resources required for the other phases of the investigation such as analysis, interpretation, and presentation of the results. How much of the limited research resources should be allocated to each of the research phases? In such a frame of reference, assuming both experimental and real-world data are feasible, the choice may lie between (a) expending a small quantity of resources on real-world data collection with a large quantity of resources left for analysis and interpretation, and (b) conducting an expensive experiment which would leave only a modicum of resources for other phases of the research. Given the uncertainties inherent in research, precise rules for deciding such choices are impossible. The researcher can only attempt to make an informed guess as to what pattern of allocation of the scarce research resources might serve best in the light of the information available to him.

EXPERIMENTAL DESIGN

Relative to real-world data, the major advantage of experimental data is its generation under the researcher's control. Not only may he decide which variables are to be examined. He also has the opportunity of controlling the levels at which many of the variables will be allowed to operate, and of deciding which particular combinations of levels he will examine. The problem of experimental design lies in deciding how this control should be exercised.[6] For instance, in what particular

[6] See Cochran, W. G. and Cox, G. M. Experimental Designs. John Wiley and Sons, Inc., New York. 1957. Chs. 1, 2, and 8A; Box, G. E. P. Fitting empirical data. Annals New York Academy of Sciences, 86: 792-816. 1960; Dykstra, O. Response Surface Designs. Technometrics, 2: 185-96. 1960.

pattern should the observations shown in Figure 5.1.c be distributed over the production surface? In making such decisions there are a number of factors the researcher must bear in mind: the limitations on available research resources; the purposes for which knowledge of the surface is required; the current state of knowledge about the surface and its functional representation; the peculiarities of the particular production process being examined; the estimation techniques to be used; and, if pertinent, the sequential nature of the experimentation. As well, the researcher must take account of the accuracy with which it is desired to estimate the production function: should a small section of the surface be fitted with a high degree of accuracy or should a larger portion be fitted roughly? Obviously, for an experiment to be fruitful, intensive planning is essential. Advantage should be taken of whatever competent advice is available. In particular, if a statistician is available, he should be consulted during the planning stage and not, as so often happens, after the experiment has been carried out.[7] Indeed, the justification for the discussion of experimental design given here is not to show the production function researcher how he might plan experiments independently of a statistician. Rather, it is to give him a sufficient awareness of the possibilities and problems so that he might derive maximum benefit from the statistical services at his disposal.

Basic Concepts

For an appreciation of the experimental techniques useful in production function research, some understanding is necessary of a few concepts that are basic to experimental design. In any experiment, *treatments* are applied to *experimental units*. If the experiment is a cross-sectional study, there will be one treatment per experimental unit. Time series experiments, on the other hand, imply the application of a number of treatments, each at a different point in time, to each experimental unit. Some experiments may involve both cross-sectional and sequential treatment applications. The treatments applied, and the response they induce, are recorded as the *observations*. Thus in a fertilizer-crop response experiment, one of the many treatments used might consist of 80 pounds of nitrogenous fertilizer and 100 pounds of phosphatic fertilizer per acre. The experimental units might be 0.1 acre plots, the response being the amount of crop grown on each experimental unit converted to a per acre equivalent. With such a cross-sectional study there would be a single set of observations for each experimental unit. In the present instance, the set of observations

[7] For some comments on the statistician's role, see Finney, D. J. The statistician and the planning of field research. Journal of the Royal Statistical Society, Series A, 119: 1-17. Also, Ostle, B. Statistics in research. Iowa State University Press, Ames. 1956. Pp. 421-26; and Box, G. E. P. The exploration and exploitation of response surfaces: some general considerations and examples. Biometrics, 10: 26-7. 1954.

would be a triplet showing the amount of each fertilizer applied to the experimental unit and the response obtained. Just what form the experimental unit takes depends on the design of the experiment. It may be a single object such as a plot of ground, a pot of soil, a plant or an animal; or it may be a group of such objects. The essential feature is that the experimental unit is the subject of a treatment, and that each set of observations refers to a particular experimental unit. As discussed in Chapter 4, sets of observations made on the same experimental unit are likely to be autocorrelated. Such observations will also contain an *experimental error* insofar as there is any lack of exactness or uniformity in the experimental technique used. When a number of experimental units are involved, as in cross-sectional studies; experimental error will also arise to the extent that the experimental units are not exactly similar. If they are not identical, as is usually the case, variations in response will not be wholly attributable to differences between treatments; some of the variation will be due to the differences between the experimental units. Since the effects of the various treatments are the object of study, it is extremely desirable to have the experimental units as similar as possible. However, experimental error can never be completely eliminated in agricultural production function research. There will always be some lack of uniformity in technique and dissimilarities between the experimental units, either intrinsically or in terms of the environment surrounding the experimental unit.

Since experimental error is always present, the technique of *randomization* has been devised to prevent this error from affecting the observations systematically. Randomization implies the allocation of the treatments to the experimental units in a random fashion. It is an insurance against experimental error causing systematic bias in the data; it does not remove the error. To gain some idea of the experimental error involved in an experiment, and hence of how reliable the data may be, *replication* is often practiced. By replication it is meant that the experiment is repeated a number of times; in part or in full; the variations in response between the same treatment applied to a number of experimental units giving an indication of the experimental error. Replication does not reduce the experimental error. It merely enables the researcher to take better account of the error.

Response Surface Designs

Historically, the experimental examination of response phenomena has usually been based on designs oriented to discrete data classification.[8] These studies have commonly failed to determine the functional

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[8]See Anderson, R. L. A comparison of discrete and continuous models in agricultural production analysis, in Baum, E. L. *et al.* (eds.), Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames. 1956. Pp. 39-47.

interrelationships between the input and output factors examined. Nor do they adequately emphasize the "fixed" conditions under which the experiments were carried out; important factors frequently being omitted from consideration altogether. Generally only one or two input factors have been examined, each at a small number of levels. Moreover, these two or three factor levels have usually covered but a small part of the range over which a response might have been elicited. Providing estimates of only two, three, or four points on a limited section of the production surface, these studies are markedly — if not totally — inadequate in terms of the generation of data for production function estimation. Similar strictures apply to current response studies of similar design.

Instead of attempting to estimate production functions such as equation 5.1, researchers have most often simply tried to ascertain such qualitative facts as whether (a) output responded to some hypothesized input, (b) one level of output induced a greater response than another, (c) any interaction occurred between inputs. Such questions were answered by way of analysis of variance, care being taken to replicate the experiments sufficiently to assure correct answers to the questions posed. In terms of the questions asked, such procedures are correct. However, from an applied research viewpoint, the questions are inadequate. The farmer needs more than qualitative information or information about one or two factors at two or three levels. He requires quantitative information of the type derivable only from empirical knowledge of the over-all production surface. For maximum benefit to the farmer, applied research on input-output phenomena should lead to the estimation of the relevant production function. Given empirical estimates, the implications of the response function may then be drawn out as illustrated in chapters 2 and 3.

That most input-output research by agricultural scientists has failed to recognize the existence of a continuous production surface, or at least the implications of such a surface, is not surprising. Why should agronomists, animal nutritionists, and their confreres, have appreciated such concepts as marginal productivities, marginal rates of substitution, marginal costs, and marginal revenues in a multifactor setting when the majority of agricultural economists did not![9] Moreover, more than a modicum of the less recent work on response phenomena by agricultural scientists was in the nature of fundamental research. Those who carried out this research cannot be wholly blamed for the fact that it was often misrepresented as applied research and seized as ready made grist for the extension mill.

[9] There were a few early agricultural economists who appreciated some of the implications of production function analysis. See, for example, Tolley, H. R. *et al.* Input as related to output in farm organization and cost of production studies. Bul. 1277. USDA, Washington, D. C. 1924; and Black, J. D. Introduction to production economics. Henry Holt and Co., New York. 1926. Ch. 12. For an appreciation of the slowness with which the conceptual framework and empirical specification of the production function was accepted, see: Case, H. C. M. and Williams, D. B. Fifty years of farm management. University of Illinois Press, Urbana. 1957; also, Black, J. D. Dr. Schultz on farm management research. Journal of Farm Economics, 22: 570-80. 1940.

Presaged by recognition of the inadequacy of the historical approach to input-output phenomena, experimental designs for production function estimation have been the subject of much discussion over the last few years.[10] A number of designs specifically oriented to the fitting of production functions have been developed. Some of the older designs have also been adapted by simply increasing the number of factors and treatments considered so that the data might be amenable to multiple regression analysis. In contrast to the experimental layouts used in discrete, "analysis of variance" type studies, the newer response designs generally involve a larger number of input factors; always with a more than proportionate increase in the number of factor levels and combinations examined. In other words, a greater number of observations are made per replicate. Moreover, these newer designs acknowledge that for satisfactory estimation the observations must be located methodically over the relevant section of the production surface. Because of the increased number of treatments and observations, experiments based on these response surface designs tend to be more expensive than their inadequate predecessors — at least *in toto*. Especially is this true for cross-sectional studies since they necessitate the use of additional experimental units. Research resources being limited, compromise becomes necessary. The usual adjustment is to reduce the extent of replication rather than to cut down the per unit size of the experiment. Of course, it is also possible to reduce the number of input factors that are allowed to vary under the experimenter's control. It must be recognized, though, that any reduction in the number of factors considered tends to decrease the practical usefulness of the subsequent analysis.

Broadly speaking, there are four classes of experimental designs that are especially pertinent to the generation of data for production function estimation. They are the complete factorials, the fractional factorials, the composite, and the rotatable designs. In contradistinction to the classical experimental designs, such as randomized blocks, Latin squares, etc., the factorial, composite, and rotatable designs relate to the arrangement of the treatments relative to one another and not to the positioning of the experimental units. Such orientation to the treatments, and hence to the observations, is what makes these designs relevant for the fitting of production functions. Of course, in using these surface-oriented designs, the concept of randomization is usually employed in allocating the treatments to the experimental units. Since

[10] For example, see Box, G. E. P. and Wilson, K. B. On the experimental attainment of optimum conditions. Journal of the Royal Statistical Society, Series B, 13: 1-45. 1951; Anderson, R. L. Recent advances in finding best operating conditions. Journal of the American Statistical Association, 48: 789-98. 1953; Fellows, I. F. Production functions in farm management, in Halcrow, H. G. (ed.). Contemporary readings in agricultural economics. Prentice-Hall, Inc., New York. 1955. Pp. 74-80; Baum, E. L. *et al.* (eds.) Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames. 1956. Pp. 37-98; Chew, V. (ed.) Experimental designs in industry. John Wiley and Sons, Inc., New York. 1958; Box, G. E. P. and Lucas, H. L. Design of experiments in nonlinear situations. Biometrika, 46: 77-90. 1959.

the factorial, composite, and rotatable designs relate to the arrangement of the treatments, it is possible within limits to superimpose them on some of the classical experimental designs. For instance, the treatments in a randomized block design might be arranged to form a fractional factorial. Some of the classical designs have been used for response surface estimation without the superimposition of planned arrangements of the treatments. In this regard, completely randomized, randomized block, and split-plot designs have been most popular.

Together with their broad advantages and disadvantages, the general features of the above-mentioned designs are outlined below so far as they relate to production function research. Particular stress is given to the factorial designs and the rotatable designs because of their greater relevance for production function estimation. Except where otherwise stated, it is assumed that a single equation model satisfactorily represents the production surface and that estimation is to be via least squares multiple regression. If a simultaneous equations model were pertinent, the experimental procedures would not necessarily be drastically altered. In most cases, the only change required would be an extension of the experiment's scope to enable measurement of the effects of those variables present in the system of equations but not entering the production relation embedded in the system.

Randomized Block Designs

Probably no design has been more frequently used in production function research than the randomized block. In this design, as usually used in cross-sectional studies, the selected treatments are allocated at random among the experimental units. Under such circumstances, the number of experimental units required per replicate is equal to the number of treatments being studied. In a time series study, where the interest centers on a number of treatment sequences, the treatment sequences would be allocated at random among the experimental units. Each replicate would require a number of experimental units equal to the number of treatment sequences being studied. An advantageous feature of the randomized block design is that it allows the experimental units to be allocated so as to make those within each replicate (block) as homogeneous as possible. Such grouping of the experimental units tends to minimize the experimental error within each block. If only a single block of experimental units is used, the randomized block design reduces to a completely randomized design. For discrete "analysis of variance" type studies based on randomized blocks, only a few treatments consisting of two or three levels of a single factor are usually examined. Such a restricted layout is of limited use for production function research. This fault, however, is not an intrinsic feature of the randomized block design. It can easily be remedied by extending the experiment to allow continuous multifactor analysis oriented to the fitting of a response surface. All that is required is an increase in the

number of factors and factor levels studied so as to provide sufficient observations for the fitting of whatever production function model is hypothesized.

As a simple example, suppose a randomized block design is to be used in a fertilizer-crop response study. Assume two fertilizers, P and K, are to be studied. Each of these factors is to be held once at each of five levels: P at 0, 20, 40, 60, and 80 lbs. per acre and K at 0, 30, 60, 90, and 120 lbs. per acre. Such a range of factor levels implies that the surface region of interest is bounded by $0 \leqslant P \leqslant 80$ and $0 \leqslant K \leqslant 120$. All told, there would be 25 possible treatments. One set of five drawn at random might be as follows:

Treatment	Level of P	Level of K
1	0	60
2	20	30
3	40	0
4	60	120
5	80	90

These (P, K) treatments would be allocated at random within blocks of five experimental units. The units might be pots, fields, or plots, depending on the scope of the experiment. Suppose there were only a single replicate of the experiment. It would generate five sets of cross-sectional observations for estimation of the surface. For estimational purposes, five is a very small number of observations. Only models containing four or less parameters could be fitted, otherwise there would be no degrees of freedom available for statistical testing of the estimated parameters. For instance, a five parameter polynomial model would be forced to fit the data perfectly but the resultant estimates of the parameters could not be tested statistically. Still, models of quite diverse logical implications, such as equations 5.7, 5.8, or 5.9 to give a few examples, could be fitted and tested via such a small experiment.

$$Y = \beta_0 P^{\beta_1} K^{\beta_2} \tag{5.7}$$

$$Y = \beta_0 + \beta_1 P + \beta_2 KP \tag{5.8}$$

$$Y = \beta_0 + \beta_1 P + \beta_2 K^2 + \beta_3 KP \tag{5.9}$$

With no replication, it would be impossible to obtain an estimate of the experimental error from the experiment itself. If the error were known from previous experiments of the same type, assuming the experimental error to be fairly constant between experiments, this a

Table 5.1. Analysis of Variance for Multiple Regression Based on a Replicated Experiment

Source of Variation in Y	Degrees of Freedom	Sum of Squares
Replicates	r-1	$\left[\sum_{j=1}^{r} (\sum_{i=1}^{t} Y_{ij})^2 / t\right] - \left[(\sum_{j=1}^{tr} Y_j)^2 / tr\right]$
Regression	k	SSR
Deviations from regression	r(t-1) - k	$\sum_{j=1}^{tr} (Y_j - \hat{Y}_j)^2$
Lack of fit	t-k-1	SST-SSR
Experimental error	(t-1) (r-1)	Found by subtraction
Total	tr-1	$\sum_{j=1}^{tr} y_j^2$

priori estimate of the error could be used. Without knowledge of the experimental error, the researcher could not satisfactorily compare the goodness of fit of the various hypothesized functions he might fit. He would have no way of knowing what proportion of the deviations from regression were due to experimental error and what to the intrinsic unsuitability of the fitted function. Analysis of variance for multiple regression could only be carried out as illustrated in Table 4.1. Now suppose that the experiment had been replicated twice. This would entail the use of ten experimental units split into two blocks of five units each. Completion of the experiment would give ten sets of observations, made up of two response readings for each of the five treatments used. The number of degrees of freedom available for estimational and testing purposes would be doubled. Just as importantly, an estimate could now be made of the experimental error; making it possible to

Table 5.2. Alternative Presentation of Analysis of Variance for Multiple Regression Based on a Replicated Experiment

Source of Variation in Y	Degrees of Freedom	Sum of Squares
Replications	r-1	$\left[\sum_{j=1}^{r} (\sum_{i=1}^{t} Y_{ij})^2 / t\right] - \left[(\sum_{j=1}^{tr} Y_j)^2 / tr\right]$
Treatments	t-1	SST
Regression	k	SSR
Lack of fit	t-k-1	SST-SSR
Experimental error	(t-1) (r-1)	Found by subtraction
Total	tr-1	$\sum_{j=1}^{tr} y_j^2$

carry out an F test of the goodness of fit to the data of the fitted functional form. Assuming a $k + 1$ parameter production function model such as equation 4.1 or 4.46, the analysis of variance for multiple regression of Table 4.1 could be extended as shown in Table 5.1 for the general case with r replications and t treatments. In format, this table corresponds to Table 4.1. An alternative method of presenting the same material that is frequently used is shown in Table 5.2. If orthogonal polynomial regression is used, the regression sum of squares of tables 5.1 and 5.2 can be further subdivided as shown in Table 4.2.

As Table 5.2 implies, the treatment sum of squares (SST) equals the sum of squares explained by regression (SSR) plus the lack of fit sum of squares. Moreover, SST can be calculated as shown in equation 5.10. Hence the lack of fit sum of squares can be obtained by subtracting SSR from SST. SSR is the numerator of equation 4.39.

$$SST = \left[\sum_{i=1}^{t} \left(\sum_{j=1}^{r} Y_{ij}\right)^2 / r\right] - \left[\left(\sum_{j=1}^{tr} Y_j\right)^2 / tr\right] \tag{5.10}$$

The experimental error sum of squares can be found by subtracting the replicate sum of squares and the treatment sum of squares from the total sum of squares. As is implied in Table 5.1, the deviations from regression are made up of deviations due to the failure of the fitted model to correctly represent the surface and to experimental errors. Mean squares are not shown in tables 5.1 and 5.2. They may be obtained by dividing the sums of squares by their respective degrees of freedom. An F test of the regression mean square against the error mean square provides an over-all test of the significance of the regression analysis, as discussed in relation to Table 4.1. The lack of fit mean square may also be tested against the error mean square. If this F value is significant at the selected probability level, it indicates that the model does not adequately describe the production relationship. Mason has suggested a stronger criterion: if the lack of fit mean square is of the same order of magnitude (or less) than the error mean square, then the fitted functional form adequately characterizes the data.[11] Of course, if the lack of fit mean square were zero it would imply that the function used fitted the surface exactly. When alternative functional forms are fitted to the data, the only changes that occur in the analysis of variance are in the regression and lack of fit mean squares. The total, replicate, treatment, and error mean squares are independent of the regression analysis. Hence the lack of fit mean squares of alternative functions can be compared as a statistical guide to the adequacy of the various functions. Such a procedure is, of course, only a rule of thumb to be used in conjunction with other criteria for assessing the correctness of a particular model. These other criteria may be

[11] Mason, D. D. Functional models and experimental designs for characterizing response curves and surfaces, in Baum, E. L. *et al.* (eds.) Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames. 1956. P. 80.

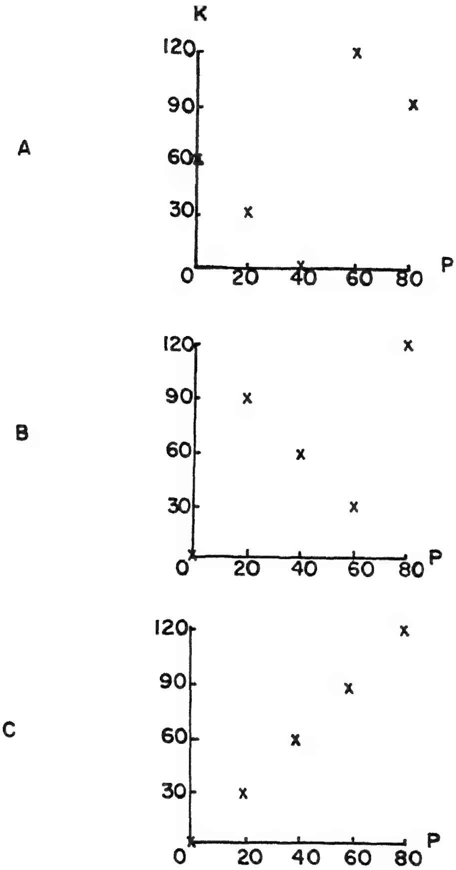

Figure 5.2. Bird's-eye view of observed points on production surface for $Y = f(P,K)$ with a randomized block design.

statistical in nature or they may be based on the underlying logic of the production process being studied.

In the above hypothetical experiment, which of the available levels of P and K entered each treatment was decided at random. The resultant pattern of observed points is shown, from a bird's-eye view, in Figure 5.2.a. It is but one of the 5! observational patterns that could have arisen. As the figure shows, it is not an intuitively satisfactory pattern since the observations are scattered rather unevenly over the surface. For this reason it would be advisable for the researcher to select the treatments — and hence the pattern of observations — purposively rather than at random. Thus, assuming only five treatments are possible, a systematic selection of factor combinations as illustrated in Figure 5.2.b might be satisfactory. Certainly it gives a much more

evenly distributed scattering of the observations over the surface area bounded by $0 \leqslant P \leqslant 80$, $0 \leqslant K \leqslant 120$.

Figure 5.2.c indicates the worst possible distribution of the five experimental points. Because of the equal spacings between factor levels, treatments made up of corresponding rankings of the factor levels are multicollinear. Of course, multicollinearity would not have resulted had the factor levels been unequally spaced. However, unequal spacings should not be used without a particular reason, for they imply greater interest in some parts of the surface than in others. Moreover, equal spacings permit orthogonal polynomial regression to be used conveniently and hence can lead to computational simplicity. Even without using orthogonal polynomials, a very worthwhile computational saving can be introduced by coding the input data if it is equally spaced. Thus, for the experiment considered above, a unit of P might be taken as 20 lbs. and a unit of K as 30 lbs. The five (P, K) treatments of Figure 5.2.a might then be written as (0,2), (1,1), (2,0), (3,4) and (4,3). An alternative coding of the factor levels that is still more convenient computationally is that given in equation 5.11,

$$X'_{ij} = \frac{X_{ij} - \bar{X}_i}{h} \tag{5.11}$$

where X'_{ij} is the coded value of the j-th level of the i-th factor, $\bar{X}_i$ is the mean level of X_i and h is the spacing between one level of X_i and the next. With this system of coding, the levels of P and K used in the randomized block experiment would be -2, -1, 0, +1, and +2. For instance, the 30 lb. level of K becomes (30-60)/30 or -1 units. Use of the above coding procedures is not limited to randomized block designs. They may be used with any design.

As an example of an experiment planned as a randomized block design with both cross-sectional and time series features, consider the following Iowa study. It was aimed at estimating the production function for liveweight beef from feed consisting only of corn grain and brome-alfalfa forage (plus a constant negligible quantity of essential feed elements not present in the corn or forage). The production function model might be represented as in equation 5.12 where Y is the cumulative quantity of liveweight beef produced from the cumulative quantity of

$$Y = f(F, C) + \epsilon \tag{5.12}$$

forage, F, and corn, C, consumed. The error, ϵ, is assumed to be negligible and arises because of the noninclusion of other relevant variable factors such as temperature, humidity, water consumption, etc.

Eighty-four steers of initial weight between 500-600 lbs. were available for the experiment. They were divided into two blocks of 42 animals each, providing for two replicates. Each block was divided into six pens of seven animals. An attempt was made to allocate the animals so that they were as similar as possible within each pen and block.

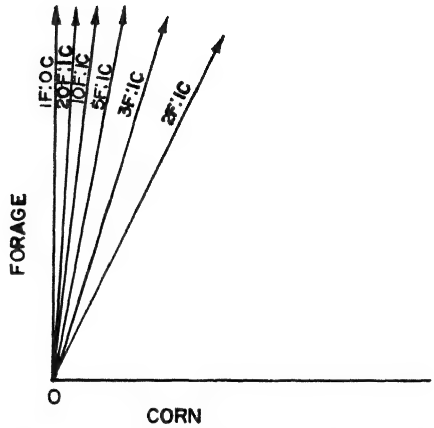

Figure 5.3. Bird's-eye view of forage-corn rations followed in a beef production study.

Each pen constituted an experimental unit. Forage and corn inputs were controlled by using six rations with F/C ratios as follows: 1:0; 20:1; 10:1; 5:1; 3:1; and 2:1. These six rations were allocated at random to the six pens in each replicate, each pen being fullfed the one ration throughout the course of the experiment. By keeping each group of animals on a constant feed mixture the possibility of carry-over effects, as would occur if the animals had to adjust to varying feed mixtures, was precluded. From a bird's-eye view, as shown in Figure 5.3, each pen followed a particular F/C ration line out from the origin over the period of the experiment. Rations containing more than one-third corn were not considered since more corn than this could not be handled by the experimental animals. Of course, with conditioning over a suitable period of time, cattle could be fed a higher proportion of corn than one-third. However, under the particular conditions of the experiment, it was not possible to condition the animals. For this reason, the feeding of a fixed feed mixture to each experimental unit was a deficiency in the design. It precluded the taking of observations over part of the surface. Observations were recorded every two weeks for 20 weeks. Counting the initial observation of zero feed intake and zero gain, there were thus 11 sets of observations on each unit or 66 sets of observations per replicate in all. These observations consisted of the cumulative weight gain of the cattle in each pen and the cumulative quantity of corn and forage consumed by each pen, tabulated for analysis on a per head basis. From a production function research viewpoint, the treatments are not the six rations that were used, but the actual feed quantities consumed. Since these intake quantities were determined by the animals, this experiment illustrates a situation where the treatments (except the initial one of zero feed) are not known in advance. Given the design of the experiment, the treatments were

only partially under the researcher's control; their location was determined only insofar as they were forced to lie on the ration lines of Figure 5.3. Nonetheless, given the necessity for the livestock to eat, the feeding of a constant mixture to each pen at least guaranteed a perhaps not too unsatisfactory scattering of observations over the relevant section of the surface. At the same time, the feeding procedure tended to intensify the number of observations close to the origin. Most importantly, with the treatments being determined by the animals, there is a difficulty in that true replication of such an experiment is well-nigh impossible despite the fact that two "replications" were carried out. The experiment was therefore not a randomized block design with two replicates but a completely randomized design using 12 experimental units. It would be virtually impossible to have groups of fullfed animals consume the same cumulative quantities of various feed types within the same time periods. For an experiment such as the above it is, therefore, not possible to carry out analysis of variance oriented to comparison of the "lack of fit" and experimental error mean squares as illustrated in tables 5.1 and 5.2. Of course, an approximate replication might be possible; the data between blocks being adjusted in some fashion so as to simulate replication. To do this, however, without strong grounds for the particular method of adjustment used, would be unsatisfactory. An alternative procedure would be to weigh the animals after they have consumed preselected quantities of feed. To some extent this would lead to mechanical difficulties in conducting the experiment since the time to consume a given quantity would vary between animals. Statistical analysis would also be complicated by the necessity to take account of the time factor.

The misinterpretation between ration types and feed quantities as treatments is not uncommon. Basically, it stems from a failure to recognize (a) that a continuous response surface is involved, and (b) that the marginal rates of substitution between feed components are not constant. An example of such misinterpretation is the suggestion that replicated cross-over designs be used in livestock response studies.[12] These designs are closely related to the randomized block design. Their essential feature is that the rations (mistakenly regarded as treatments) are interchanged among the experimental units during the experiment. For discrete "analysis of variance" type studies, such interchange makes it possible to take better account of the variation between experimental units; thereby reducing the experimental error. However, the benefits for continuous multifactor production surface estimation are slight, given the infeasibility of replicating the feed intake treatments.

Just as with any experimental design, the decision to use randomized blocks poses three questions. They are: How many treatments should be used? What particular treatments should be studied? How

[12] See Hoglund, C. R. *et al.* (eds.) Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames. 1959. Pp. 167-92.

many replications should be carried out? Given that research resources are not free goods, these questions are related. For instance, with only a limited number of experimental units available, any increase in the number of simultaneous replications must be at the expense of the number of experimental units in each replicate. The result is a reduction in the number of treatments that might be studied per replicate. Concomitantly, the question of which particular combinations of factor levels should be used as treatments becomes far more important. There are no clearly defined answers to these questions. Still, the number of treatments, replications and types of treatments that should be used can be partly decided in terms of the minimal requirements to be met. Thus, to determine the experimental error and lack of fit mean squares, at least two replicates (blocks) are necessary—unless the researcher has prior knowledge of the experimental error to be expected. So far as treatments are concerned, there should always be at least sufficient to provide a few more sets of observations than there are parameters to be estimated in the production function model. Also, while replication generally tends to display rapidly diminishing returns (in terms of the satisfactoriness of the estimate of the error), increasing the number of treatments studied does not lead too quickly to diminishing returns—at least within the range of experimental resources usually available. With a larger number of treatments it is possible to obtain a more complete distribution of observed points over the response surface, as well as the provision of additional degrees of freedom for statistical testing of the estimated parameters. It is therefore best to have only two or, at most, three blocks so that as many different treatments as are necessary for estimation of the production function parameters may be studied. Of course, if the experimental error is extremely variable from block to block, it may be necessary to use more than three replicates. The question of which particular treatments should be studied when the randomized block is used has already been discussed in part with reference to Figure 5.2. The minimal requirement is that the selected treatments should not be multicollinear. So far as possible, an even distribution of the treatments over the surface section of interest is desirable—assuming the researcher is equally interested in all parts of this portion of the surface. If he is not, the scatter of the observed points should roughly correspond to the intensity of his interest in different sections of the surface. With respect to the factor levels to be used, a handy rule of thumb is to arrange the treatments so that the mean level of each factor approximates what is thought to be its most profitable level. The advantage of such a procedure lies in the fact that regression analysis provides estimates of highest precision for points closest to the means of the explanatory variables. If the range of each factor is fairly large, this procedure is to be highly recommended.

When a sequential program of experimentation using randomized blocks is being followed, which treatments to use at each phase will in large part be determined by the stage of the study. Thus, in the early

phases, a small number of treatments involving a large number of factors and covering a wide range of the surface might be examined. After ascertaining in this way the broad shape of the surface, as well as screening out unimportant factors, subsequent experiments might investigate particular sections of the surface in more detail. For instance, once the approximate positions of the surface's ridge lines have been ascertained, treatments for any experimental design should be confined to the section of the surface between the ridgelines. Outside of this region, the surface is of no economic interest. Response there is always inefficient, regardless of input-output price ratios.

Complete Factorial Designs

In studying Figure 5.2, the reader may have noticed that the distribution of the experimental points over the surface was limited because each input level was only studied once. Had each level been used more than once, in combination with a number of different levels of the other factor, it would have been possible to obtain a more complete scattering of observations over the surface. Perhaps the most intuitively appealing distribution of the experimental points is that which would have resulted if each level of one input had been studied in conjunction with each level of the other input. The resultant distribution of the observations would have been akin to that of Figure 5.1.c. Such experimental layouts, with treatments consisting of all possible combinations between factors of the within-factor levels selected for study, are known as complete factorial designs. They are frequently used to generate data for response surface estimation. The factorial design of an experiment says nothing of the way in which the experimental units should be organized. The word "factorial" refers only to the arrangement of the treatments. Thus a factorial design may be superimposed on any of the ordinary experimental designs. In response surface experiments, the factorial arrangement is most often used with a completely randomized, randomized block, or split-plot arrangement of the experimental units. Of course, when the factor levels entering the treatments are not fully determined by the researcher — as is often the case in livestock feeding experiments — a factorial arrangement is infeasible because factor levels cannot be conveniently replicated.

Had the fertilizer-crop randomized block experiment relative to Figure 5.2 been arranged as a complete factorial, the necessary (P, K) treatments would have been as listed below. The treatments are coded by way of equation 5.11.

(-2,-2)	(-1,-2)	(0,-2)	(+1,-2)	(+2,-2)
(-2,-1)	(-1,-1)	(0,-1)	(+1,-1)	(+2,-1)
(-2, 0)	(-1, 0)	(0, 0)	(+1, 0)	(+2, 0)
(-2,+1)	(-1,+1)	(0,+1)	(+1,+1)	(+2,+1)
(-2,+2)	(-1,+2)	(0,+2)	(+1,+2)	(+2,+2)

The design would be described as a randomized block with the treatments arranged as a 5x5 or 5^2 factorial, there being 25 treatments or experimental points per block. Likewise, if three factors were being investigated in a complete factorial design, one at three levels, one at four levels and the other at five levels, the experiment would be described as a 3x4x5 factorial. It would involve 60 treatments. The location over the surface of the experimental points of a two factor factorial may be shown graphically in two dimensions. Such a planar view can only be given when two inputs are involved. With three factors, a three dimensional diagram would be necessary. For more than three inputs, geometric representation of all the experimental points is impossible so that it is extremely difficult to conceptualize the location of suitable experimental points. This being so, it is useful to have rules of thumb — such as factorialization provides — for deriving factor levels to enter the various treatments.

So far as they are oriented to the simultaneous examination of multifactor effects and provide sufficient observations for the fitting of a continuous function to the response surface, factorial designs are suited to production function research.[13] The symmetric arrangement of the treatments guarantees a relatively satisfactory distribution of experimental points over the surface, especially if equal spacings are used between factor levels. Of course, the researcher using a complete factorial still has to decide on how many levels of each factor to use and the size of these factor levels, as well as the extent of replication to be carried out. While there are no strict rules for answering these questions, the situation is no more difficult than for a nonfactorial randomized block design except insofar as more treatments and experimental units are required per trial. Nor does the factorial arrangement complicate the estimational procedures in any way. Still, there are a number of disadvantages to complete factorial designs. From an estimational point of view, they are unsatisfactory in that they lead to estimation of pure higher order effects, such as β_{11} of equation 5.13, with less precision than the interaction effects of the same order, such as β_{13} of equation 5.13. Also, and most importantly when research materials are limited, they require too many treatments, and hence too many experimental units, relative to the number of parameters to be estimated. As the number of factors or factor levels is increased, complete factorials become increasingly wasteful of research resources. In the estimation of response surfaces by polynomial type functions, as frequently used in crop and livestock studies, it has rarely been found worthwhile to use a function involving terms of higher

[13] Historically, factorial experiments have been analyzed in discrete terms, with the effects of each factor alone (pure or main effects) and the joint effects of a number of factors (interaction effects) considered as single-valued phenomena. The significance of these various effects were tested by way of analysis of variance. Such an approach is, of course, misdirected in a continuous response surface is relevant. For a contrast of the two approaches to factorial analysis, and the extent of their congruity, see Anderson, R. L. A comparison of discrete and continuous models in agricultural production analysis, in Baum, E. L. *et al.* (eds.) Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames. 1956.

degree than the second. Such a quadratic polynomial for three factors is given in equation 5.13. For the fitting of a quadratic polynomial, at least three levels of each factor must be studied. If four levels are examined, a cubic polynomial could be fitted; with five levels a quintic polynomial; and so on as the number of factor levels is increased. Thus with more than three levels, the use of a complete factorial arrangement provides for the estimation of higher order effects that the researcher generally knows to be negligible. This emphasis on effects that are of little interest could be reduced if the number of replications of each factor level were decreased, substituting an incomplete factorial arrangement for complete factorialization. Such designs are discussed later.

Even when only three levels of each factor are used, complete factorialization is an inefficient method of generating data. Consider Table 5.3. It shows the number of experimental points or sets of observations implied by 3^n and 4^n complete factorials relative to the number of parameters to be estimated in an n factor quadratic polynomial for $2 \leq n \leq 7$. With two or three factors at three levels in a single replicate, a 3^n experiment would not be too large. For instance, with three factors there would be 27 sets of observations to estimate the 10 parameters of equation 5.13. However, with more than three

$$(5.13) \quad Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_{11} X_1^2 + \beta_{22} X_2^2 + \beta_{33} X_3^2 + \beta_{12} X_1 X_2 + \beta_{13} X_1 X_3 + \beta_{23} X_2 X_3 + \epsilon$$

factors the number of observed points becomes increasingly disproportionate to the number of parameters to be estimated. Even if sufficient research resources were available to conduct such large experiments, it is simply not necessary to have so many degrees of freedom available for statistical testing; quite apart from the well-nigh impossible mechanical complexity of conducting such large experiments. With more than three levels of each factor, these effects are increased more than proportionately, as the figures of Table 5.3 for 4^n designs indicate. Since even a single replicate of a complete factorial involving more than three factors or factor levels requires a rather impractical number of treatments and experimental units, it is undesirable to obtain an estimate of the experimental error by replicating the whole experiment. For instance, a 3^3 complete factorial with two replicates would require 54 treatments and experimental units, generating 54 sets of observations for the estimation of 10 parameters. Obviously, to obtain an estimate of experimental error in this way compounds the wastefulness of the complete factorial design. If an estimate of the error is available from previous experimentation, assuming the experimental error is relatively stable between experiments, it may be compared with the mean square of the deviations from regression relevant to the unreplicated factorial data. If the deviations from regression mean square is larger than the error, it is an indication that the fitted function probably does not adequately describe the surface. An alternative that is far

Table 5.3. Number of Experimental Points in 3^n and 4^n (n = 2, . . . , 7) Complete Factorials Contrasted With the Number of Parameters To Be Estimated in an n Factor Quadratic Polynomial

Number of Factors (n)	Number of Experimental Points (3^n)	(4^n)	Number of Parameters To Be Estimated ($2n + 1 + n\frac{n(n-1)}{2}$)
2	9	16	6
3	27	64	10
4	81	256	15
5	243	1,024	21
6	729	4,096	28
7	2,187	16,384	36

more economical than complete duplication of the experiment is also available. It is to replicate only the treatments at the center of the experiment, i.e., those treatments in which all the factors have a coded value of 0. If this treatment is replicated r times, an estimate of the experimental error mean square is given by the sum of squares of the deviations of these r central responses from their mean, $\sum_{j=1}^{r}\left[Y_j - \left(\sum_{j=1}^{r} Y_j / r\right)\right]^2$, divided by their degrees of freedom, r-1. An estimate of the lack of fit mean square for a first order function is given by $qr(\bar{Y}^* - \bar{Y}_j)^2 / (q + r)$ where $\bar{Y}^*$ is the mean response for the q noncentral treatments and $\bar{Y}_j$ is the mean response of the r applications of the central treatment. These lack of fit and error mean squares may then be compared as a guide to the adequacy of the fitted function, as discussed previously.

Another approach to the estimation of experimental error in unreplicated factorial designs is possible. It is to use the apparent effects of those effects known a priori to be zero or negligible as the basis of an estimate of the experimental error. In the absence of prior knowledge, this procedure is dangerous in that misleading results will be given if the effects assumed to be negligible do, in fact, play some role.[14] Generally it is the higher order interactions that are so used, but any effect that is justifiably thought to be really nonexistent may be used. For example, with a 4^2 factorial it would be possible to estimate a third degree polynomial such as equation 5.14.

$$(5.14)\quad Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_{11} X_1^2 + \beta_{22} X_2^2 + \beta_{12} X_1 X_2 + \beta_{111} X_1^3 + \beta_{222} X_2^3 + \beta_{112} X_1^2 X_2 + \beta_{122} X_1 X_2^2 + \epsilon$$

[14] See Davies, O. L. (ed.) The design and analysis of industrial experiments. Oliver and Boyd, London. 1956. Pp. 286-89.

Suppose β_{112} and β_{122} coefficients could be assumed to be zero. Then the sums of squares associated with the $X_1^2X_2$ and $X_1X_2^2$ effects in the analysis of variance for multiple regression could be summed and divided by the sum of their degrees of freedom, which is two, to give an estimate of the error mean square. These interaction sums of squares could only be conveniently obtained if orthogonal polynomial regression were used as in Table 4.2. The resultant analysis of variance table would be as shown in Table 5.4 which, except for the absence of replication, corresponds to Table 5.1. Note that with the third order interaction coefficients of equation 5.14 assumed to be zero, there are only eight parameters of equation 5.14 to be estimated, including the constant term β_0. The sum of squares for regression therefore has seven degrees of freedom. Given the information of Table 5.4, comparison of the lack of fit and error mean squares may be made in the usual way.

A concept related to the use of factorial designs is that known as confounding. Because of the large number of experimental units necessary for complete factorialization, it may be impossible to study all of the treatments under uniform conditions. The most efficient method of overcoming this difficulty, granted the unlikely assumption that a complete factorial has to be used, is to divide the experimental units into blocks in such a manner that some or all of the less important interaction effects are confounded or "lost." Such confounding then allows the more important effects, either pure or interaction, to be investigated under uniform conditions. The construction and implications of such designs will not be discussed here.[15] The factor levels and the treatment arrangements required for the more useful factorial designs with interaction effects confounded have been listed by Cochran and Cox.[16] It is also possible to confound main effects. Such confounding occurs when split-plot designs are used. In general, this fact makes split-plot designs rather unsuitable for production function estimation since the main effects in response studies are usually not of negligible importance.

Fractional Factorial Designs

Complete factorials of greater size than 3^2, 3^3, and 4^2, even without replication, generally tend to be very wasteful of research resources. This fact has stimulated the development of a number of designs related to the factorial but specifically oriented to the fitting of continuous response or production functions. Before considering these response surface designs, however, mention should be made of two compromise factorial designs that have frequently been used in production function research. The first of these is the rather informal

[15] For a full discussion of the confounding of interaction effects, see Davies, O. L. (ed.) The design and analysis of industrial experiments. Oliver and Boyd, London. 1956. Ch. 9.

[16] Cochran, W. G. and Cox, G. M. Experimental designs. John Wiley and Sons, Inc., New York. 1957. Pp. 234-43.

Table 5.4. Analysis of Variance for the Multiple Regression Estimate of a Second Degree Polynomial From a 4^2 Factorial With Experimental Error Estimated From Third Order Interaction Effects

Source of Variation in Y	Degrees of Freedom	Sum of Squares
Regression	7	SSR
Deviations from regression	8	$\sum_{j=1}^{16} (Y_j - \hat{Y}_j)^2$
Lack of fit	6	Found by subtraction
Experimental error	2	From third order interactions
Total	15	$\sum_{j=1}^{16} y_j^2$

selection by the researcher of some of the experimental points of a complete factorial. Instead of studying all possible combinations of input factor levels, only some would be studied. Thus if 24 experimental units were available they might be used in an incomplete 4^3 design by selecting for study only 24 of the 64 experimental points in the complete design.

Presumably, researchers using this procedure have been guided by the desire to obtain observations adequate both in terms of their coverage of the relevant surface section, and in terms of having sufficient observations to enable estimation of the hypothesized response model. From a statistical point of view, such an informal approach to the design of an experiment for production function estimation is extremely naive. It takes no account of the precision or efficiency of estimation, nor of untoward confounding that might occur. Still, such an informal selection of treatments does recognize the major aspects of quantitative response phenomena. It is obviously a more satisfactory procedure than the classical experimental approach of considering but one or two factors at a few levels with analysis aimed at the derivation of one or two point estimates of supposedly discrete phenomena.

The other method of circumventing the large size of a complete factorial when five or more factors are studied is to use fractional factorials. In essence, a fractional factorial design is equivalent to a single block of a confounded complete factorial. The effect(s) confounded should be one(s) that the researcher believes to be negligible. Such designs make it possible to gather sufficient data to investigate the more important factor effects from an experiment a fraction of the size of a complete factorial. Thus a 3^4 complete factorial would involve 81 treatments and sets of observations. By confounding the higher order interactions, blocks of 27 units constituting a one-third replicate could be used. One such block would generate 27 sets of observations for the estimation of the 15 parameters in a second degree four factor polynomial model. To gain maximum benefit from such fractional factorials it is essential that the particular treatments necessary for correct

confounding be used. The researcher does not have to work these out for himself; he may obtain them from the tables of designs presented in the standard texts on experimental design.[17]

From a statistical viewpoint, the use of fractional factorials for response surface estimation is not entirely satisfactory. Like complete factorials, they lead to the estimation of higher order pure effects with relatively low precision. In addition, if the effects that are confounded are in reality not negligible, fractional factorials may lead to a false picture of the structure of the production process. The parameter estimates will be biased by the omission of the significant higher order effects that are lost by confounding. Nor, without replication, do fractional factorials directly provide an estimate of the experimental error; although they do provide some degrees of freedom for lack of fit analysis. However, an estimate of the experimental error may be obtained by replicating the central treatment as outlined above for the case of a complete factorial. Alternatively some of the unconfounded higher order interaction effects thought to be of no importance may be used to estimate the error, as previously explained. Or, again, any a priori knowledge of the error may be utilized. In these ways it may be possible to carry out lack of fit analysis.

To summarize, while fractional factorials are not as satisfactory as the more specialized response surface designs, they do have a large order of merit. This is especially true insofar as they can often be used in co-operative research with other scientists not interested in the fitting of a surface function. For these other scientists the use of a specialized response surface design of the type discussed below may be quite unsatisfactory. In other words, fractional factorials may often enable a compromise to be reached in the generation of data for a variety of research purposes.

Composite Designs

Although fractional factorials are far less cumbersome than complete factorials, they may still require an inordinate number of treatments and experimental units. Moreover, if a polynomial production function model is to be fitted, there appears to be no compelling reason why the factorial approach should be adopted. Indeed, other designs are possible which (a) require considerably fewer treatments than fractional factorials in order to estimate a given number of polynomial coefficients and (b) provide estimates that are statistically just as good, or better, in terms of the number of observations on which they are based. The designs constructed so far that have these characteristics are the composite and the rotatable designs. Both these designs were

[17] See, for instance, Cochran, W. G. and Cox, G. M. Experimental designs. John Wiley and Sons, Inc., New York. 1957. Pp. 244-92 and 341. Also Davies, O. L. (ed.) The design and analysis of industrial experiments. Oliver and Boyd, London. 1956. Pp. 431-90; De Baun, R. M. Response surface designs for three factors at three levels. Technometrics, 1: 1-8. 1959.

developed for use in industrial chemical experimentation. For such research, the experimental conditions differ markedly from those generally found in biological and agricultural research. First, the experimental error in chemical response studies is generally small and relatively constant for experiments of the same type. Secondly, sequential experimentation is far more feasible in chemical studies since the time required for a single trial is but a few days, at most. Consequently, both the composite and the rotatable designs are oriented to sequential experimentation. Nonetheless, both are eminently suited to the generation of data for the fitting of agricultural production functions from either a single experiment or a sequence of experiments. The most important proviso to their use is that the hypothesized model should be a polynomial. As will be pointed out later, this restriction is not too severe. Except where otherwise noted, a polynomial model of the production process will be assumed in the following discussion of the composite and rotatable designs. Equivalently, it might be said that it is assumed that the researcher is satisfied with a polynomial approximation of the true response function. These assumptions relate to the economic specification of the production function and will be discussed in Chapter 6.

Composite designs, as most frequently used, consist of a complete or fractional 2^k factorial supplemented with additional experimental points or treatments in such manner that all the coefficients of a second order polynomial in k factors may be estimated. If these extra treatments are arranged symmetrically about the center of the factorial, a central composite design results. By adjoining the additional experimental points at a distance from a "corner" of the 2^k factorial, a noncentral composite design is obtained.

To construct a central composite design, $2k+1$ supplementary treatments have to be added to the basic 2^k factorial. One of these additional treatments is placed at the center of the design and its coded value, in terms of equation 5.11, is therefore $(0, 0, \ldots, 0)$. The other 2k additional treatments are placed in pairs along the coordinate axes at distances of $\pm\alpha_1$, $\pm\alpha_2$, $\ldots$, $\pm\alpha_k$ from the center, respectively. They thus have the coded values $(\pm\alpha_1, 0, 0, \ldots, 0)$, $(0, \pm\alpha_2, 0, \ldots, 0)$, $\ldots$, $(0, 0, 0, \ldots, \pm\alpha_k)$. For convenience it is desirable that the α values be equal or, what is the same thing, that the 2k additional noncentral experimental points be equidistant from the center of the design. Such an arrangement is assumed in the following discussion. As an example, consider the central composite design for the study of three factors, X_1, X_2, and X_3, shown in Table 5.5.

The first 8 treatments are those of a 2^3 complete factorial. Treatments 9 to 15 augment this factorial, the ninth being at its center while treatments 10 to 15 are the noncentral supplementary experimental points. A single replicate of the experiment would give 15 sets of observations which would be sufficient to enable estimation of the 10 parameters in a three factor polynomial of second degree. If a 3^3 complete factorial were used to estimate such a polynomial, 27 experimental

units and treatments would be required. In this instance the composite design is thus far less demanding of research resources than the factorial. As Table 5.6 shows, this fact is always true when more than two factors are involved. Moreover, as k increases the saving that results from using a central composite design becomes greater.

So far, nothing has been said of the α values that might be used in a central composite design. They may take any value except zero. Moreover α can be chosen to give any desired compromise between precision and bias since both the bias that results if the true representation of the surface is not quadratic, and the precision of estimation, increase as α becomes larger.[18] As well, the value of α can be chosen so as to make the design orthogonal. This, of course, leads to great computational savings and enables the sum of squares due to regression to be conveniently separated into a component due to each effect as in Table 4.2. To obtain orthogonality, α should be 1.000, 1.215, 1.414, or 1.547 for the case of 2, 3, 4, or 5 factors respectively; each factor being held at levels of $\pm$ 1 in the basic 2^k factorial. These α values do not make all the regression estimates orthogonal directly. To take advantage of the orthogonality, the production function parameters have to be estimated by fitting a function such as equation 5.15 which corresponds to the three factor case.

$$(5.15) \quad Y = \eta + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_{11} X_{11} + \beta_{22} X_{22} + \beta_{33} X_{33} + \beta_{12} X_1 X_2 + \beta_{13} X_1 X_3 + \beta_{23} X_2 X_3$$

This equation corresponds to equation 5.13 where X_{ii} of equation 5.15 is related to X_i^2 of equation 5.13 as shown in equation 5.16, N being the number of sets of observations.

$$(5.16) \quad X_{ii} = X_i^2 - \frac{\Sigma X_i^2}{N}$$

The estimated β values of equation 5.15 correspond to those of equation 5.13 except for the η term. To obtain b_0, the estimate of β_0 of equation 5.13, the relation shown in equation 5.17 is used.

$$(5.17) \quad b_0 = \bar{Y} - \left(\frac{b_{11} \Sigma X_1^2 + b_{22} \Sigma X_2^2 + b_{33} \Sigma X_3^2}{N}\right)$$

The same general procedure holds for any number of factors.

Another alternative choice of α values is that which gives the design the property of rotatability; the estimated response at any point on the surface having the same variance as any other estimated response

[18] Discussion of the statistical technicalities of composite designs is to be found in Box, G. E. P. and Wilson, K. B. On the experimental attainment of optimum conditions. Journal of the Royal Statistical Society, Series B, 13: 1-45. 1951.

Table 5.5. Central Composite Design for Three Factors

Treatment	X_1	X_2	X_3
1	1	1	1
2	1	1	-1
3	1	-1	1
4	1	-1	-1
5	-1	1	1
6	-1	1	-1
7	-1	-1	1
8	-1	-1	-1
9	0	0	0
10	α	0	0
11	$-\alpha$	0	0
12	0	α	0
13	0	$-\alpha$	0
14	0	0	α
15	0	0	$-\alpha$

the same distance from the center of the design. Such designs are discussed in the next section.

In addition to alternative selections of the value of α, various extensions of the central composite design are possible. One modification is that obtained by using a number of 2^k factorials instead of a single 2^k factorial as the nucleus of the design. Such designs have been studied by Tramel.[19] He found an increase in the precision of estimation resulted from using a nest of three 2^k factorials with each factor held at levels of ± 1 in the first, ± 2 in the second, and ± 3 in the third

Table 5.6. Number of Experimental Points per Replicate Required for Estimation of a k Factor Second Degree Polynomial Using Factorial and Central Composite Designs

Design	k				
	2	3	4	5	6
3^k factorial	9	27	81	243	729
$(1/3)3^k$ factorial			27	81	243
Central composite	9	15	25	27*	43*

*Based on augmentation of a $(1/2)2^k$ factorial

[19] Tramel, T. E. A suggested procedure for agronomic-economic fertilizer experiments, in Baum, E. L. *et al.* (eds.) Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames. 1957. Pp. 168-75.

factorial and an α value of four. Relative to the central composite design having factor levels of ± 1 and α equal to two, Tramel found that the triple 2^k design gave worthwhile decreases in the variance of the estimate of the constant term, i.e., β_0 of equation 5.13 for instance, and of the quadratic and interaction coefficients. The disadvantage of this design relative to the ordinary central composite design is the number of experimental units and treatments it requires. Thus while the central composite for k factors requires only $2^k + 2k + 1$ experimental points, the triple 2^k design requires $3(2^k) + 2k + 1$ points, an increase of 2^{k+1}. To this extent the triple 2^k central composite designs are self-defeating. Relative to the ordinary central composite design, they do not markedly reduce the size of experiment required to estimate second order polynomial approximations to the production surface as compared with complete or fractional 3^k factorials.

The estimation of surface models by multiple regression from central composite design data is quite straightforward. For lack of fit analysis of variance, an a priori estimate of the experimental error may be utilized, or an estimate of the error may be obtained by replicating the central treatment a number of times. Alternatively, the whole experiment may be replicated. Whichever procedure is used, the lack of fit analysis proceeds as illustrated previously for the complete factorial design.

Noncentral composite designs are obtained by adding k additional points at a distance from a "corner" of the basic 2^k factorial. They are most suitable for sequential experimentation when the researcher has no prior knowledge of the response surface.[20] Of course, in agricultural or biological response studies, sequential experimentation is only possible under closely controlled conditions. Used in this way, the first stage experiment would consist of the basic 2^k factorial. In the second stage, k supplementary treatments would be added about the "corner" of the original factorial at which the greatest response was obtained. Thus if there were three factors under consideration and the first stage factorial experiment indicated that the surface region of most interest extended away from the treatment $(-1, +1, -1)$, then three additional treatments, $(-1 -\alpha, +1, -1)$, $(-1, 1+ \alpha, -1)$ and $(-1, +1, -1 -\alpha)$, would be carried out. In all, the researcher would then have 11 sets of observations from which a three factor polynomial of second degree could be fitted. If no a priori estimate of experimental error were available, the central treatment might be replicated a number of times at either stage of the experiment to give an estimate of error for lack of fit analysis.

Noncentral composite designs may also be of some use when

[20] Noncentral composite designs were developed for the experimental determination (by the sequential "method of steepest ascent") of the point on a response surface at which some feature, such as response or cost or profit, attains its maximum (or minimum) value. For such purposes it is not necessary to estimate a response function. However, in order to draw out the economic implications of a given response process under varying input-output price ratios it is essential that a production function be estimated.

sequential experimentation is not possible. For instance, if the approximate location of the ridge lines is known, the noncentral composite design provides an algorithm for locating relatively efficient treatment locations within the ridge lines. However, noncentral composite designs are not to be recommended for nonsequential experiments when the researcher is equally interested in all surface points within some range of each factor that is independent of the range of other factors.

Rotatable Designs

There are a number of criteria that might be used as a basis of comparison between response surface designs. In a practical research context, the most important feature of a design is perhaps the quantity of research resources required to derive satisfactory estimates of the production parameters; the definition of satisfactory estimates varying at the discretion of the researcher. Another criterion is the amount of information used in estimating the less important parameters of the surface function. It would be desirable to "allocate" the information used in estimation among the parameters in proportions corresponding to their importance. On the above grounds, the composite, fractional, and complete factorial designs are generally to be preferred, in that order. Two riders must be attached to this statement: the designs must be feasible in terms of both (a) the treatments possible and, roughly speaking, (b) the model to be fitted.

A third principle that a good response surface design should fulfill has been enunciated by Box and Hunter.[21] It is that of rotatability: the variance of an estimated response should depend only on its distance from the center of the design and not on its direction from the center. Stated another way, rotatable designs are such that all estimated responses an equal distance from the design's center lie on the same variance contour. This principle is intuitively appealing, especially in exploratory response surface studies where the researcher has a uniform intensity of interest in all parts of the surface section within the range of factor levels being studied. Moreover, rotatable designs do not necessarily involve more experimental points than nonrotatable designs. To date, the concept of rotatability has only been developed in relation to the estimation of polynomial approximations of the response surface. This is a disadvantage in that a design that is rotatable for response estimates based on a polynomial function will not be rotatable in terms of a nonpolynomial production function. However, since the use of multiple regression in production function research generally implies the use of a polynomial in one disguise or another, this defect is not too serious.

[21] Box, G. E. P. and Hunter, J. S. Multi-factor experimental designs for exploring response surfaces. Annals of Mathematical Statistics, 28: 195-241. 1957. For a less technical discussion, see Box, G. E. P. and Hunter, J. S. Experimental designs for the exploration and exploitation of response surfaces, in Chew, V. (ed.) Experimental designs in industry. John Wiley and Sons, Inc., New York. 1958. Pp. 138-90.

For the fitting of production surfaces, the rotatable designs of most importance are those of second order, i.e., rotatable designs from which all quadratic factor effects can be estimated. To a lesser extent, third order rotatable designs may also be of consequence. It is of interest to note that complete or fractional 2^k factorials, with or without the inclusion of central treatments, are rotatable first order designs. The construction of rotatable designs of any order is based on the selection of treatments located at the vertices and center of certain regular figures or combinations of figures in the space of k dimensions where k is the number of factors being studied. The finer details of construction are of no concern here; they lie within the province of the statistician rather than of the production function researcher. It should be noted, though, that rotatable designs of any order can be arranged in blocks so that the estimates of the parameters of the polynomial fitted to the total array of data are uninfluenced by block effects. As might be expected, these arrangements have to be specially planned; the appropriate details are outlined below. The possibility of such orthogonality between block effects and the production parameters increases the attractiveness of the rotatable designs. It enables rotatable designs to be used in sequential experimentation without worrying about time trends. Also, for single stage experiments, such blocking gives the researcher the opportunity to segregate his experimental units into more homogeneous groupings either in terms of the units per se or of the experimental environment; so reducing the experimental error without introducing added effects due to blocks. With or without blocking, the treatments in rotatable designs should be allocated at random among the experimental units to be used so as to reduce the biasing effect of unidentified factors.

Second order rotatable designs

Consider first the second order rotatable designs. A relatively efficient group of such designs is obtained by using the central composite design with α, the distance of the supplementary points from the center of the design, selected so as to make the design rotatable. Using a complete 2^k factorial as the nucleus of the central composite design, a rotatable arrangement results if α is chosen to satisfy equation 5.18.

$$\alpha = 2^{k/4} \tag{5.18}$$

Thus for two, three, four, five, or six factors, with a complete 2^k design as core, rotatability would result with α values of 1.414, 1.682, 2.000, 2.378, or 2.828, respectively. If a fractional 2^k factorial is used as nucleus, α must be selected to satisfy equation 5.19 in order to achieve rotatability

$$\alpha = 2^{\frac{k-p}{4}} \tag{5.19}$$

where $(\frac{1}{2})^p$ is the fraction of a complete 2^k factorial used as the nucleus. Thus with five factors and using a $(\frac{1}{2})2^5$ factorial as nucleus, a rotatable central composite design would result if α were set equal to 2.000. To avoid confounding of second order effects, k should be greater than four if a $(\frac{1}{2})2^k$ nucleus is used and greater than seven if a $(\frac{1}{4})2^k$ nucleus is used.

A general feature of second order rotatable designs that should be noted at this point is the role of central treatments. With none or few replications of the central treatment, the variance of the estimated response tends to be larger at the center than at points adjacent to the center. With many replications of the central treatment, the variance profile of the estimated response has a deep trough at the center. As a compromise between these two extremes, it has been suggested that the central treatment be replicated that number of times which would make the variance of the estimated response uniform within a radius of one unit, in terms of equation 5.11, from the center. Designs arranged in such fashion are known as uniform information rotatable designs. The replications at the center are also advantageous in themselves. If there are n_c such replications of the basal treatment, they provide $n_c - 1$ degrees of freedom for the estimation of experimental error. Lack of fit analysis may then be carried out. Plans for two to six factor rotatable central composite designs, with the required number of central points added to give a constant variance of estimation within a distance of one unit from the center of the design, have been tabled by Cochran and Cox.[22] They also list second order designs in blocks arranged so that the estimated parameters of the fitted parameters are independent of block effects.

In addition to the rotatable designs obtained by appropriate selection of α in the central composite arrangement, numerous other rotatable designs of second order are possible. With only two factors any distribution of treatments equally spaced on a circle around the central treatment is rotatable, provided that the number of experimental points on the circle is greater than four and that there is at least one observation using the central treatment. With three factors, a rotatable design that requires only 12 noncentral treatments in addition to replicates of the basal treatment is available. The design matrix for this arrangement is shown in Table 5.7. In using this design, α must be chosen to satisfy equation 5.20, N being the total number of experimental points including replications of the central treatment. It compares favorably with the three factor rotatable central composite design which requires 14 noncentral treatments.

$$\sum_{j=1}^{n} X_{ij}^2 = 4\alpha^2(1.176^2 + 1.902^2) = N \; ; \; i = 1, 2, \text{ or } 3 \tag{5.20}$$

[22] Cochran, W. G. and Cox, G. M. Experimental designs. John Wiley and Sons, Inc., New York. 1957. Pp. 348-50 and 370-75; also, Box, G. E. P. and Hunter, J. S. Experimental designs for the exploration and exploitation of response surfaces, in Chew, V. (ed.). Experimental designs in industry. John Wiley and Sons, Inc., New York. 1958. Pp. 187-89.

For more than three factors, the simplest rotatable designs are those obtained by using the central composite design with an appropriate α value based on equation 5.18 or 5.19.

Ordinary least squares multiple regression procedures may be used to fit a production function to data generated by second order rotatable designs.[23] Because of the symmetrical arrangement of the treatments, the regression computations are greatly simplified. This is especially true for the rotatable central composite designs where the factorial nucleus has equal spacings between factor levels. A short-cut procedure for estimating the coefficients of the second degree polynomial relevant to second order rotatable designs is also available.[24] This procedure involves the following calculations where Y_j is the observed response for the j-th treatment; this treatment being denoted by $(X_{1j}, X_{2j}, \cdots, X_{kj})$ for k factors.

$$(5.21)\quad b_0 = AN^{-1}\left[2\lambda^2(k+2)\left(\sum_{j=1}^{N} Y_j\right) - 2\lambda c \sum_{i=1}^{k}\sum_{j=1}^{N} X_{ij}^2 Y_j\right]$$

$$(5.22)\quad b_i = cN^{-1}\sum_{j=1}^{N} X_{ij} Y_j$$

$$(5.23)\quad b_{ii} = AN^{-1}\left\{[(k+2)\lambda - k]\,c^2 \sum_{j=1}^{N} X_{ij}^2 Y_j - (1-\lambda)\,c^2 \sum_{i=1}^{k}\sum_{j=1}^{N} X_{ij}^2 Y_j - 2\lambda c \sum_{j=1}^{N} Y_j\right\}$$

$$(5.24)\quad b_{ih} = \lambda^{-1} N^{-1} c^2 \sum_{j=1}^{N} X_{ij} X_{hj} Y_j\ ;\ i \neq h = 1, 2, \ldots, k$$

where

$$(5.25)\quad A = [2\lambda\{(k+2)\lambda - k\}]^{-1}$$

$$(5.26)\quad c = N\left(\sum_{j=1}^{N} X_{ij}^2\right)^{-1}$$

$$(5.27)\quad \lambda = kN[(k+2)(N - n_c)]$$

The b values are the estimates of the β coefficients in the second order polynomial representation of the surface. N is the total number of treatments, including n_c replications of the central treatment.

The analysis of variance for multiple regression may also be simplified when a second order rotatable design is used; Table 5.8 shows the appropriate paradigm. The sum of squares, S_0, attributable to fitting a zero order polynomial is calculated as indicated in equation 5.28.

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[23] See, for example, Hader, R. J. *et al.* An investigation of some of the relationships between copper, iron and molybdenum in the growth and nutrition of lettuce: I. Soil Science Society of America Proceedings, 21: 59-75. 1957. This reference also contains a valuable discussion of the general experimental procedures relevant to composite and rotatable designs.

[24] Hunter, J. S. Applications of statistics to experimentation. Statistical Techniques Research Group, Princeton University. 1958.

Table 5.7. Rotatable Design of Second Order for Three Factors

Treatment	X_1	X_2	X_3
1	0	1.902α	1.176α
2	0	1.902α	-1.176α
3	0	-1.902α	1.176α
4	0	-1.902α	-1.176α
5	1.176α	0	1.902α
6	1.176α	0	-1.902α
7	-1.176α	0	1.902α
8	-1.176α	0	-1.902α
9	1.902α	1.176α	0
10	1.902α	-1.176α	0
11	-1.902α	1.176α	0
12	-1.902α	-1.176α	0
13	0	0	0
14	0	0	0
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
N	0	0	0

Table 5.8. Analysis of Variance for Multiple Regression of Data From a Second Order Rotatable Design With k Factors

Source of Variation in Y	Degrees of Freedom	Sum of Squares
Regression	$(k + 2)(k + 1)/2$	$S_0 + S_1 + S_2$
b_0 term	1	S_0
b_i terms	k	S_1
b_{ii} and b_{ih} terms	$k(k + 1)/2$	S_2
Deviations from regression	$N - (k + 2)(k + 1)/2$	$\sum_{j=1}^{N} Y_j^2 - S_0 - S_1 - S_2$
Lack of fit	$N - (k + 2)(k + 1)/2 - (n_c - 1)$	By subtraction
Experimental error	$n_c - 1$	$\sum_{c=1}^{n_c} (Y_c - \bar{Y}^*)^2$
Total	N	$\sum_{j=1}^{N} Y_j^2$

(5.28)
$$S_0 = \left(\sum_{j=1}^{N} Y_j\right)^2/N$$

The extra sum of squares attributable to the fitting of a first order polynomial, i.e., associated with the b_i terms, is given by equation 5.29.

(5.29)
$$S_1 = \sum_{i=1}^{k} b_i \sum_{j=1}^{N} X_{ij} Y_j$$

Likewise the additional sum of squares explained by fitting the second order polynomial, i.e., associated with the squared and interaction terms, is given by equation 5.30.

(5.30)
$$S_2 = b_0 \sum_{j=1}^{N} Y_j + \sum_{i=1}^{k} \sum_{h=1}^{k} b_{ih} \sum_{j=1}^{N} X_{ij} X_{hj} Y_j - \left(\sum_{j=1}^{N} Y_j\right)^2/N$$

The experimental error sum of squares is obtained as the sum of the squared deviations of the n_c central treatment responses from their mean, $\bar{Y}^*$, as indicated in Table 5.8. Relative to tables 5.1 and 5.2, the sums of squares of Table 5.8 are in uncorrected terms. However, this does not interfere with the subsequent analysis. Mean squares for each of the listed sources of variation may be obtained by dividing each sum of squares by its associated degrees of freedom. An F test for lack of fit of the second degree polynomial model may be carried out in the usual way. The order of the model may also be tested by calculating the F value with $k(k + 1)/2$ and $(n_c - 1)$ degrees of freedom for the mean square of the second order coefficients, b_{ii} and b_{ih}, as in equation 5.31.

(5.31)
$$F = 2\, S_2 (n_c - 1)/k(k + 1) \sum_{c=1}^{n_c} (Y_c - \bar{Y}^*)^2$$

If this F value is significant at the desired probability level, it is an indication that the second degree model provides a better approximation to the surface than a first degree model. Should the F value of equation 5.31 be nonsignificant, the mean square, S_1/k, of the first order coefficients may be tested to give an indication of the adequacy of a first order model.

Third order rotatable designs

A number of third order rotatable designs have been devised by Gardiner, Grandage, and Hader.[25] Since these designs are not tabled in the standard texts on experimental design, some of the more useful arrangements will be outlined below. The use of third order designs presumes, of course, that the researcher believes a third degree

[25] Gardiner, D. A. *et al.* Some third order rotatable designs. Mimeo Series No. 149. Institute of Statistics, Raleigh, N. C. 1956. Gardiner *et al.* Third order rotatable designs. Ann. Math. Stat., 30: 1082-96. 1959.

polynomial would fit the production surface more advantageously than a second degree model. In the past, production function researchers have generally been satisfied with functions of second order. However, as they become more facile with the experimental procedures and problems involved, and with the advent of better experimental facilities making possible greater control of the experimental conditions, it is likely that third order designs will become increasingly useful.[26]

With two factors, the simplest set of third order rotatable designs consist of experimental arrangements with seven or more treatments spaced equally around each of two concentric circles. In all, a minimum of 14 treatments is therefore required. The central treatment may be replicated a number of times to provide an estimate of the experimental error for lack of fit analysis. For instance, if it were desired to estimate all the 10 parameters of a two factor third order polynomial such as equation 5.14, a design might be used with seven points on each of two concentric circles lying in the surface area of interest. If the (0, 0) treatment were replicated three times, two degrees of freedom would be available for estimating the experimental error.

It is possible to arrange a third order two factor rotatable design in two blocks such that the first block constitutes a second order rotatable design. Such an arrangement is extremely attractive for sequential experimentation. If the second order polynomial fitted to the first block is felt to be unsatisfactory, the researcher may proceed to carry out the second block of treatments. A third order approximation to the surface could then be fitted using the data from a third order rotatable design constituted by the two blocks. For such a sequential design, the first block should consist of seven or more points equally spaced around the circumference of a circle within the surface region of interest and having its center at the experimental point corresponding to the coded treatment (0, 0). By replicating this central treatment one or more times, a rotatable second order design is obtained. The second block should have seven or more treatments arranged on a circle of different radius but with the same center as that used in the first block. It is desirable that the estimates of the parameters of the third order polynomial be free of block effects. For such to be the case, the radius, r_2, of the circle used in the second block should be chosen to satisfy equation 5.32. In this equation r_1 is the radius of the circle used in the first block, n_1 is the total number of treatments used in the first block and n_{c1} is the number of treatments at the center of the first block.

$$r_2^2 = r_1^2(n_1 - n_{c1})/n_1 \tag{5.32}$$

[26] In this regard, it is interesting to note that the Australian Commonwealth Scientific and Industrial Research Organization has recently installed a Phytotron consisting of 300 six feet by three feet cabinets. Each of these experimental units is under separate control in terms of simulated climatic conditions, in addition to the normal factors such as fertilizer, soil type, etc., that a researcher might introduce. Under such conditions the use of third order designs could not be necessarily classed as impractical; nor could the simultaneous investigation of seven or eight factors in a series of sequential experiments be dismissed as too unwieldy.

Thus it would be possible to carry out a sequential two factor third order rotatable design with a minimum of 15 treatments. Of course, to estimate experimental error a few replicates of the central treatment would be required, assuming a lack of a priori information about the error.

For three factors, perhaps the most useful design is that shown in Table 5.9 where α is chosen to satisfy equation 5.33 for a total of N treatments.

$$(5.33)\quad \sum_{j=1}^{N} X_{ij}^2 = [2(1.82969^2 + 1.16343^2) + 8(1.25992^2)]\alpha^2 = N \; ; \; i = 1, 2, \text{ or } 3$$

It requires 32 noncentral treatments and, for best results, should be supplemented by at least four or five replicates of the central treatment. Five such central replicates would give a total of 37 sets of observations from which efficient estimates of the 20 parameters of equation 5.34 could be obtained. Four degrees of freedom would be available for estimation of the experimental error.

$$(5.34)\quad Y = \beta_0 + \sum_{i=1}^{3} \beta_i X_i + \sum_{j \geq i = 1}^{3} \beta_{ij} X_i X_j + \sum_{k \geq j \geq i \geq 1}^{3} \beta_{ijk} X_i X_j X_k + \epsilon$$

Three factor rotatable designs of third order that may be carried out sequentially in two blocks with the first block constituting a second order rotatable design are available. They require an over-all larger number of treatments and experimental units than does the nonsequential design of Table 5.9. However, fewer treatments are required within each stage of the experiment. Table 5.10 gives the design matrix for such a three factor rotatable arrangement of third order for α satisfying equation 5.35.

$$(5.35)\quad \sum_{j=1}^{N} X_{ij}^2 = N \; ; \; i = 1, 2, \text{ or } 3$$

The values of d, e, f, and g in Table 5.10 are 1.414214, 0.341564, 1.286527, and 1.985406, respectively. The first stage requires 20 noncentral treatments while the second block has 30. An arbitrary number of replications of the central treatment may be used in each block to give estimates of the experimental error. To make the estimated coefficients of the fitted third order polynomial free of block effects, the number of central treatments used in each block, denoted by n_{c1} and n_{c2}, respectively, should approximately satisfy equation 5.36.

$$(5.36)\quad n_{c2} = 2.2n_{c1} + 14.1$$

Thus for n_{c1} equal to two, three, or four, n_{c2} should be 19, 21, or 23, respectively. The total number of treatments would then be 71, 74, or 77, respectively. Using the two blocks sequentially, the researcher would

Table 5.9. Rotatable Design of Third Order for Three Factors

Treatment	X_1	X_2	X_3
1	α	α	α
2	α	α	$-\alpha$
3	α	$-\alpha$	α
4	α	$-\alpha$	$-\alpha$
5	$-\alpha$	α	α
6	$-\alpha$	α	$-\alpha$
7	$-\alpha$	$-\alpha$	α
8	$-\alpha$	$-\alpha$	$-\alpha$
9	1.82969α	0	0
10	-1.82969α	0	0
11	0	1.82969α	0
12	0	-1.82969α	0
13	0	0	1.82969α
14	0	0	-1.82969α
15	1.25992α	0	1.25992α
16	0	-1.25992α	1.25992α
17	-1.25992α	0	1.25992α
18	0	1.25992α	1.25992α
19	1.25992α	0	-1.25992α
20	0	-1.25992α	-1.25992α
21	-1.25992α	0	-1.25992α
22	0	1.25992α	-1.25992α
23	1.25992α	1.25992α	0
24	1.25992α	-1.25992α	0
25	-1.25992α	-1.25992α	0
26	-1.25992α	1.25992α	0
27	1.16343α	0	0
28	-1.16343α	0	0
29	0	1.16343α	0
30	0	-1.16343α	0
31	0	0	1.16343α
32	0	0	-1.16343α
33	0	0	0
34	0	0	0
.	.	.	.
.	.	.	.
.	.	.	.
.	.	.	.
N	0	0	0

Table 5.10. Noncentral Treatments for Rotatable Design of Third Order for Three Factors in Two Blocks

Block	Treatment	X_1	X_2	X_3
1	1	α	α	α
	2	α	α	$-\alpha$
	3	α	$-\alpha$	α
	4	α	$-\alpha$	$-\alpha$
	5	$-\alpha$	α	α
	6	$-\alpha$	α	$-\alpha$
	7	$-\alpha$	$-\alpha$	α
	8	$-\alpha$	$-\alpha$	$-\alpha$
	9	$d\alpha$	0	0
	10	$d\alpha$	0	0
	11	$-d\alpha$	0	0
	12	$-d\alpha$	0	0
	13	0	$d\alpha$	0
	14	0	$d\alpha$	0
	15	0	$-d\alpha$	0
	16	0	$-d\alpha$	0
	17	0	0	$d\alpha$
	18	0	0	$d\alpha$
	19	0	0	$-d\alpha$
	20	0	0	$-d\alpha$
2	21	$e\alpha$	$f\alpha$	$f\alpha$
	22	$e\alpha$	$f\alpha$	$-f\alpha$
	23	$e\alpha$	$-f\alpha$	$f\alpha$
	24	$e\alpha$	$-f\alpha$	$-f\alpha$
	25	$-e\alpha$	$f\alpha$	$f\alpha$
	26	$-e\alpha$	$f\alpha$	$-f\alpha$
	27	$-e\alpha$	$-f\alpha$	$f\alpha$
	28	$-e\alpha$	$-f\alpha$	$-f\alpha$
	29	$f\alpha$	$e\alpha$	$f\alpha$
	30	$f\alpha$	$e\alpha$	$-f\alpha$
	31	$-f\alpha$	$e\alpha$	$f\alpha$
	32	$-f\alpha$	$e\alpha$	$-f\alpha$
	33	$f\alpha$	$-e\alpha$	$f\alpha$
	34	$f\alpha$	$-e\alpha$	$-f\alpha$
	35	$-f\alpha$	$-e\alpha$	$f\alpha$

Table 5.10 (Continued)

Block	Treatment	X_1	X_2	X_3
2	36	$-f\alpha$	$-e\alpha$	$-f\alpha$
	37	$f\alpha$	$f\alpha$	$e\alpha$
	38	$-f\alpha$	$f\alpha$	$e\alpha$
	39	$f\alpha$	$-f\alpha$	$e\alpha$
	40	$-f\alpha$	$-f\alpha$	$e\alpha$
	41	$f\alpha$	$f\alpha$	$-e\alpha$
	42	$-f\alpha$	$f\alpha$	$-e\alpha$
	43	$f\alpha$	$-f\alpha$	$-e\alpha$
	44	$-f\alpha$	$-f\alpha$	$-e\alpha$
	45	$g\alpha$	0	0
	46	$-g\alpha$	0	0
	47	0	$g\alpha$	0
	48	0	$-g\alpha$	0
	49	0	0	$g\alpha$
	50	0	0	$-g\alpha$

not need to carry out the second block of treatments if the second order polynomial fitted to the first stage block should prove satisfactory.

With four factors the most practicable third order rotatable design is that shown in Table 5.11. Replications of the central treatment are not listed. The values of d, e, and f are 1.200919, 0.256303, and 1.736604. Again α must be chosen so as to satisfy equation 5.35 for a total of N experimental points. As indicated in the table, the design matrix may be split into two blocks. The first block is a second order rotatable design. Hence, this experimental arrangement may be used effectively for sequential experimentation. To eliminate block effects, n_{c1} and n_{c2} should be selected to satisfy equation 5.37 with, of course, $n_{c2} \geq 0$.

(5.37) $$n_{c2} = 3.3n_{c1} - 26.7$$

Thus with n_{c1} equal to nine, the central treatment of the second block should be replicated four times. Such an experiment would provide 141 sets of observations from which each of the 35 effects in a third order four factor polynomial could be estimated.

For more than four factors, third order rotatable designs require far too many experimental points to be useful. While some degree of fractional replication is possible with rotatable central-composite designs for five or more factors, it is not feasible with third order designs for less than seven factors. Fractional replication with less than seven factors is impossible without confounding some of the second

Table 5.11. Noncentral Treatments for Rotatable Design of Third Order for Four Factors in Two Blocks

Block	Treatment	X_1	X_2	X_3	X_4
1	1	α	α	α	α
	2	$-\alpha$	α	α	α
	⋮	⋮	⋮	⋮	⋮
	16	$-\alpha$	$-\alpha$	$-\alpha$	$-\alpha$
	17	2α	0	0	0
	18	-2α	0	0	0
	⋮	⋮	⋮	⋮	⋮
	23	0	0	0	2α
	24	0	0	0	-2α
2	25	$d\alpha$	$d\alpha$	$e\alpha$	$e\alpha$
	26	$d\alpha$	$d\alpha$	$e\alpha$	$-e\alpha$
	⋮	⋮	⋮	⋮	⋮
	40	$-d\alpha$	$-d\alpha$	$-e\alpha$	$-e\alpha$
	41	$d\alpha$	$e\alpha$	$d\alpha$	$e\alpha$
	⋮	⋮	⋮	⋮	⋮
	56	$-d\alpha$	$-e\alpha$	$-d\alpha$	$-e\alpha$
	57	$d\alpha$	$e\alpha$	$e\alpha$	$d\alpha$
	⋮	⋮	⋮	⋮	⋮
	120	$-e\alpha$	$-e\alpha$	$-d\alpha$	$-d\alpha$
	121	$f\alpha$	0	0	0
	122	$-f\alpha$	0	0	0
	⋮	⋮	⋮	⋮	⋮
	127	0	0	0	$f\alpha$
	128	0	0	0	$-f\alpha$

and third order effects. Still, if the researcher believed some of the third order effects to be negligible, he could confound them and so reduce the size of the experiment by fractional replication. Assuming interest in all effects up to and including third order terms, a five factor rotatable design would require nearly 400 treatments. For six factors, approximately 1,000 treatments would be required. On the other hand, complete 4^5 and 4^6 factorials would necessitate 1,024 and 4,096 treatments respectively! By comparison, therefore, the rotatable designs of third order are still relatively efficient. Likewise, rotatable designs of higher than the third order would appear attractive in comparison with the corresponding factorial design. Nonetheless, they would be impractical in terms of the experimental resources required. Moreover, it appears most unlikely that polynomials of higher order than three will ever be warranted as approximations to the production surfaces found in agriculture.

So far as estimational procedures are concerned, data from third order rotatable designs may be fitted by ordinary least squares procedures. Analysis of variance oriented to lack of fit analysis may be carried out in the usual way. If desired, the additional sum of squares explained by fitting a third degree polynomial instead of a second order function may be calculated by subtraction and tested for significance.

COLLECTION OF NONEXPERIMENTAL DATA

Under experimental conditions the researcher has the opportunity of exercising *ex ante* control over the generation of data. The decisions as to which factors will be allowed to vary, and the extent of their variation, are made by the experimentalist. Moreover, he can choose the experimental units through which the effects of these factors and their variation are to be studied. Such direct control over the generation of data is impossible under nonexperimental conditions. With real-world data the researcher can generally only exercise *ex post* procedures in purposively selecting his data. The primary decisions as to input factor levels are made by the real-world controllers of the production process and not by the researcher. Still, there is no sharp line of demarcation between experimental and nonexperimental data. By exercising a sufficient degree of *ex post* control, a researcher might sometimes be able to obtain a set of real-world data comparable to experimental data in terms of the factor levels and production units to which the data relates. Conversely, some experimental data is only partially generated under the researcher's control. For instance, the factor levels studied in many livestock response experiments are determined by the animals and not by the researcher.

As previously stressed, experimental data is in many ways more preferable for the fitting of production functions than nonexperimental material. Paradoxically, the major disadvantage of experimental data lies in the rigid control that it implies. Such control often leads to

estimates having much lower variances than the variance ruling in the real-world, thereby reducing the usefulness of the fitted function for extension purposes. Still, this disadvantage could be overcome if experiments of sufficient size to take account of all sources of real-world variation were used. Such experiments, however, tend to be infeasible because of restrictions on research resource availability. Real-world data may then have to be used, balancing its probable inaccurate specification of the function against the enhanced usefulness of the estimates in an extension context. Also, under many circumstances, an experimental approach even on a small scale is impossible because of the expenses involved. Thus the estimation of firm production functions, unlike the estimation of functions for individual physical technologies, necessitates the use of real-world data. No research program would have sufficient resources to set up an experiment with firms as experimental units let alone provide inputs on the scale required for a range of experimental treatments. Hence it is not surprising that the most common use of real-world data in production function estimation occurs in the fitting of functions for whole firms or enterprises within a firm. In discussing nonexperimental data, therefore, the general orientation will be towards the estimation of farm-firm production functions. Examples used will relate to these situations. It must be pointed out, however, that the procedures and problems mentioned also relate to the estimation of production functions for particular technologies from real-world data.[27]

Just as with experimental data, nonexperimental observations may be either time series or cross-sectional. Sometimes a mixture of both types may be used. So far as collecting nonexperimental data, two approaches are possible. Use may be made of data already recorded for some other purpose. Alternatively, data may be collected directly from the real-world decision makers associated with the production process. Either of these procedures may be used with both the time series and cross-sectional approaches. For obvious reasons, time series data have usually been transcribed from data already recorded in such form as census tabulations and farm management accounts and records. Conversely, cross-sectional data have most frequently been obtained by survey procedures, usually of the personal interview type.[28] Inherently, both these methods of collecting data are similar. Both imply the sampling of some population relevant to the production process under study.

Whichever method of collecting real-world data is used, the recorded observations must satisfy a number of criteria. Chief among these is that the data be relevant to the production function that it is

[27] For examples of the use of real-world data in the derivation of production relationships for particular technologies, see Clark, J. and Bessell, J. E. Profits from dairy farming. Bulletin No. 7. Imperial Chemical Industries Ltd., Central Agricultural Control, London. 1956.

[28] If worthwhile data is to be obtained, it is essential that the survey be conducted with care. For a discussion of the mechanics of survey procedures, see Cannell, C. F. and Kahn, R. L. Collection of data by interviewing, in Festinger, L. and Katz, D. (eds.) Research methods in the social sciences. Dryden Press. New York. 1953. Pp. 327-80.

desired to estimate. For a given production function, there will be a population of situations where the function is to be found in operation. Only data from these situations should be used to estimate the production function. Quite obviously, to use data relevant to some other production function can only vitiate the analysis. Stated in such fashion, this primary requirement of the data appears rather trite. As a practical matter, however, it is quite a stumbling block; rarely, if ever, can it be fully satisfied empirically. The difficulty stems from the myriad of input factors relevant to any real-world response phenomena. Suppose it were possible to take account in a single continuous function of all the inputs X_1 to X_k relevant to some product Y. Provided that each factor was recorded at a sufficient number of levels, an over-all production function could be fitted. Such a function is given as equation 5.38.

$$(5.38) \qquad Y = f(X_1, X_2, \ldots, X_k)$$

As a practical matter, though, account cannot be taken of the totality of input factors. Not only are they too numerous, but some may as yet have no satisfactory scale of measurement. So far as the number of inputs is concerned, researchers have frequently aggregated various types of factors into a small number of categories so as to simplify the computational problem. However, as shown in Chapter 6, aggregation is not without danger. It may lead to a meaningless specification of the production function. For the moment, to simplify the exposition, we assume no aggregation of the inputs. Since account cannot be taken of all the k inputs of equation 5.38, the researcher has to compromise by attempting to fit a function such as equation 5.39 with factors X_{g+1} to X_k fixed at known or unknown levels.

$$(5.39) \qquad Y = f(X_1, X_2, \ldots, X_g / X_{g+1}, \ldots, X_k)$$

But the function that is actually fitted will always be akin to equation 5.40 because the researcher using real-world data can never exercise sufficient control to prevent some of the unrecorded factors, say X_{h+1} to X_k, from varying.

$$(5.40) \quad Y = f(X_1, X_2, \ldots, X_g / X_{g+1}, \ldots, X_h // X_{h+1}, \ldots, X_k) + \epsilon$$

While equation 5.39 specifies a particular production function, equation 5.40 does not. There are as many functions corresponding to equation 5.40 as there are possible levels of the variable but unrecorded factors X_{h+1} to X_k, i.e., an infinite number of functions assuming these factors to be continuous. In a given sample of real-world data, either time series or cross-sectional, each vector of observations $(Y, X_1, X_2, \ldots, X_g)$ will be associated with some particular set of values of the factors X_{h+1} to X_k. Each such set of values locates a different production surface in the space with axes $Y, X_1, X_2, \ldots, X_g$. Fitting a function to the sets of

observations on $Y, X_1, \ldots, X_g$ therefore results in the fitting of a hybrid surface. The same result would ensue if some of the factors X_1 to X_g had been aggregated.

Figure 5.4 illustrates the problem of hybridity when only variations in a single input, X_1, are recorded and analyzed. For simplicity, suppose there are only two other inputs, X_2 and X_3, relevant to Y. Assume X_2 remains fixed between observations on Y and X_1, but that X_3 varies between observations. If three sets of observations were taken, they might locate the points D, E, and F in the YX_1 plane, as shown in the figure. Fitting a function to these observations gives the curve DEF as an estimate of the production function. However, since X_3 varies between observations, the points D, E, and F lie on three different production functions in the YX_1 plane. These are shown as the curves OA, OB, and OC. The fitted function is a hybrid incorporating points lying not on a single production function but on a number of them. Of course, if the levels of X_3 were recorded and incorporated in the analysis, there would be no difficulty. The sets of observations would all lie on the same production surface in the space with axes Y, X_1, and X_3.

The extent to which the fitted hybrid function misinterprets the structure of the production process depends upon the importance of the unobserved variable factors relative to the observed factors, and also on the extent of variation in the unrecorded variables. If these factors vary but little, or are only of negligible import, then the production surfaces associated with the sets of observations will lie close together. The hybrid function might then be taken as providing a satisfactory estimate.[29] But if the influence of the factors X_{h+1} to X_k is not negligible, then the true surfaces associated with the vectors of observations will not lie close together and the fitted hybrid may be quite atypical of the true response process. The problem of hybridity also arises with experimental data. However, the essence of the experimental approach is the element of control that the researcher may exercise. The experimentalist can, in large measure, ensure that only unimportant factors occur in the subset X_{h+1} to X_k of unrecorded variable factors. Such control cannot be conveniently exercised in the collection of real-world data. Especially is this true when the cross-sectional approach is used. With time series data the situation is not so bad because within a single production unit there is less tendency for the "fixed" factors to vary over time, at least in terms of the periods usually covered by time series observations. Also, it should be noted that even if the function of equation 5.39 could be estimated, or if a satisfactory hybrid function corresponding to equation 5.40 was estimated, these functions would not be of wide applicability. They refer only to the levels at which the variables X_{g+1} to X_k are fixed or approximately fixed. If these variables have nonnegligible influence when varied, then the fitted function would

[29] For an empirical study in which such an interpretation is used consciously, see Jawetz, M. B. Farm size, farming intensity and the input-output relationships of some Welsh and West of England dairy farms. University College of Wales, Aberystwyth. 1957.

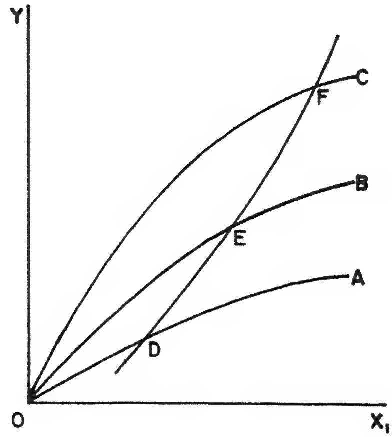

Figure 5.4. Relation between time and hybrid production functions for $Y = f(X_1/\ldots.)$.

not serve satisfactorily for estimates involving markedly different levels of the variables X_{h+1} to X_k. For this reason, it is desirable that in collecting real-world data attempts should be made, so far as possible, to have the fixed variables at levels close to their modal values. The fitted function might then be applied more extensively in a policy or extension context than would otherwise be the case.

Another problem pertinent to the collection of real-world response data is that of ensuring that information is recorded on all the variables thought relevant. Like the previous problem, this one also appears rather trite. Most often, the difficulty is not that the more important input factors are unknown, but that their level of use is not available in sufficiently disaggregated form. The researcher may be forced to use aggregated input categories rather than analyzing response in terms of the individual factors. This problem is most acute with data abstracted from such sources as census tabulations or farm records. With survey data gathered especially for production function estimation it may be overcome, at least to some extent. It is only necessary that a sufficiently detailed survey be undertaken and that the respondents be capable of supplying the data required. However, even if information is available on all the important factors taken individually, there may be too many of them from a computational viewpoint. The researcher may then have to aggregate the data, just as if disaggregated information were not available. Of course, it is preferable for the researcher to be able to aggregate the data at his own discretion rather than to have it forced upon him, in perhaps undesirable fashion, by the data itself.

The third criterion that nonexperimental data must meet is that it adequately cover the section of the production surface in which the researcher is interested. Certainly, the observations should be scattered

sufficiently to avoid problems of multicollinearity. Should some of the inputs be collinear per se, then the redundant variables must be omitted. If the correlation between certain inputs is extremely high throughout the relevant population, there is no means of selecting a meaningful sample so that the correlation is low. However, it is far more likely to find multicollinearity arising because of institutional factors — such as the traditional use of inputs in fixed proportions. The researcher must then seek observations to include in his data from production units free of the institutional factors causing multicollinearity. More usually, the problem will be one of approximate rather than exact multicollinearity. For instance, suppose a random sample were taken of the population of firms relevant to a given production function. It is likely that a scattering of input observations akin to those of Figure 5.1.b would result. To avoid such concentration of the observations within a small segment of the production surface, sampling must be purposive rather than random. The best procedure is to decide on the range of interest for each factor, divide this range into a number of equal segments and for each combination of factor ranges obtain an equal number of sets of observations. The procedure is illustrated schematically in Table 5.12 for the case of three factors A, B, and C with each factor range divided into three segments. These sub-ranges are denoted by the subscripts 1, 2, and 3. For each section of the surface corresponding to the factor range $A_iB_jC_k$, $(i,j,k = 1,2,3)$, an equal number of observations would be sought.

Such procedures are obviously related to the arrangement of a factorial experiment. The difference is that instead of using definite preselected input quantities as the experimental treatments, the real-world "treatments" are chosen so as to lie within some range of combinations of the input factors. The result of such a procedure, known as an analytic survey, is to produce a pattern of observations roughly corresponding to Figure 5.1.c. Just as in the case of experimentation, the design of an analytic survey may be modified to give a pattern of observations analogous to the fractional factorials. The essential feature of the design of an analytic survey is the emphasis on all parts of the possible range of the variables examined and not just on point estimates of the means of the variables.[30] The procedure recognizes that a sample which is designed to estimate mean values is not an efficient one for the estimation of relationships covering the entire relevant range of the variables. Controls, analogous to those used in experiments, may also be introduced in survey procedures. For instance, a further restriction might be introduced into the design of Table 5.12 that the additional factors D and E be at a constant level throughout the sample. Such controlled sampling leads to far more meaningful estimates of the production parameters. However, they have the disadvantage of reducing the

[30] See Anderson, R. L. Proper planning reduces research errors. Journal of Farm Economics, 35: 572-81. 1953. For an example of the collection of data from previously recorded material by means of an analytic survey, see Antill, A. G. Towards a production function for dairy farms. The Farm Economist, 8: 1-11. 1955.

Table 5.12. Design for an Analytic Survey Involving Three Factors — A, B, and C

	A_1			A_2			A_3		
	B_1	B_2	B_3	B_1	B_2	B_3	B_1	B_2	B_3
C_1									
C_2									
C_3									

size of the population to which the estimates are relevant. From this point of view, the rougher estimates from an "uncontrolled" analytical sample may be of more use in a policy or extension context. Whether or not this is so, depends on the importance of the factors such as D and E above.[31]

Analytic survey procedures are, of course, infeasible for the collection of time series data unless a number of production units are to be observed through time. For a single production unit it is difficult to obtain a satisfactory scattering of time series observations since input levels in succeeding production periods tend to be highly correlated; leading, of course, to the estimational problems associated with serial and autocorrelation. In addition, there may be time trends in the recorded input levels that have to be taken into account.[32]

With real-world data, the researcher has also to be particularly wary of errors in the data. Unlike the experimentalist, he has no direct record of the input and output quantities. The data come to him in second- or even third-hand fashion. As emphasized in discussing estimational procedures, error-ridden data can only lead to shabby estimates. Moreover there may be errors of measurement due to lack of homogeneity, i.e., quality differences between and within observations on what is regarded as a single input or output factor. The within-observation problem of quality differences is really a forced type of aggregation — ideally each separate quality of a factor should be regarded as a separate factor. Aggregation of inputs, if it is necessary, should then ideally be based on an index of the comparative production contribution of each quality level. Between observations, each observation might likewise be adjusted in terms of some standard quality of the factor, whether it be an input or an output. The problem, of course, is that the required standards are not available; nor, in most instances, are the quality differences ascertainable — either within or between observations of a "single" factor. Such problems of equating the theoretical

[31] An example of controlled sampling is to be found in Heady, E. O. Resource productivity and returns on 160-acre farms in north central Iowa. Iowa State University Research Bulletin 412. Ames. July, 1954.

[32] See Phelps Brown, E. H. The meaning of the fitted Cobb-Douglas function. Quarterly Journal of Economics, 71: 546-60. 1957.

product or production factors with observed outputs or inputs arise most frequently in the estimation of production functions for farms or for enterprises within farms — especially for the land, machinery, and labor inputs. Since these are really problems of economic specification, further discussion of the influence of such measurement problems on the derived estimates is deferred till the next chapter. It might be noted, though, that as elaborated in Chapter 4, the use of weighted regression procedures may enable these problems to be overcome; provided that the additional necessary assumptions can be made.

Another problem that arises in the collection of real-world data relates to the basis of measurement to be used for inputs and outputs. The best approach is to gather observations in terms of physical quantities. If necessary, and feasible, these may then be adjusted in terms of standard physical indices to take account of quality differences within factors. However, real-world data is frequently only available in money terms. The recorded observations are then relative to the prices ruling at the time the product or factors were sold or bought. If the data cannot be converted back into physical terms, the researcher may have no alternative but to base his estimates on money values. Use of the derived estimates is then limited to situations in which comparable price relationships exist. Moreover, if the researcher does not know the original quantity-price relationship, he will never know when a comparable price relationship exists between the variables!

Finally, the number of real-world observations to be taken must be determined with regard paid to the number of parameters to be estimated. At least this many sets of observations should be obtained, and preferably more so that sufficient degrees of freedom will be available for testing purposes.

CHAPTER 6

Economic Specification of the Production Function

IN PLANNING production function research, the first step must be an attempt to specify the production function in an economic sense. Only then is it appropriate to consider the questions of data collection and statistical estimation, all the while cycling back to the economic specification hypotheses to adjust them to what is feasible in terms of data collection and statistical estimation. Indeed, none of these aspects of production function research — economic and statistical specification and data collection — is independent; each influences the others. It is only for expository convenience that we have rather arbitrarily segregated them. That we have left the discussion of economic specification till last may seem rather perverse. However, such treatment is justified — to the uninitiated reader the problems of economic specification are probably the hardest with which to grapple, unless he has some understanding of the associated problems involved in statistical estimation and data collection.

The problem of economic specification in econometric models is a broad one. For production function research, some of the questions it raises are the following: What specific production function is it desired to estimate? Is this function independent of other economically meaningful relations in the milieu of the production process? Or is it embedded in a system of economic relations? If the latter, do the relative time relationships imply a causal chain type representation or a set of simultaneous relations? What variables are relevant to the production function? Must some of these variables be aggregated and how might such aggregation be carried out? What is the algebraic form of the production function — continuous or discontinuous, linear or nonlinear? What economic implications should the function sustain? Is there any a priori information available about the economic and physical logic of the production process that may help to answer the above questions? Such are some of the questions pertinent to the economic specification of the production function. To the extent that they are answered incorrectly, the estimated production function will be biased. It will tend not to reflect the true structure, in an analytical economic sense, of the production process.

INTER AND INTRA PRODUCTION UNIT FUNCTIONS

Perhaps the commonest problem of economic specification in empirical production function research is that of hybridity. The major elements of this problem have already been discussed with reference to the collection of real-world data in Chapter 5. To reiterate, the problem of hybridity arises because of the impossibility of measuring all of the input variables relevant to a given production process. Relative to a sample of observations, either time series or cross-sectional, there will nearly always be a number of relevant factors that are unmeasured but variable throughout the sample. Because of his neglect of these factors, the researcher cannot help but fit a hybrid function. Of course, the neglect of these variable but unobserved factors is generally forced upon the researcher because of the limited availability of research resources.

The problem of hybridity in empirical production function research was the subject of debate, in the economic literature, involving the validity of Douglas' pioneering research on macro-production functions.[1] To a degree, this literature is rather confusing, the discussion being in terms of inter and intra firm functions. In the present context, it is better to think in terms of production units rather than firms. The production unit might be a firm, a region, or a glasshouse pot — or any other vehicle for the production process. "Intra" or "inter" then refer respectively to the production function estimated from time series observations on a given production unit or to the function estimated from cross-sectional observations on a number of units. More importantly, the discussion in the literature is blighted by the fact that only very broad aggregates of input factors — capital and labor — are considered. In reality, both of these input categories encompass a large number of essentially different input factors, especially the capital category. Because of this level of aggregation, it appears meaningful to postulate that each of the firms in a cross-sectional study is most likely to be operating on a different production function. However, so long as the output and inputs are each measured in some homogeneous terms, there are no grounds for saying that cross-sectionally estimated production functions must be hybrid unless unobserved variations occur in some relevant factors. Hybridity will not be caused by imperfect aggregation in the absence of unobserved variable factors. Rather, imperfect aggregation per se might be said more strictly to lead to other types of bias in the derived estimates.

The implication in the literature that time series data may not lead to the problem of hybridity is also not entirely true. For each set of time series data that is used, the researcher must decide whether there has been significant variation over time in any of the nonnegligible

[1] See Reder, M. W. An alternative interpretation of the Cobb-Douglas function. Econometrica, 11: 259-64. 1943; and Bronfenbrenner, M. Production functions: Cobb-Douglas, interfirm, intrafirm. Econometrica, 12: 35-44. 1944.

unobserved factors, at least insofar as he can, granted that the factors are unobserved!

It is perhaps worth noting that the estimated production function of a firm, if hybrid, may still be of interest. Intuitively, it seems that firms are probably far more likely to tend to move along the hybrid surface as time passes than they are to move along the theoretically correct production surface. In terms of policy implications, therefore, a fitted hybrid function may be of use insofar as it depicts the growth path over time of firms' input-output relationships. Still, the hybrid function is of no use in terms of the usual implications that may be drawn from correctly estimated production functions, unless the hybrid function is sufficiently akin to the correct functions. If the correct functions do depict surfaces lying close together, then the implication is that the time path of firms' input-output relations tends to follow a movement rather similar to that along a single production surface.

As noted previously, the problem of hybridity may also arise under experimental conditions. There, however, it is not so important since the researcher has greater opportunity to ensure that the uncontrolled factors are only unimportant ones.

SPECIFICATION OF THE MODEL

In formulating an economic model of the production process he wishes to study, the researcher faces three main tasks. First, he has to decide whether a single equation or a system of equations is appropriate. Secondly, he has to choose the set of variables that are relevant to the model. Lastly, hypotheses have to be made, and tested, as to the most appropriate algebraic form of the equation(s). The ideally correct answers to these questions lie in the logic — economic, biological, or physical — underlying the production process. In such terms, a given production process has some "true" functional representation, perhaps of a stochastic nature, involving some definite set of variables either in a single equation or a set of equations. However, in an empirical research context, it is generally impossible to answer these questions of economic specification. On the one hand, the economic, physical, and biological logic of the production process is usually in large degree unknown. This is especially true of biological processes where the underlying logic involves such phenomena as cell mechanisms, photosynthesis, nutrient absorption, hormone, and enzymatic processes. For purely mechanical production processes involving no biological elements, the problem of economic specification is not so great — at least ideally. As a practical matter, however, difficulties arise from a lack of knowledge of the logic of entrepreneurial decision-making processes.

Apart from such general lacunae in understanding the production logic, there is another difficulty that besets the researcher in his attempt to formulate a model of the production process. For empirical research, the model has to be not only logically sound but also

computationally feasible. Otherwise, the model can only be an ivory tower plaything. Generally, the construction of a computationally feasible model implies that the researcher has to compromise with the theoretically ideal model — two types of adjustment usually being necessary. First, the number of separate variables that may be considered has to be determined in terms of data availability, and also with regard to the resources available for estimation. As the number of variables increases, the work involved in computation grows more than proportionately. Secondly, it is necessary to use a functional representation that is statistically manageable, both in terms of estimation and of testing. For these reasons, the construction of an economic model of a given production phenomenon is not independent of the problems of data collection and statistical estimation.

Single and Multi-Equation Models

A little has already been said in Chapter 4, from an estimational viewpoint, of the question of single versus multi-equation models in production function research. The crux of the problem is a question of fact. Can the production process be satisfactorily represented by a single unilateral causal relationship between output and inputs or are these variables so mutually interdependent that the process can only be satisfactorily represented by a system of simultaneous equations? Or should the production process be depicted as a chain of causal relationships, the dependent variable in each prior equation entering later equations as a predetermined causal factor?[2] In a sense, the latter interpretation lies between the single equation and the simultaneous equations representations. If the levels of the explanatory variables generated by the prior equations of a causal chain system are taken as given, then only the final equation depicting output as a function of the predetermined input variables remains to be estimated. This final equation corresponds to the single equation used in the single equation model. Should the independent variables of this single equation be generated exogenously relative to the production unit under study, it would be rather irrelevant to attempt to specify the prior equations. To do so would imply specifying a causal chain model when a single equation model would do just as well. However, should the causal factors in the final production function be determined within the production system, the causal chain model is to be preferred — at least theoretically if not in terms of data availability. Whether a causal chain or a simultaneous system of equations should be used depends largely on the unit time period to which the production function estimation is oriented. Figures 6.1 and 6.2 illustrate how this is so. Consider Figure 6.1. It

[2] See Wold, H. Causal inference from observational data. Journal of the Royal Statistical Society, Series A, 119: 41-49. Also Wright, S. The interpretation of multi-variate systems. In Kempthorne, O. *et al.* (eds.) Statistics and mathematics in biology, Iowa State University Press, Ames, 1954. Pp. 11-33; Wold, H. Ends and means. In Grenander, U. (ed.) Probability and statistics. Almgrist and Wiksell, Stockholm, 1959. Pp. 355-434.

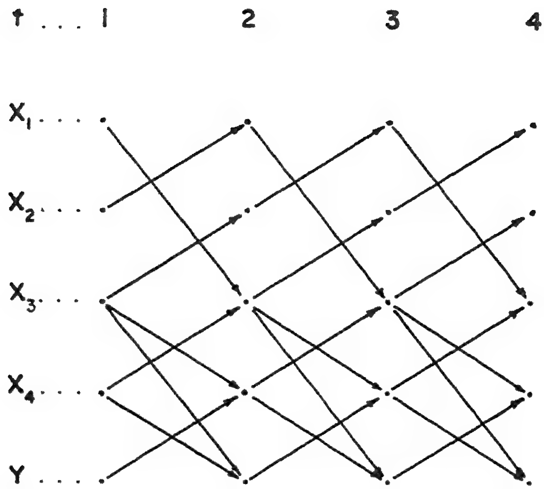

Figure 6.1. Arrow scheme for the causal chain model of equations 6.1 to 6.5.

depicts a hypothetical causal network between four "inputs," X_1, X_2, X_3, and X_4 and an "output" Y over a period of four time units, t(1), t(2), t(3), and t(4). These time units are basic relative to the production process. As the arrows indicating causation show, a complete cycle of the process requires three basic time units. It will not necessarily be true that the three time periods within each cycle will be equal. However, corresponding periods within different cycles of the system would be of equal length.

Figure 6.1 may be represented by equations 6.1 to 6.5. In these equations the second subscript denotes the time period in which the value of the variable is determined, except for Y which has only a time subscript. Each of the equations represents a unilateral causal chain with the variable on the left-hand side determined by the variables on the right. Moreover, the values of all the right-hand side variables are predetermined in a prior time period and are therefore exogenous. After choosing the appropriate algebraic form and provided that it be suited to least squares analysis, unbiased estimates of the parameters may be obtained by the ordinary multiple regression procedures.

$$X_{1,t} = f(X_{2,t-1}) \tag{6.1}$$

$$X_{2,t} = f(X_{3,t-1}) \tag{6.2}$$

$$X_{3,t} = f(X_{1,t-1},\ X_{4,t-1}) \tag{6.3}$$

$$X_{4,t} = f(X_{3,t-1},\ Y_{t-1}) \tag{6.4}$$

$$Y_t = f(X_{4,t-1},\ X_{3,t-1}) \tag{6.5}$$

Such a model is perfectly satisfactory from an estimational viewpoint so long as the observations that are used relate to the basic time periods, t(1), t(2) . . . . However, data for production function estimation is usually not available in such form but only for some aggregation of the basic time units. For instance, for a firm, data is usually only available for monthly or yearly periods while the basic time units might be but a few days. Likewise, for biological phenomena, the basic time periods may be only a matter of seconds whereas the only mechanically feasible observations might relate to months in the life of the plant or animal. Thus, suppose that for the production process depicted by Figure 6.1 it is only possible to gather observations for a time span T where T covers a sequence of basic time units. For instance, T might span t(1), t(2), and t(3). To depict the production process diagrammatically in terms of the time unit T, it is then necessary to aggregate the basic periods t(1), t(2), and t(3) into a single period. Within this single period the variables then become simultaneously determined. They can only be interpreted as being mutually interdependent, as indicated by the arrow network of Figure 6.2. This figure depicts the periods T(1), T(2), and T(3) and therefore corresponds to nine of the basic time units of the type depicted in Figure 6.1. As in Figure 6.1, the arrows connecting different time periods in Figure 6.2 denote the influence of predetermined exogenous variables.

Figure 6.2 may be represented by equations 6.6 to 6.10, the last equation being the production function of interest. As perusal of the equations indicates, they constitute a system of simultaneous equations, with both endogenous and exogenous variables appearing in each equation. The identification of these equations does not concern us here, although, as indicated in Chapter 4, it must be taken into account in deciding on the appropriate estimational procedures to be used.

$$X_{1,T} = f(X_{2,T-1}, X_{2,T}) \tag{6.6}$$

$$X_{2,T} = f(X_{3,T-1}, X_{3,T}) \tag{6.7}$$

$$X_{3,T} = f(X_{1,T-1}, X_{1,T}, X_{4,T-1}, X_{4,T}) \tag{6.8}$$

$$X_{4,T} = f(Y_{T-1}, Y_T, X_{3,T-1}, X_{3,T}) \tag{6.9}$$

$$Y_T = f(X_{3,T-1}, X_{4,T-1}, X_{3,T}, X_{4,T}) \tag{6.10}$$

It should be pointed out that all of the relations in a causal chain or a simultaneous equations model need not be between inputs or input and output. They may also relate to such phenomena as entrepreneurial decision processes and supply and demand relations. Such phenomena are, of course, particularly relevant for models incorporating the production function for a firm. Another point that must be noted is that some production phenomena may only be representable by a simultaneous system, it being impossible to postulate any meaningful basic time

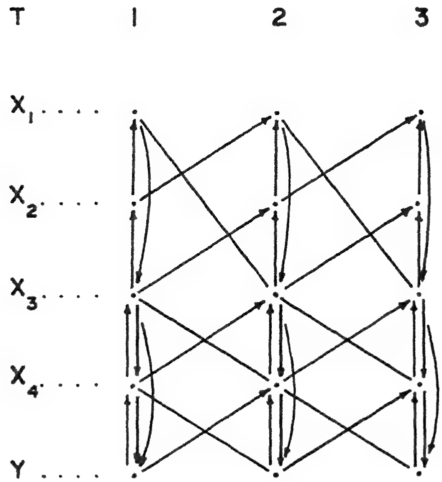

Figure 6.2. Arrow scheme for the simultaneous equations model of equations 6.6 to 6.10.

periods in terms of which a causal chain model might be hypothesized. In large degree, whether or not such is the case is a philosophical question. It suffices to note here that so far as empirical research is concerned, simultaneous models are theoretically correct whenever data cannot be collected in terms of the basic time units. It does not matter whether the data cannot be collected because the basic time periods do not in fact exist or whether the collection of such data is mechanically infeasible.

Obviously, the simplest type of production process — at least in economic terms — is one that can be fully represented by a single unilateral causal relation. However, few real-world production processes of import can be truly represented by a single equation model. Probably only processes of a purely physical-engineering type that encompass only a single technology and a single product can be so represented. Despite this, it is a fact that most production function research, whether it has involved plants, animals, firms, or enterprises within a firm, has been based on single equation models. To a large extent, the single equation approach has been used because of its computational simplicity, the implicit hope of the researcher being that the single equation estimates are not greatly biased. Whether or not they are depends on the mechanics of the process being studied. In general, however, the researchers have had no idea of the extent of the bias introduced by not using a multi-equation model. To a lesser extent, the single equation approach has been used without any appreciation by the researcher that a system of equations might be more appropriate — at least theoretically although perhaps not computationally. Such a situation is evident with regard to nearly all of the work that has been carried out with

plants and animals.[3] Thus with plants, researchers have estimated numerous production functions based on a single equation model involving such inputs as various types of fertilizers and moisture with grain or forage as output. No explicit attention has been paid to the fact that the plant produces many things apart from the particular part of it that is harvested. For instance, the relation between fertilizer and corn grain production is in reality but part of a simultaneous system of equations representing, say, the production of grain, leaves, stalks, husks, and roots. Likewise with animals, feed intakes enter a system of interdependent production relations involving muscle, bone, excreta, and so on, all of which react with one another in some causal or simultaneous network. Indeed, insofar as these plant and animal production processes are amenable to experimental control, it is likely that the use of the more correct multi-equation models would be fruitful, at least in comparison with the mediocre success they have attained when used in macro-economic studies in which nonexperimental data has had to be used by necessity.

From a theoretical viewpoint, there is no doubt that the use of a single equation approach when a system of equations model is appropriate must, under usual circumstances, lead to biased estimates of the production parameters.[4] While the single equation approach allows theoretically satisfactory predictions of output to be made when the process is allowed to operate in its normal fashion, it need not allow predictions to be made of the effect of deliberate interventions with the production process. In other words, if the single equation approach is used instead of the multi-equation approach, the underlying structure of the production process will most likely remain unknown. As a practical matter, however, as pointed out in Chapter 4, single equation estimates have generally been found to be just as logical and meaningful in an economic sense as those derived at much more expense by the use of simultaneous equations models. Relative to recursive systems, too few studies have been made to justify an opinion.

Choice of Variables

Whatever type of model is used to depict the production process, the researcher has to decide on the variables to be used. Should any relevant variables be omitted, the fitted model will be biased in an economic sense; it could not be expected to truly depict the production process, either structurally or predictively. Likewise, the unwarranted inclusion of variables will lead to bias.

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[3] For an exception, see Lucas, H. L. Experimental designs and analyses for feeding efficiency trials with dairy cattle, in Hoglund, C. R. *et al.* (eds.) Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames, 1959. Pp. 187-91.

[4] See Marschak, J. and Andrews, W. H. Random simultaneous equations and the theory of production. Econometrica, 12: 143-205. 1944. Also, Haavelmo, T. The statistical implications of a system of simultaneous equations. Econometrica, 11: 1-12. 1943; Wold, H. Interdependent versus causal chain systems. International Review of Statistics, 26: 5-25. 1958.

Ideally, the choice of variables should be made in terms of the underlying mechanics of the production process. Given such knowledge and sufficient research resources, the researcher could estimate a function with all the relevant variables included. Such is never the case, however. Research resources are always limited and the underlying production logic can only be hypothesized. In consequence, (a) some of the relevant variables may be unknown and may, perhaps, only be discovered through fundamental research and (b) some of those variables known to be relevant may be unobservable. Still, the researcher must draw on all aspects of his knowledge of the production process in attempting to enumerate as many as possible of the relevant variables, compromising where necessary in terms of the resources available for collecting observations and carrying out analyses. Such compromises must be made with reference to what is known, from prior experience, of the relative importance of the variables known to be relevant. So far as possible, the variables omitted because of limited research resources should be those of least importance.

To some extent, also, the researcher may be able to use trial and error methods in deciding on the variables that are relevant. A given algebraic form of the production function may be tried with a variety of combinations of variables. That combination which best accounts for the observed output may be selected — provided that the influence of the included variables is not contrary to any of the physical, biological, or economic logic known to underlie the production process. The latter criterion also especially applies to the problem of choosing an algebraic form for the production function. Indeed, the selection of variables and the choice of a functional form are closely related topics. The following discussion of alternative algebraic forms is therefore also pertinent to the selection of variables.

Choice of Algebraic Form

In choosing an algebraic form for the production function to be estimated, the researcher must attempt to take account of whatever is known of the logic or basic mechanics of the production process. Also, the selected function must be computationally manageable, both for estimation and testing, as stressed in Chapter 4. The true algebraic form will never be known. In general, the only knowledge of the production process available to the researcher will be broad information on the shape of the surface. Oftentimes, this data will be sufficient to indicate the type of returns to scale to be expected, and the approximate location and shape of the isoquants and isoclines. The choice of an appropriate algebraic form when such information is available has been discussed at length in Chapter 3.[5]

[5] See also: Heady, E. O. Organization activities and criteria in obtaining and fitting technical production functions. Journal of Farm Economics, 39: 360-69; Baum, E. L. *et al.* (eds.) Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames, 1956. Pp. 39-98; and Hald, A. Statistical theory with engineering applications. John Wiley and Sons, Inc., New York, 1952. Pp. 658-62.

Still, there will often be no strong guides as to what algebraic form might be appropriate. In such circumstances, the ideal procedure would be to use the method proposed by Hildreth.[6] He suggests the use of a discrete model, account being taken of the qualitative restrictions, such as diminishing returns, that are known to exist. With such an approach, no attempt is made to specify the production function in complete algebraic detail as is done in attempting to estimate a continuous model. As yet, these form-free procedures are not sufficiently developed to be of any general use. This is especially true for situations involving more than one input factor. The alternative approach is to attempt to approximate the response surface by means of a polynomial.

Polynomial approximation

Frequently, production functions are estimated as polynomials without appreciation of the reason why the response surface may be approximated by a polynomial. The reason is that if the algebraic form of a function is unknown, it may be approximated over the range of interest by the standard mathematical procedure known as a Taylor series expansion. Such expansions can be reduced to a polynomial form. While the approximation will differ in algebraic form from the true function, its implications will be quite similar over the relevant range.[7]

Thus, suppose $y = f(x)$, the function being a smooth curve of unknown algebraic form. Via a Taylor series expansion, it is possible to estimate the value of y for x values in the neighborhood of any point $x = a$. The actual expansion is shown in equation 6.11 where $f'(a)$ denotes the value of the first derivative of $f(x)$ at $x = a$, $f''(a)$ the second derivative, and so on.

$$(6.11) \qquad y = f(x) = f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + \cdots$$

The more terms evaluated in this expansion, the better the values of y can be estimated for x values near $x = a$. As inspection shows, grouping of x terms of like power reduces the Taylor series of equation (6.11) to a polynomial in x as in equation (6.12).

$$(6.12) \qquad y = f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \cdots$$

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[6] Hildreth, C. G. Discrete models with qualitative restrictions. In Baum, E. L. *et al.* (eds.) Economic analysis of fertilizer use data. Iowa State University Press, Ames, 1956. Ch. 4. Also, Stemberger, A. P. North Carolina Agr. Exp. Sta. Bul. 126, 1957.

[7] If the aim is to study the basic logic as well as the implications of a response process, a differential equations approach should be used. See Box, G. E. P. The elucidation of basic mechanisms. Bul. Inst. Int. Statist., 36: 215-25, 1958.

An analogous result occurs when y is a function of more than one variable. Thus if $y = f(x_1, x_2)$, the polynomial resulting from a Taylor's series expansion would be as shown in equation 6.13.

$$(6.13) \quad y = \beta_0 + (\beta_1 x_1 + \beta_2 x_2) + (\beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{12} x_1 x_2) + (\beta_{111} x_1^3 + \beta_{222} x_2^3 + \beta_{112} x_1^2 x_2 + \beta_{122} x_1 x_2^2) + \ldots$$

In consequence, the fitting of a polynomial type production function is nothing other than the evaluation of the first few powers in a Taylor series expansion of the unknown true production function. Because the Taylor series expansion only holds for a region in the neighborhood of the selected point on the surface, such as $x = a$ in equation 6.11, the estimated polynomial will only adequately fit the production surface within a limited region of the surface. Within this region, a satisfactory fit can generally be obtained by evaluating only the first few powers in the Taylor series. For this reason, a first or second degree polynomial usually fits the production surface adequately. Sometimes it will prove worthwhile to estimate terms of the third degree. Such will tend to be the case the greater the precision required and the larger the area of the production surface that the researcher is trying to fit.

How well a polynomial of given degree fits the production surface also depends on the choice of scale used for the response and for the independent variables. Thus instead of using the simple first degree polynomial shown in equation 6.14, it has often been found better to use a log transformation of the variables as in equation 6.15.

$$(6.14) \quad y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k$$

$$(6.15) \quad \log y = \log \beta_0 + \beta_1 \log x_1 + \beta_2 \log x_2 + \ldots + \beta_k \log x_k$$

Equation 6.15 is simply the ordinary Cobb-Douglas function. Probably few who have used it have recognized that it may be regarded as a very simple Taylor series estimate. An interesting point is that recognition of the Cobb-Douglas function in these terms leads to consideration of higher degree polynomials in the logarithms. Thus, for the two input variable case, a second degree polynomial of the form shown in equation 6.16 might be estimated by conventional least-squares procedures.

$$(6.16) \quad \log y = \log \beta_0 + \beta_1 \log x_1 + \beta_2 \log x_2 + \beta_{11} (\log x_1)^2 + \beta_{22} (\log x_2)^2 + \beta_{12} \log x_1 \log x_2$$

In more conventional fashion, equation 6.16 might be written as in equation 6.17,

$$(6.17) \qquad y = \beta_0 x_1^{\beta_1} x_2^{\beta_2} x_1^{\beta_{11} \log x_1} x_2^{\beta_{22} \log x_2} x_2^{\beta_{12} \log x_1}$$

which reduces to

$$(6.18) \qquad y = \beta_0 x_1^{\beta_1 + \beta_{11} \log x_1} x_2^{\beta_2 + \beta_{22} \log x_2 + \beta_{12} \log x_1}$$

There are innumerable transformations that might be applied to the variables. One type that has been used extensively is the square root transformation of the independent variables. It has proved very successful in fertilizer studies. Polynomials of first and second degree using the square root transformation for two variables are shown in equations 6.19 and 6.20.

$$(6.19) \qquad y = \beta_0 + \beta_1 \sqrt{x_1} + \beta_2 \sqrt{x_2}$$

$$(6.20) \qquad y = \beta_0 + \beta_1 \sqrt{x_1} + \beta_2 \sqrt{x_2} + \beta_{11} (\sqrt{x_1})^2 + \beta_{22} (\sqrt{x_2})^2 + \beta_{12} \sqrt{x_1} \sqrt{x_2}$$

The latter equation reduces to equation 6.21 in which form it is not so easily recognized as a Taylor series expansion.

$$(6.21) \quad y = \beta_0 + \beta_{11} x_1 + \beta_{22} x_2 + \beta_1 \sqrt{x_1} + \beta_2 \sqrt{x_2} + \beta_{12} \sqrt{x_1} \sqrt{x_2}$$

Such square root transformations are members of a far larger family of functions obtained by using the general transformation shown in equation 6.22.

$$(6.22) \qquad x_i^* = x_i^{m/n}$$

For the square root transformation m is 1 and n is 2. For m = 3 and n = 5, the two factor second degree polynomial would be as shown in equation 6.23; with m = -1 and n = 1, equation 6.24 results.

$$(6.23) \quad y = \beta_0 + \beta_1 x_1^{3/5} + \beta_2 x_2^{3/5} + \beta_{11} x_1^{6/5} + \beta_{22} x_2^{6/5} + \beta_{12} x_1^{3/5} x_2^{3/5}$$

$$(6.24) \qquad y = \beta_0 + \frac{\beta_1}{x_1} + \frac{\beta_2}{x_2} + \frac{\beta_{11}}{x_1^2} + \frac{\beta_{22}}{x_2^2} + \frac{\beta_{12}}{x_1 x_2}$$

The square root transformation having proved useful, it would seem that many of the other functions derivable by use of the power transformation of equation 6.22 would also have utility. However, they have not been tried extensively. One difficulty in their use is that they lead to extremely complicated equations for the derivation of isoquants and isoclines.

Another type of transformation that may sometimes prove useful is to use the logarithm of the new variable obtained by adding a constant to

the independent variables.[8] The resultant function for two variables in a first degree polynomial is given in equation 6.25.

$$(6.25) \qquad y = \log \beta_0 + \beta_1 \log (a_1 + x_1) + \beta_2 \log (a_2 + x_2)$$

With such a function, the isoclines are straight lines not passing through the origin. In consequence, this function allows the marginal rate of substitution between inputs to change as the level of output changes. However a difficulty exists in that the constants, such as a_1 and a_2 of equation 6.25, which allow the best fit can only be ascertained iteratively.

From a mathematical point of view, there is no reason why more than one transformation should not be used within the same polynomial expansion. By using a variety of transformations it is very easy to build odd looking functions. Thus if a logarithmic and a power transformation are used on x_1 and x_2 respectively, the second degree polynomial appears as shown in equation 6.26.

$$(6.26) \qquad y = \beta_0 + \beta_1 \log x_1 + \beta_2 x_2^{m/n} + \beta_{11} (\log x_1)^2 + \beta_{22} x_2^{2m/n} + \beta_{12} x_2^{m/n} \log x_1$$

In part, the choice as to which type of transformation to use in a polynomial approximation to the surface may be decided with reference to the known logic of the production process. For example, it is known with respect to animals that as they grow a greater proportion of the feed intake is used for maintenance and less for growth. The production function must therefore allow decreasing productivity to each unit of feed input, either on an individual feed type basis or over-all. It is also known that as the proportion of protein concentrate in the ration is increased, it replaces fewer units of low-protein carbohydrate. Thus a second requirement of the fitted function is that it must allow diminishing marginal rates of substitution between, say, corn and soybean oilmeal. Another biological fact is that, for rapid growth, young animals generally require a greater percentage of protein in the ration than older animals. Therefore, a third requirement of the production function is that it must allow substitution rates between feed types to change as the animals gain in weight. Equivalently, the isoclines must either be curved, or be linear and not pass through the origin. Functions 6.27 and 6.28 below fulfill all of the requirements outlined above, the symbols C, S, and Y referring to, say, pounds of corn, soybean oilmeal, and animal liveweight gain respectively.

$$(6.27) \qquad Y = \beta_0 + \beta_1 C + \beta_2 S + \beta_{11} C^2 + \beta_{22} S^2 + \beta_{12} CS$$

$$(6.28) \qquad Y = \beta_0 + \beta_{11} C + \beta_{22} S + \beta_1 \sqrt{C} + \beta_2 \sqrt{S} + \beta_{12} \sqrt{CS}$$

[8] See Heady, E. O. Organization activities and criteria in obtaining and fitting technical production functions. Journal of Farm Economics. 39: 366-67. 1957.

The Cobb-Douglas function, equation 6.29, permits decreasing productivity to the feeds and diminishing rates of substitution between feed types, but does not allow substitution rates to change for a particular type of ration as the animals increase in weight. However, the logarithmic function, equation 6.30, does satisfy all three requirements of the biological logic.

$$\log Y = \beta_0 + \beta_1 \log C + \beta_2 \log S \tag{6.29}$$

$$\log Y = \beta_0 + \beta_1 \log (a_1 + C) + \beta_2 \log (a_2 + S) \tag{6.30}$$

Still, the logarithmic function, equation 6.29, may be useful. For example, an animal feeder may wish to feed only one ration which averages lowest in cost over the entire production period. Function 6.29 is appropriate for this purpose because the substitution rates between feeds along any ration line are then constant. In other words, the isoclines are then regarded as linear and passing through the origin of the input axes. Equating the feed price ratio with the slope of the gain isoquants gives an average least-cost ration for the whole growing period. Should the producer be interested in frequent changes of rations during the production period, however, equations 6.27, 6.28, and 6.30 allow better estimates of the least-cost rations than does the logarithmic function 6.29.

The use of polynomial functions in production function research is generally beset by two difficulties. The first stems from the fact that while the consideration of higher terms in the Taylor series generally leads to a better fit to the surface, it also leads to greater difficulties in estimating isoquants and isoclines. This difficulty is generally compounded by the use of a transformation of the variables, and by any increase in the number of input factors considered. Thus the derivation of isoquants and isoclines for polynomials involving other than square root and squared factors in addition to first degree terms is very time consuming. As mentioned previously this difficulty arises when by use of the power transformation in equation 6.22, the resultant function has terms raised to peculiar powers, as for instance in equations 6.23 and 6.26. Moreover, it must be stressed that to draw out the economic implications of the production surface, the researcher has to do more than simply fit a function to the surface — as can easily be done by way of a Taylor's expansion over the relevant ranges of the inputs. For the economic implications of the production surface to be recognized and acted upon, the isoquants and isoclines have to be calculated. It is only through their evaluation that account is taken of the price and cost conditions ruling in the real world. The manipulations involved in deriving isoquants and isoclines and their subsequent interpretation have been outlined in Chapter 3.

The second difficulty with the use of polynomial functions is that as the degree of the polynomial increases, and as the number of factors considered rises, the number of terms involved in the expansion rapidly

Table 6.1. Number of Coefficients To Be Estimated for Polynomials of Various Degree

Degree of Polynomial	Number of Input Factors					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	6	10	15	21	28
3	4	10	20	35	56	84
4	5	15	35	70	126	210

increases. In consequence, the number of degrees of freedom available for statistical testing is decreased. The reason is that for each term in the polynomial there is a coefficient that has to be estimated at the cost of a degree of freedom. The number of coefficients, say c, in a polynomial of degree d for k factors is given by the combinatorial formula shown in equation 6.31.

$$c = \frac{(k + d)!}{d!k!} \tag{6.31}$$

The number of coefficients to be estimated for polynomials of various degrees involving various numbers of factors are shown in Table 6.1. Perusal of this table indicates how rapidly the number of degrees of freedom available for testing, out of a given total, may be diminished as the polynomial is extended.

ADEQUACY OF THE FITTED FUNCTION

Having fitted a particular function, two questions arise. First, in an over-all sense, does the function adequately characterize the data? Secondly, are some of the individual terms in the function of such negligible import that they should be excluded? These questions are, of course, related. Moreover, it must be stressed that, granted the relevance of the input factors considered, there is no sure answer to these questions. Researchers may quite rationally differ in the weight they attach to the various criteria used in assessing the suitability of a given or alternative functions.

Consider first the question of the adequacy of the over-all function. The major question to be asked is: Does the function adequately characterize the data both in terms of the known logic about the production process and in statistical terms (which may or may not be the same thing)? The former question will generally have been raised and answered at the initial stage of choosing the function. For testing the statistical adequacy, although not necessarily the basic logical adequacy, of the fitted function, three procedures are generally available when experimental data has been fitted by regression procedures. They

are based on the size of the coefficient of multiple determination, an F test of the regression mean square and a comparison of the lack of fit and error mean squares. For nonexperimental data, only the first two approaches are feasible, it not being possible to calculate a lack of fit term from nonexperimental data. Replications are needed to segregate the deviations from regression into lack of fit and error components.

It is desirable that the coefficient of multiple determination, R^2, be as close to unity as possible. Generally, quite low values of R^2 will be statistically significant. However, such statistical significance merely denotes that there is *some* relation between the independent and dependent variables. In production function research we desire more than this; we want to explain the *major* part of the response in terms of the considered inputs. If we cannot do so, then the fitted function will be of little operational use for real-world decision making. If the number of parameters estimated is large relative to the number of observations used, then the appropriate term for looking at the percentage of the variation in output explained by the regression model is $\bar{R}^2$, the adjusted coefficient of multiple determination. The derivations of R^2 and $\bar{R}^2$ are given in equations 4.40 and 4.41 of Chapter 4.

An over-all test of the significance of the fitted regression model may be carried out by calculating the F ratio, regression mean square divided by error mean square, as shown in equation 4.42 of Chapter 4. This ratio provides a test of the null hypothesis that all the regression coefficients are equal to zero. If the F value is larger than the tabled value of F at the desired probability level, the null hypothesis is probably not true.

With experimental data it is also possible to calculate a lack of fit mean square. This term can be compared with the error mean square to test whether the variation of the observations about the fitted model are like those that might be expected given the errors natural to the experimental environment. The latter errors are estimated by the error mean square and are independent of the fitted model. They are ascertained by repeated observation, i.e., by replication. A statistically significant lack of fit F ratio testifies that the variation around the fitted function is larger than that which might be expected from experimental error alone. In such cases, the model cannot be said to adequately fit the data. A stronger criterion sometimes used is that the fitted function should only be regarded as providing an adequate fit if the lack of fit mean square is of the same order of magnitude (or less) than the error mean square. The calculation of the lack of fit mean square for alternative experimental designs is discussed in Chapter 4.

When these various tests indicate that the fitted model is statistically satisfactory, and the model does not infringe the basic logic of the production process, no problem arises. However, having fitted a number of functions, it may be found that some satisfy some of the criteria better than others and vice versa. The function chosen as best will then depend on the weight the researcher attaches to the various criteria, statistical and logical. At such a stage, the selection of a function is more of an art than a science.

As for the over-all function, a number of criteria provide guides as to whether or not individual terms should be dropped from the fitted function. One criteria used is the testing of the statistical significance of the individual regression coefficients. The method is shown in equations 4.34, 4.35, and 4.36 of Chapter 4. However, the dropping of individual nonsignificant terms is not to be recommended; by its nature a significance test merely takes account of the strength of evidence against the worst possible result. Even if the evidence against the regression coefficient being zero is slight, the best estimate of its size is still that obtained from the data. Indeed it is most unlikely that the true value of the coefficient is exactly zero. Hence Anderson[9] has suggested a more lenient criterion: a variable should be dropped only if the standard error of the regression coefficient exceeds the absolute size of the estimated coefficient, and then only if there are no strong logical grounds for including the variable.

When a polynomial function has been fitted, it is possible to calculate F ratios whereby the contribution of each group of terms — linear, quadratic, cubic, etc. — to the regression sum of squares can be tested for significance. If orthogonal polynomial regression has been used, this testing can be done very conveniently as shown in relation to Table 4.2 of Chapter 4. If the data is not amenable to orthogonal polynomial regression, more tedious procedures have to be used.[10] Suppose the fitted function is of second degree in three variables, as in equation 6.32, and that an estimate of the error mean square is available from

$$(6.32)\quad Y = b_0 + b_1 X_1 + b_2 X_2 + b_3 X_3 + b_{11} X_1^2 + b_{22} X_2^2 + b_{33} X_3^2 + b_{12} X_1 X_2 + b_{13} X_1 X_3 + b_{23} X_2 X_3$$

replication or prior experience. Working in terms of the raw data, as in equation 4.9 of Chapter 4, the sum of squares associated with the linear regression on X_1, X_2, and X_3 may be calculated as follows. First fit equation 6.33.

$$(6.33)\qquad Y = b_0^* + b_1^* X_1 + b_2^* X_2 + b_3^* X_3$$

With the b^*'s of equation 6.33, calculate the linear sum of squares as shown in equation 6.34

$$(6.34)\qquad \text{Linear S.S.} = \sum_{i=0}^{3} b_i^* \sum_{j}^{n} X_{ij} Y_j - \frac{(\sum_{j}^{n} Y_j)^2}{n}$$

where X_0 is one and $\Sigma X_{ij} Y_j$ is the sum of crossproducts over the n corresponding observations on X_i and Y. The linear mean square,

[9] Anderson, R. L. Some statistical problems in the analysis of fertilizer response data. In Baum, E. L. *et al.* (eds.) Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames, 1957. Pp. 187-206.

[10] See Anderson, R. L. and Bancroft, T. A. Statistical Theory in Research. McGraw-Hill, Inc., New York. 1952. P. 172.

obtained by dividing the linear sum of squares by its degrees of freedom, may be tested against the error mean square as an F ratio. If the F value is significant, then the linear regression portion of equation 6.32 has made a worthwhile contribution.

The additional variation in Y accounted for by the quadratic terms of equation 6.32, i.e., by X_1^2, X_2^2, X_3^2, X_1X_2, X_1X_3, and X_2X_3, may be found as shown in equation 6.35.

$$(6.35)\quad \text{Quadratic S.S.} = \sum_{i=0}^{3} b_i \sum_{j=1}^{n} X_{ij}Y_j + \sum_{i \geq h \geq 1}^{3} b_{ih} \sum_{j=1}^{n} X_{ij}X_{hj}Y_j - \frac{\left(\sum_{j}^{n} Y_j\right)^2}{n} - \text{Linear S.S.}$$

The quadratic contribution may then be tested by an F ratio in the same way as suggested for the linear mean square. The calculations involved in such testing of groups of variables of like degree become quite simple for experimental data from factorial, composite, or rotatable designs because of their semiorthogonal nature.

The majority of the criteria discussed above for assessing the adequacy of a particular function, and the contribution of individual terms in the function, are statistical. Nonetheless, the selection of a particular fitted function is the final act of economic specification. Once a particular fitted function is selected, the economic implications of the relevant surface region are set. They follow automatically from the fitted function, for any given set of price conditions. This fact is the justification for the inclusion of these statistical tests under the topic of economic specification. Moreover, it is worth emphasizing again that most weight should be attached to whatever information is available about the logic underlying the production process. Given such information, and being sure of its reliability, only secondary importance need be attached to the statistical tests. Still, more often than not, information about the basic logic will be lacking. The role of the statistical tests must then be dominant.

SPECIFICATION ERRORS

It is never possible to specify and fit the true production function that is relevant to a given production process. Generally the true functional form and the complete range of variables that it should contain are unknown. Some variables may be known to be relevant but be impossible to include because data about them is not available. As well, approximations and abridgements may have to be made due to limited research resources. All these exclusions, approximations, and other empirical compromises result in specification errors. The fitted function cannot help but be biased to some extent. Indeed, it might be said

that a large part of chapters 4 and 5 and the preceding portion of the current chapter are largely concerned with various procedures oriented to the minimization of these specification errors. We now elaborate a method whereby the bias that does exist may be formally recognized and, to some extent, its consequences evaluated.[11] There is a proviso, however. The framework of analysis assumes (a) there is no error in specifying or measuring the output, (b) that the input variables are non-stochastic, and (c) that the production function is linear in the parameters. Insofar as least-squares procedures are most frequently used in estimating production functions, the framework will be outlined in relation to the ordinary regression procedures.

Suppose the *true* production function is

$$Y = \beta_0 X_0 + \beta_1 X_1 + \dots + \beta_k X_k \quad . \tag{6.36}$$

The general case will be that the X's are not specified correctly. The actual function estimated will be

$$Y = \bar{\beta}_0 \bar{X}_0 + \bar{\beta}_1 \bar{X}_1 + \dots + \bar{\beta}_h \bar{X}_h \tag{6.37}$$

where $\bar{X}_i$ may or may not be exactly the same as X_i and h may be greater or less than k. What we would like to do is estimate equation 6.36. Instead, we are forced by the exigencies of the real world to estimate equation 6.37. The question that arises is what is the relationship between the $\bar{\beta}$ estimates of equation 6.37 and the true production coefficients which are the β's of equation 6.36. Following the matrix notation outlined in equations 4.20 to 4.24 of Chapter 4, the $\bar{\beta}$ estimates are given by

$$[\bar{\beta}] = ([\bar{X}]' [\bar{X}])^{-1} [\bar{X}]' [Y] \ . \tag{6.38}$$

the true regression model is

$$[Y] = [X] [\beta] + [\epsilon] \ . \tag{6.39}$$

Substituting equation 6.39 in equation 6.38 and taking the expectation of $[\bar{\beta}]$ we have, since $E[\epsilon]$ is zero,

$$E[\bar{\beta}] = [P] [\beta] \tag{6.40}$$

where

$$[P] = ([\bar{X}]' [\bar{X}])^{-1} [\bar{X}]' [X] \ . \tag{6.41}$$

[11] In doing so, we mainly follow the development of an idea of Theil's as ably presented by Griliches. See Griliches, Z. Specification bias in estimates of production functions. Journal of Farm Economics, 39: 8-20, 1957; Theil, H. Linear Aggregation of economic relations. North Holland Publishing Company, Amsterdam, 1954.

In equation 6.40, the expected value of the estimates actually derived are expressed in terms of the true parameters, $[\beta]$, and the available data, $[\bar{X}]$. Each $\bar{\beta}_i$ is influenced by all the β's. Now equation 6.41 is analogous to equation 6.38 or 4.24. Hence the elements of [P] can be regarded as regression coefficients of the regression of each X_i on all of the included $\bar{X}$'s. For example, the i-th column of [P] is composed of the regression coefficients p_{0i} to p_{hi} that would be obtained if equation 6.42 could be calculated.

$$(6.42) \quad X_i = p_{0i}\bar{X}_0 + p_{1i}\bar{X}_1 + \ldots + p_{hi}\bar{X}_h + \bar{\epsilon}_i \; ; \; i = 0, 1 \ldots, k$$

The set of regressions denoted by equation 6.42 has been termed "auxiliary" regressions. Together, equations 6.40 and 6.42 indicate that the effect of the true parameters on the actually estimated coefficient $\bar{\beta}_i$ for a particular $\bar{X}_i$ will depend on the sign and size of the p's attached to that $\bar{X}_i$ in the various auxiliary regressions. If, in some fashion, the signs and magnitudes of the p's were known, it would be possible to ascertain the size and direction of bias in the actually estimated $\bar{\beta}$ values.

Omission of Variables

Suppose all of the X's, the true inputs, are measurable except one, say X_k. So long as this variable is uncorrelated with any of the other X's, its omission will not bias the estimates of $\beta_0, \beta_1, \ldots, \beta_{k-1}$. Still, the assumption of zero correlation between the excluded factor and any of the other inputs is not likely to hold in the real world. In general, a certain degree of positive correlation is to be expected. The result will be a tendency to overestimate one or more of the coefficients of the included variables. Such an effect can be deduced from equation 6.43 which gives the expected value of $\bar{\beta}_i$ when X_k is excluded.

$$(6.43) \qquad E(\bar{\beta}_i) = \beta_i + p_{ik}\beta_k \; ; \; i = 0, 1, \ldots, k\text{-}1$$

The p's of equation 6.43 are from equation 6.44 which is the only non-trivial auxiliary regression. All the other auxiliary regressions are trivial since, in the case under consideration, $\bar{X}_i = X_i$.

$$(6.44) \qquad X_k = p_{0k}X_0 + p_{1k}X_1 + \ldots + p_{k-1,k}X_{k-1} + \bar{\epsilon}$$

If X_k should be negatively associated with some of the other inputs, unlikely though it be, the p_{ik} term of equation 6.43 will be negative and $E(\bar{\beta}_i)$ will be less than β_i; there would be a bias towards underestimation.

To generalize, if the last K input factors are omitted from the estimation, the expected value of $\bar{\beta}_i$ is given by equation 6.45.

$$(6.45) \qquad E(\bar{\beta}_i) = \beta_i + \sum_{j=k-K+1}^{k} p_{ij}\,\beta_j \; ; \; i = 0, 1, \ldots, k\text{-}K$$

Again, it is generally to be expected that $\bar{\beta}_i$ will be biased upwards, given the fact that positive correlations among inputs are most common.

The economic implications of the bias in the $\bar{\beta}$ values will be traced out through the economically oriented manipulations and interpretations of the production function. For instance, such biases will be carried over into the isocline and isoquant derivations. Moreover, it should be noted that the above discussion of the omission of relevant variables applies equally to the derivation of production functions from experimental and nonexperimental data.

Aggregation Within Inputs

Within a single input category, no matter how finely we define the category, there will usually be quality differences. Only under experimental conditions where the inputs can be made of uniform quality through the application of chemical and other standards are these quality differences likely to be negligible. But with factors such as land and labor, quite large quality differentials will be the rule. Generally, little account is taken of these differences — one acre of land is regarded as being much the same as any other acre. If adjustments are made, they can only be approximate in the absence of precise knowledge of the quality differentials. How does the failure to take account of the true quality differences affect the specification of the production function?

Suppose quality differences occur only in X_k of equation 6.36. The quantity of X_k used in estimation is taken as $\bar{X}_k$. But, in truth, we may say $\bar{X}_k$ is made up of m portions each differing in quality, the j-th such portion being denoted by ${}_j\bar{X}_k$. The use of $\bar{X}_k$ without taking account of these quality differences assumes that equation 6.46 holds.

$$(6.46) \qquad X_k = {}_1\bar{X}_k + {}_2\bar{X}_k + \ldots + {}_m\bar{X}_k$$

Denote by g_j the weight that should be attached to ${}_j\bar{X}_k$ to transform it into equivalent units. Hence X_k may be expressed in terms of ${}_j\bar{X}_k$ as shown in equation 6.47.

$$(6.47) \qquad X_k = g_1\,{}_1\bar{X}_k + g_2\,{}_2\bar{X}_k + \ldots + g_m\,{}_m\bar{X}_k$$

Thus, for the case under consideration, equation 6.36 may be rewritten as in equation 6.48.

$$(6.48) \qquad Y = \beta_0 X_0 + \beta_1 X_1 + \ldots + \beta_{k-1} X_{k-1} + \beta_k \sum_{j=i}^{m} g_j\,{}_j\bar{X}_k$$

The ignoring of quality differences within the one factor is thus equivalent to the omission of a number of variables together with the inclusion of an additional variable, $\bar{X}_k$. $\bar{X}_k$ is an imperfect aggregate, being based on the assumption that g_j is always unity. The exclusion of the variables $g_j\ {}_j\bar{X}_k$ will generally tend to bias the β values upward, as outlined in relation to equations 6.43 and 6.45. However, the net result will depend on whether the influence of the inclusion of the imperfect aggregate $\bar{X}_k$ is to complement or counteract the bias resulting from the omissions.

Aggregation Over Inputs

The more general case of the situation examined above is that in which several distinctive inputs are aggregated into a single input category. Suppose all inputs are aggregated into two categories $\bar{X}_1$ and $\bar{X}_2$, and the estimated function is as in equation 6.49.

$$Y = \bar{\beta}_0 + \bar{\beta}_1 \bar{X}_1 + \bar{\beta}_2 \bar{X}_2 \tag{6.49}$$

while equation 6.50 denotes the true production function.

$$Y = \beta_0 + \sum_i \beta_{1i} X_{1i} + \sum_j \beta_{2j} X_{2j}, \quad i + j = k \tag{6.50}$$

Hence the aggregation is such that

$$\bar{X}_1 = \sum_i X_{1i} \quad \text{and} \quad \bar{X}_2 = \sum_j X_{2j}. \tag{6.51}$$

By virtue of equation 6.40, the expected values of the $\bar{\beta}$'s of equation 6.49 are given, in scalar notation, by equations 6.52 and 6.53.

$$E\ (\bar{\beta}_1) = \sum_i p_{i1} \beta_{1i} + \sum_j p_{j1} \beta_{2j} \tag{6.52}$$

$$E\ (\bar{\beta}_2) = \sum_i p_{i2} \beta_{1i} + \sum_j p_{j2} \beta_{2j} \tag{6.53}$$

Each estimated $\bar{\beta}$ thus consists of a weighted average of the true parameters associated with each of the basic unaggregated inputs that are included in the corresponding aggregate plus a weighted sum of the parameters not included in the aggregate relevant to the given $\bar{\beta}$. The weights, as before, are the auxiliary regression coefficients. If the auxiliary regression coefficients and the true production function parameters, i.e., the β's, are uncorrelated, there will be no bias due to aggregation. Such a situation is not to be expected.

The problems associated with aggregation over inputs are most likely to arise in the estimation of production functions for firms. In such cases the number of input categories is large and quality differences in inputs between and within firms are to be expected. To minimize specification bias due to aggregation in such cases, two working

rules should be used. First, perfect complements, i.e., resource categories that have to be used in fixed proportions, should be treated as a single input. The use of one such resource implies the use of its complements. To include each of the complementary categories would lead to multicollinearity (see Chapter 4) because of the perfect correlation between levels of the complementary inputs. Secondly, perfect substitutes should also be aggregated into a single input category. Ideally this can only be done by the use of standard units as in equation 6.47. The difficulty is that the standardizing weights will generally be unknown. Should equation 6.46 hold true, the aggregation problem is trivial—the input categories are perfect substitutes simply because they are but portions of identically the same input!

EQUILIBRIUM PROBLEMS IN SURVEY DATA

It has been argued that if producers maximize profits, identification of the production function from single period survey data will be impossible.[12]

Thus if entrepreneurs maximize profits, and all use the same production function and there is no resource fixity, all sample firms would have the same input-output array and the production function could not be estimated. No real-world data has ever evidenced such concurrence. However, suppose entrepreneurs are only prevented from maximizing profits by resource fixity and there is a common proportionate element of resource fixity across the sample. Then all observations will lie on the one isocline (assuming a common production function) and the data will (usually) be multicollinear. However, the assumptions of certainty, omniscience, proportionate resource fixity, and a common production function are too strong to apply to real-world data. As Konijn[13] has shown, if they do not hold true, it is possible to derive a meaningful production function estimate.

[12] See Soper, C. S. Production functions and cross-section surveys. Economic Record, 34: 111-17, 1958. Also, Marschak, J. and Andrews, W. H. Random simultaneous equations and the theory of production. Econometrica, 12: 143-205, 1944.

[13] Konijn, H. S. Estimation of an average production function from surveys. Economic Record, 35: 118-25, 1959. Also, Soper, C. S. Production functions: A reply to Dr. Konijn. Economic Record, 35: 434-35, 1959.

CHAPTER 7

Miscellaneous Empirical Problems Relating to Estimation of Production Functions

Actual estimation of production functions and eliciting of their real-world implications is an empirical task. A variety of unavoidable empirical problems thus arise. These do not fit neatly under the theoretical labels discussed in Chapters 3, 4, 5, and 6. We consider the more important of these miscellaneous empirical problems in the present chapter. Emphasis is on the compromises and approximations that may be expected to serve best in data collection and analysis. The research worker is faced with the necessity of striking a balance between the theoretically desirable and the empirically feasible, in terms of data availability and computational resources. Frequently, it is possible that a worthwhile compromise cannot be attained.

The majority of the problems discussed relate to the estimation of farm-firm production functions. Farm-firm problems predominate in the discussion because their estimation is more difficult than the derivation of production functions for individual technologies.[1] The latter can be studied under experimental conditions while the farm-firm production process is complicated by its relation to entrepreneurial decision making.

CLASSIFICATION AND MEASUREMENT OF INPUTS AND OUTPUTS

The applicability of an empirically derived function depends on the way in which the input and output factors are defined and measured, and on the use to which the fitted function is to be put. If a high degree of aggregation is used, the implications of the resultant function may be of

[1] For general discussion of the estimation of farm-firm functions, see Heady, Earl O. Elementary models in farm production economics research. Jour. Farm Econ., 30: 1-18; Plaxico, J. S. Problems of factor-product aggregation in Cobb-Douglas productivity analysis. Jour. Farm Econ., 37: 664-75; Parish, R. M. and Dillon, J. L. Recent applications of the production function in farm management research. Review of Marketing and Agricultural Economics, 23: 215-36; Jarrett, F. G. Resource productivities and production functions. Australian Jour. Agr. Econ., 1: 67-78; and Konijn, H. S. Estimation of an average production function from surveys. Econ. Record, 35: 118-25; Wolfson, R. J. An econometric investigation of regional differentials in American agricultural wages. Econometrica, 26: 225-56.

little relevance in decision making. For the farm operator, knowledge that the marginal return to the broad category "capital" exceeds its marginal cost is insufficient. Returns may not exceed costs for some capital items within the aggregate; for others, the opposite will be true. On the other hand, the information derived from a production function based on aggregative input and output categories may be quite useful to a government policy maker.

Ideally, input and output variables should be measured in physical units of a homogeneous nature; quality differences in a given factor being absolved by a standardization procedure based on the relative contribution to production of each quality level of the particular factor Largely, this requirement is attained in experimental studies of particular technologies involving individual fertilizer and feed elements. The situation differs for farm-firm studies based on either time series or cross-sectional from the world of decision making. Then it is impossible to measure all items in physical terms, especially capital goods and services. Heterogeneous capital forms have no common physical unit. They must be aggregated to some extent and measured in value terms for computational purposes. While a tractor gives inputs of work hours at a given intensity, this physical input cannot be added to that from, say, a welding kit. Faced with a myriad of different types of these inputs, the research worker must aggregate to some extent. The practical basis of aggregation ordinarily is in value terms. Similarly, there are various grades of output which can be aggregated feasibly only in value terms. Consequently, the theoretical distinction between a physical production function and a value function is generally blurred in farm-firm analysis. Also, the generality of the fitted function is reduced since it strictly applies only to the particular price regime on which the value estimates are based.[2]

The major broad resource categories are land, labor, capital, and management. The peculiarities associated with classification and measurement for farm-firms of these categories will now be outlined.

Capital

A statement that capital inputs have, based on farm sample estimates, a marginal return of so many dollars indicates nothing about the productivity of particular forms of capital inputs. The statement is meaningful only under the assumption that the composition of capital increments corresponds to the average capital composition employed in the sample firms. This problem can be minimized, however, by classifying capital inputs into a number of value categories. The particular categories chosen will depend on the investigator's assessment

[2] For a procedure oriented to overcoming these difficulties see Trant, G. I. Adjusting for price levels in production function studies, in Heady, E. O. *et al.* (eds.) Resource productivity, returns to scale, and farm size. Iowa State University Press, Ames. 1956. Pp. 162-67.

of the strategic inputs in the production process, the bookkeeping habits of farmers, and the purpose of the investigation. Two general rules should be followed in deciding on the various individual items to be placed in each input category. They are:

1. The inputs within an individual category should be as nearly perfect substitutes or perfect complements as possible.
2. Relative to each other, the categories of inputs should be neither perfect substitutes nor perfect complements.

Quite apart from the fact that these two rules lead to theoretical correctness, they are quite functional in that they tend to specify the production problem in a fashion meaningful to farmers. The more important real-world problems of resource allocation fall under the second rule.

To demonstrate the variety of ways in which the capital factor has been approached, the classification of factors used in a small number of published farm-function analyses are listed below in historical sequence.[3]

Kamiya (1941)[4]: Land; labor.

Tintner (1944)[5]: Land; labor; farm improvements (buildings, fences, etc.); liquid assets (livestock, feed, seed, fertilizers); working assets (machinery, breeding stock, equipment); cash operating expenses (repairs, fuel, oil, purchased feed).

Heady (1946)[6]: Real estate (land and improvements); labor; machinery and equipment (inventory value plus value of repairs; fuel and lubricants); livestock and feed (stock on hand and purchased, livestock expense and feed fed); miscellaneous operating expenses.

Johnson (1952)[7]: Land; labor; machinery investment (inventory value); livestock (inventory value) and forage production investment (replacement value of hay and pasture stands plus investment in structures or land clearing necessary to establish such crops); cash operating expenses.

Heady and Shaw (1954)[8]: Crop function: Cropland; crop labor; capital services used on crops plus annual cash expenses (including depreciation on all items used directly or indirectly in crop production). Livestock function: Labor; capital services used on livestock (including depreciation on buildings, etc., and depreciation on breeding stock, and purchase value of feeding stock).

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[3] In all cases, land is measured in acres, real estate in dollars or pounds, labor in man-months or man-years, all capital inputs in dollars or pounds.

[4] Kamiya, K. On productivity of labor. Jour. Rur. Econ., 17: 89-98.

[5] Tintner, G. A note on the derivation of production functions from farm records. Econometrica, 12: 26-34.

[6] Heady, E. O. Production functions from a random sample of farms. Jour. Farm Econ., 28: 989-1004.

[7] Bradford, L. A. and Johnson, G. L. Farm management analysis. John Wiley and Sons, Inc., New York. 1953. Pp. 145-47.

[8] Heady, E. O. and Shaw, R. Resource returns and productivity coefficients in selected farming areas. Jour. Farm Econ., 36: 243-57.

Heady (1954)[9]: Land fixed. Crop function: Labor, machinery expenses (depreciation, repairs, fuel, oil, etc.); annual crop expenses (seed, fertilizer, lime seed treatment, etc.).

Antill (1955)[10]: Land; labor; purchased feeds; other capital inputs (including interest on crops, livestock, machinery and equipment inventories, depreciation on machinery and equipment, rent or rental value of real estate, cost of salaried management).

Schapper and Mauldon (1957)[11]: Labor (excluding part-time and contract); purchased feed; fertilizer purchased; miscellaneous operating expenses (part-time labor, fuel, oil, machinery repairs and maintenance, freight).

Wang (1958)[12]: Cropland; labor; capital services (seed, fertilizers, rent for working cattle and equipment, miscellaneous crop outlays).

This listing illustrates the diversity of approaches that have been used in classifying capital inputs. Neglecting Kamiya's study (1941) which did not consider capital, the historical tendency has been (a) to reduce the number of capital input categories and (b) to measure the input of durable assets by the actual maintenance and depreciation costs associated with their use rather than by their capital value on an inventory basis. For the reasons of interpretation already stressed, the grouping of capital items into a single category must be regarded as unwise unless they are good substitutes or complements. On the other hand, some level of aggregation is required for computational feasibility. However, caution must be exercised in inferences since the diverse micro-inputs may have varying productivity magnitudes.

As usually oriented, the production function study for firms is concerned with the short run of a single production period or year. Measurement of the input of durable assets by maintenance and depreciation costs is certainly to be recommended over that of capital investment or inventory value. Durable items such as machinery constitute a stock resource which gives rise to a flow of input services over the life of the asset. Only the flow of services within the selected production period is relevant to the production function for that period. The actual input flow relevant to the period under consideration should be used. Measurement of this flow as a depreciation rate based on an arbitrary bookkeeping basis often is not satisfactory. For instance, on one farm a substantial amount of depreciation might be allowed as capital service input on a relatively new tractor used but little during the production period. On another farm in the same sample, a small amount of depreciation might be taken as the capital service input on an old tractor used

[9] Heady, E. O. Resource productivity and returns on 160-acre farms in north central Iowa. Iowa Agr. Exp. Sta. Bul. 412. Ames. 1954.

[10] Antill, A. G. Towards a production function for dairy farms. The Farm Economist, 8: 1-11.

[11] Schapper, H. P. and Mauldon, R. G. A production function from farms in the whole-milk region of Western Australia. Econ. Record, 33: 52-59.

[12] Wang, Y. Resource returns and productivity coefficients for selected crop systems in Taiwan area. Proceedings of Agr. Econ. Seminars. National Taiwan University, Taipei. Sept. 16-20, 1958. Pp. 90-98.

extensively. In each case, a substantial observation error may occur. In the absence of detailed records on the actual use of capital items, however, little can be done to overcome these measurement deficiencies. Still, they should be recognized and necessary caution exercised in interpretation. The effect of such measurement errors can be ascertained via the method outlined in Chapter 6, relative to errors of specification, provided that the necessary auxiliary information is available.

The classifications used by Antill, and Schapper and Mauldon, as listed above, illustrate separate treatment of particular categories of capital input (purchased feed and fertilizer) which are considered to be of key importance in the particular production process being examined. Heady's study (1946) illustrates a method of treatment for complementary factors: a preliminary analysis showed high correlations between machinery and equipment inputs and fuel and lubricants. These items were therefore grouped together to form a single input category. Livestock input and feed expenses were lumped together for the same reason.

Care needs to be taken in delimiting the cash operating expenses category. Items of input which are determined directly by the volume of production should not be included. Selling charges, where they constitute a fixed commission on the amount of the sale, are an example. Included as a separate input they would have an elasticity of production of unity and a very high marginal productivity. But the results would be meaningless; selling expenses do not determine farm production but are determined by production. Other examples are packing and handling costs. Perhaps such expenses substantially explain the high marginal productivity of cash operating expenses derived in some early farm-firm studies. This possibility is indicated by the results of an Australian study. In a wool growing area, annual capital expenditure including shearing and wool selling expenses was found to have a production elasticity of 0.55 and a marginal productivity of \$1.73 per \$1 spent. When shearing and wool selling expenses were excluded, the production elasticity of capital fell to 0.43 and its marginal productivity to \$1.59.[13] Of course, the extent to which particular cost items are determined by output will vary by industry and region, and judgment is required to determine whether the decision makers vary particular inputs independently of output.[14]

Labor

In measuring the labor input, two factors should be borne in mind. First, what is required for estimational purposes is a measure of the

[13] Parish, R. M. and Dillon, J. L. Recent applications of the production function in farm management research. Review of Marketing and Agricultural Economics, 23: 215-36.

[14] See Jawetz, M. B. Farm size, farming intensity and the input-output relationships of some Welsh and West of England dairy farms. University College of Wales, Aberystwyth. 1957. P. 9.

labor input actually used in deriving the given production, not a measure of total labor, utilized and unutilized, available during the production period. (Unless detailed records have been kept by the farmer, the required information may not be available.) Second, account should be taken, so far as possible, of variations in labor quality. Work performance not only varies with age but also with other factors. The above comments apply especially to farm family labor. Hired employees work under supervision and receive wages presumably corresponding to their input. Where hired labor is a substantial portion of total input, it might best be measured in a separate category to make full use of the more reliable measurements. But if it is used directly as a substitute for family labor, the two might best be grouped together.

Land

The difficulties associated with measuring the land input are not so intractable as those associated with labor and capital. First, the sample of observations can be confined to farms that are relatively homogeneous with respect to land quality.[15] Secondly, the flow of services from the land input seems to be reasonably indexed, both between and within farms, by measurement in acreage or market value terms so long as only land actually used in production is included as input. Thirdly, differences in land quality are likely to be reasonably reflected in market values so that a standardized measure of the land input may be used to at least take account of gross differences in land quality. Still, the effects of differences in land quality may be serious. They are likely to be systematically related to differences in other inputs since, usually, the better quality land tends to be farmed more intensively. The resulting marginal productivity estimates derived from a fitted production function suggest that labor and capital, if employed in larger quantities on the poorer land, would give returns similar to those achieved on the better land. Similarly, the production elasticities may be overestimated. Failure to standardize the land input observations provides fitted functions which are hybrids relating to different land qualities.[16]

Management

The most difficult input to measure in farm-firm studies is the management or entrepreneurial factor. Unlike experimental studies where the managerial function is exercised by the investigator and remains fixed, the farm-firm research worker is faced with observations whose genesis rests not with him but with many decision makers, each

[15] See Heady, E. O. Productivity and income of labor and capital on Marshall silt loam farms in relation to conservation farming. Iowa Agr. Exp. Sta. Bul. 401. Ames. 1953.

[16] Heady, Earl O. Elementary models in production economics research. Jour. Farm Econ., 30: 1-18.

possessing a different degree of entrepreneurial ability. Such differences would be unimportant if management could be measured directly in physical terms. It could then be handled in the manner of other physical inputs. However, a generally applicable scale for measuring management has not yet been devised. Likely it never will be, except in terms of interaction of such factors as time and intensity of cerebration, intelligence level, problem handling ability, etc. Still, it may be feasible to scale the management factor as it relates to a given production process. Such scaling techniques have sometimes proved fruitful in handling nonphysical variables in psychological and sociological analyses.

Granted that differences in management input are likely to exist in a cross-sectional sample of farm data, danger exists in the possibility that differences in inputs of management are systematically related to variations in the input of other factors. However, perfect correlations are not to be expected between management and other inputs. While a large scale of operation may often be associated with a high level of management, it may also be a reflection of waste. Likewise, a small scale of production due to lack of knowledge or initiative reflects a low level of management. Still, a small scale of enterprise often results from lack of resources due to circumstances beyond the control of a capable entrepreneur. Nonetheless, it is reasonable to hypothesize that a significant positive correlation exists between managerial and capital inputs. Also, increased nonmanagerial inputs probably can be effectively utilized with less than proportionate increase in management. If these hypotheses do hold, then, as Griliches has shown, the omission of management may lead to underestimation of returns to scale and overestimation of capital returns.[17] Should these tendencies be regarded as serious disadvantages, the researcher may attempt to minimize them by (a) attempting to restrict the farm sample to a group relatively homogeneous with respect to the level of management, for instance, to good or poor managers on some subjective rating scale; (b) attempting to select a farm sample that minimizes correlations between management and other factors; or (c) by introducing some measure of management into the production function analysis.[18]

A few attempts have been made to include management as a factor in empirically fitted production functions. None of these has been reported fully in the literature, so it must be assumed that little success was attained in devising an adequate managerial scale. In such studies the usual procedure has been to rate the sample entrepreneurs on a management index relative to their knowledge of farming practices and techniques and the degree of economic rationality thought to be shown

[17] Griliches, Z. Specification bias in estimates of production functions. Jour. Farm Econ., 39: 8-20.

[18] See Johnson, G. L. Problems in studying resource productivity and size of business arising from managerial processes, in Heady, E. O. *et al.* (eds.) Resource productivity, returns to scale and farm size. Iowa State University Press, Ames. 1956. Pp. 16-23. Also, Westermarck, N. Management and success in farming. Acta Agriculturae Scandinavica. 8: 375-403; 9: 164-80.

by their current managerial decisions relative to the use of recommended practices. Such indices have three major disadvantages. First, they may not adequately distinguish between knowledge and entrepreneurial logic. Secondly, there is a danger that the management index may tend to measure the managerial potential or capability of the entrepreneur rather than his actual management input over the production period being analyzed. Lastly, such indices suffer from the fact that they incorporate subjective elements. What one assessor might regard as indicative of a large input of management might typify a low input to another assessor. Still, such difficulties of subjective judgment are not likely to be too important. For instance, in an unpublished study (1948) by Heady, a management index was used to measure managerial inputs; the managerial input ratings based on this index and scored independently by two people had a correlation coefficient of .91. With management incorporated, the Cobb-Douglas function estimated in this study is given in equation 7.1 where Y is output and X_1, X_2, X_3, and X_4 are land, labor, capital services, and management, respectively.

$$Y = .038X_1^{.273}\, X_2^{.363}\, X_3^{.383}\, X_4^{1.002} \tag{7.1}$$

Obviously, the estimate of production elasticity with respect to management is untrustworthy. Still, it is statistically true that for the farm sample studied, increasing returns to scale did appear to prevail when management was included. With management excluded, constant returns to scale seemed to prevail. The sample of farms to which equation 7.1 refers were Iowa farms which had successfully repaid and utilized a loan from the Farmers' Home Administration. This restriction may in part explain the high elasticity of production for the management factor since it is to be expected that returns on resource inputs obtained from loans may depend to a greater extent on management than on any other single factor.

Given the difficulties inherent in measuring management, a reverse approach has sometimes been suggested. It is that the residuals between production levels estimated from the fitted function and the actually observed production levels be attributed to management. These residuals would then be used as the basis for an objective management rating.[19] Thus in Figure 7.1, using the single input simplification, the fitted function might be AB. The actually observed output-input combinations might be those marked. Those lying above AB would be given a positive management rating and those below a negative rating, each rating being proportionate to the size of the residual. Such an approach has a number of disadvantages. The residuals may not be related to management but to other factors. Also, this procedure only considers one aspect of the management question — that typified by wasteful or inefficient use of a given resource bundle. The residual between estimated and

[19] See Heady, E. O. Production functions from a random sample of farms. Jour. Farm Econ., 28: 1003. Swanson, E. R. A measure of economic success in farming. Acta Agr. Scand. 9: 485-96.

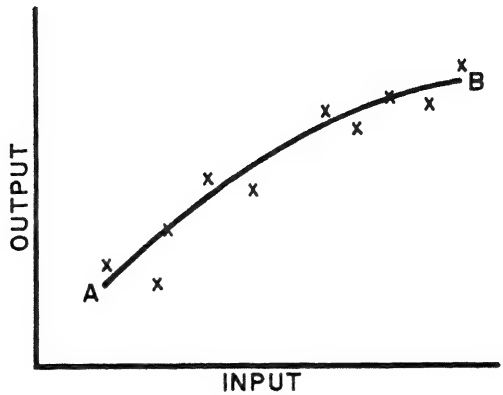

Figure 7.1. Residuals between input-output observations and the fitted production function.

observed output levels, where the observed is the smaller, indicates how much more production might, on average, be expected from the observed input levels. The other contribution of management, pertinent in farm situations when decreasing returns prevail, is to drive the entrepreneur to the optimum position on the production function as determined by the prevailing price regime. Due to exogenous resource restrictions, however, the "distance" between observed and optimal input combinations may give no indication of the management input that is being used. Relative to the management factor as it relates to wasteful resource use, it might be noted that resource waste may also be regarded as equivalent to an error of observation in the data. Thus, suppose a farmer used 15 units of a resource in such a fashion that they were only equivalent to 10 units applied efficiently. The true input would then be 10 units and not 15 units. However, by force of circumstances associated with data collection, such discrepancies cannot be detected. They remain as errors of observation in the data, closely akin to errors arising from quality differences within a single resource.

OUTPUT

Aggregation problems generally exist with respect to output in farm-firm studies. The difficulty arises because most farms produce either more than one type of product or several qualities of the same product. Where particular input quantities can be associated with each type or grade of output, distortions caused by aggregation may be avoided by fitting a separate function for each product. To this end, separate functions have been fitted for crop and livestock enterprises in a number of Iowa cross-sectional studies.[20] Iowa studies, using

[20] See Heady, E. O. Resource productivity and returns on 160 acre farms in north cen-

simultaneous equations under the assumption of joint relationships, have not proved direct interdependence between broadly grouped outputs such as crops and livestock. Provided that the productivity of resources is independent between the different products, the production function for each enterprise can be validly used to estimate marginal resource productivities, isoquants, and isoclines relative to each product and, hence, to ascertain the optimum allocation of resources between the various enterprises. However, such estimates must be interpreted cautiously if resource productivities are not independent between the different enterprises; the fitting of a separate function for each product assumes that the productivity of a resource relative to a specific type of output is uninfluenced by the level of resource use associated with the other products produced by the firm. Still, except in cases of gross interdependence between enterprises, estimates derived from individual output functions are likely to be more reliable and useful than recommendations derived from a single aggregative function with all outputs lumped together. With products aggregated on a value basis, nothing can be said relative to investments in individual enterprises.

In many instances, the researcher has no alternative but to use an aggregative measure of value output. Estimation of functions for an individual enterprise is infeasible unless records are available indicating the quantity of each input associated with each output. Such detailed records are rarely available for a cross-sectional sample although they may be available in time series form for some individual farms. If an aggregative measure of output has to be used, the distortions caused by output aggregation may be minimized by deriving separate functions for groups of firms producing the various outputs in approximately the same proportions. This approach requires a fairly large analytical sample to ensure a sufficient number of farms in each group for which a function is fitted. Still, the disadvantages of using large purposive samples will generally be outweighed by the greater usefulness of the estimates, compared to the "rough and ready" function obtained by fitting a single over-all function to a group of multi-enterprise farms. The justification of the grouping procedure is that the outputs within each group of farms are treated as if they were joint products produced in fixed proportions. For such products, aggregation would be warranted since their mix is fixed and cannot be varied by the farmer.

Another approach to relationships in multi-enterprise situations has been suggested by Plaxico: A function of the form of equation 7.2 might be used.[21] Marginal

tral Iowa. Iowa Agr. Exp. Sta. Bul. 412. Ames. 1954; Heady, E. O. Productivity and income of labor and capital on Marshall silt loam farms in relation to conservation farming. Iowa Agr. Exp. Sta. Bul. 401. Ames. 1953; also Heady, E. O. and Shaw, R. Resource returns and productivity coefficients in selected farming areas. Jour. Farm Econ., 36: 243-57.

[21] Plaxico, J. S. Problems of factor-product aggregation in Cobb-Douglas value productivity analysis. Jour. Farm Econ., 37: 664-75. See also Waugh, F. V. Regression between sets of variates. Econometrica, 10: 290-310.

$$(7.2) \qquad Y_1^{\alpha_1} Y_2^{\alpha_2} \ldots Y_h^{\alpha_h} = \beta_0 X_1^{\beta_1} X_2^{\beta_2} \ldots X_k^{\beta_k}$$

productivities, isoquants, and isoclines relevant to the i-th output, Y_i, could than be estimated with other products held constant at specified levels. However, unless resource productivities are strongly interdependent among products, functions such as equation 7.2 may not be satisfactory in terms of the economic, biological, or physical logic underlying the production process. Moreover, the fitting of such functions is computationally irksome relative to the estimation of production functions involving but a single independent variable.

USE OF COBB-DOUGLAS TYPE FUNCTIONS

Of possible algebraic forms, Cobb-Douglas functions have been the most popular in farm-firm analyses. This algebraic model provides a compromise between (a) adequate fit of the data, (b) computational feasibility, and (c) sufficient degrees of freedom unused to allow for statistical testing. In other words, the Cobb-Douglas is a relatively "efficient user" of degrees of freedom. Such efficiency is important where research resources are limited and collection of farm-firm data is expensive. However, it probably has greatest use in diagnostic analyses, reflecting marginal resource productivities at mean levels of inputs. Because of the elasticity restraints outlined in Chapter 3, estimates departing above the mean may well overestimate productivities. Too, under the aggregation procedures frequently necessary, the fitted functions can tell the farm little about returns from specific investments. The Cobb-Douglas function has been used less extensively for individual technologies such as fertilizer and feed relationships. More reliable response estimates have usually been attainable from nonmultiplicative polynomial type functions. However, whenever a Cobb-Douglas type function is to be used, certain procedures should be followed in assembling and analyzing the data. If these adaptions are not made, the conclusions drawn from the analysis will not be as reliable as they might be.

Geometric Versus Arithmetic Aggregation

Since some degree of aggregation is necessary in farm-firm studies, machinery services arising from, say, tractors and milking machines may be aggregated together as a single input. The usual procedure has been to simply add the money value of such micro-inputs to provide an aggregative input. However, the arithmetic sum of the micro-inputs introduces bias in the resultant estimates, except when the micro-inputs that are summed are always used in fixed proportions.[22] Bias can be

[22] See Shephard, R. W. Cost and production functions. Princeton University Press,

reduced by using as the aggregated input, not the arithmetic sum of the micro-inputs, but their geometric sum, i.e., their product. Thus, if micro-inputs $x_1, x_2, \ldots, x_n$ are to be aggregated into the single category X for use in a Cobb-Douglas function, aggregation as specified by equation 7.3 must be regarded as less satisfactory than aggregation as in equation 7.4. Intuitive justification for the multiplicative approach to aggregation in Cobb-Douglas analyses is provided by equation 7.5 which corresponds to equation 7.4. It is obviously anomalous to use equation 7.3 when the basic estimational feature of the Cobb-Douglas function is a logarithmic transformation into a linear form.

$$X = x_1 + x_2 \ldots + x_n \tag{7.3}$$

$$X = x_1 \, x_2 \ldots x_n \tag{7.4}$$

$$\text{Log } X = \log x_1 + \log x_2 \ldots + \log x_n \tag{7.5}$$

As Griliches has shown, the possible bias arising from using the geometric sum can be determined if a little auxiliary information is available. The procedure has been outlined in Chapter 6.[23] However, if the arithmetic sum method of aggregation is followed, it is virtually impossible to estimate the possible biases. Moreover, the multiplicative basis of aggregation approaches more closely the ideal, but impractical, method of aggregation for Cobb-Douglas functions. The perfect method would be to aggregate the micro-inputs multiplicatively with each micro-input weighted proportionately to its (unknown) elasticity of production.

Zero Input Levels

The Cobb-Douglas function implies that at least some quantity of each input must be used if output is to be nonzero. In the real world, such a condition does not hold true; indeed, most samples of data, either real world or experimental, will contain observations with one or more of the inputs at zero level. The practical difficulty arises in the conversion of the raw data to logarithmic form, the logarithm of zero being minus infinity. To overcome this problem, assuming there are strong grounds for using a Cobb-Douglas model, the zero observations may be replaced by some figure of arbitrary small size. (Or, a constant can be added to all observations for the particular input category.) For instance, if a fertilizer has been applied at rates of zero,

Princeton. 1953. Pp. 61-71. (This work is an important classical study of the production function, although much of the text involves relatively abstract mathematics.) See also Solow, R. M. The production function and the theory of capital. Review of Economic Studies, 23: 101-8.

[23] Griliches, Z. Specification bias in estimates of production functions. Jour. Farm Econ., 39: 8-20.

80, and 160 pounds per acre, the zero application might be taken as being one pound per acre for the purpose of fitting a Cobb-Douglas function. Too, when aggregating, the geometric aggregate would very often be zero unless the adjustment to some arbitrary nonzero level is made. Moreover, if zero observations occur quite frequently in the sample, their piecemeal adjustment is unsatisfactory and may induce nonnegligible errors. Alternative algebraic functions should be then investigated.

Returns to Scale

Historically, in Cobb-Douglas analyses of farm-firms, the sum of the estimated input coefficients has been taken as an indication of the returns to scale. Thus, relative to equation 7.6, $\Sigma b_i < 1$ has been taken as implying decreasing returns to scale and $\Sigma b_i > 1$ increasing returns to scale.

$$\hat{Y} = aX_1^{b_1}\ X_2^{b_2}\ \ldots\ X_k^{b_k} \tag{7.6}$$

The procedure usually adopted is to perform a t test to ascertain if Σb_i is significantly different from unity at the desired probability level. However, such estimates of scale returns will be biased unless all input factors are included in the production function.[24] Returns to scale will be underestimated if the excluded inputs vary less than proportionately with changes in the included factors over the sample of observations. If the reverse situation holds true the elasticity of production or returns to scale will be overestimated. Thus omission of management from farm-firm functions can take on importance. Intuitively it seems true that as the scale of operation of a farm increases, the management factor does not need to increase to the same extent as other factors. The exclusion of management may therefore be expected to lead to underestimation of returns to scale. Likewise, neglecting quality differences in other input factors will lead to biased estimates of scale returns. For instance, it seems most likely that the ignoring of quality differences in labor and land will generally lead to overestimation of the returns to capital, underestimation of the returns to labor and land, and underestimation of returns to scale.

If the research worker believes that constant returns to scale must prevail if all input factors are included, then it is logical to test the divergence between Σb_i and unity as an indication of the importance of omitted input factors. It may be argued that such an approach is

[24] See Jarrett, F. G. Estimation of resource productivities as illustrated by a survey of the lower Murray Valley dairying area. Australian Jour. of Stat., 1: 3-11; Heady, Earl O. Production functions from a random sample of farms. Jour. Farm Econ., 27: 995-97; and Griliches, Z. Specification bias in estimates of production functions. Jour. Farm Econ., 39: 8-20.

logically more attractive than the procedure of regarding Σb_i as an estimate of the true returns to scale.

Variance of Marginal Productivities

The formal derivation of marginal productivities has been noted in Chapter 3. To reiterate, the marginal productivity of X_i, the i-th input in equation 7.6 is given by equation 7.7.

$$\frac{\delta \hat{Y}}{\delta X_i} = b_i \hat{Y}/X_i \tag{7.7}$$

The most reliable, and perhaps the most useful, estimate of marginal productivity is obtained by taking X_i at its geometric mean; i.e., at the value where log X_i assumes its arithmetic mean. Also, $\hat{Y}$ should be the estimated level of output when each input is held at its geometric mean. For further implications of the analysis, it may be desirable to estimate marginal productivities when inputs are held at levels other than their geometric means. However, the qualifications mentioned earlier should then be kept in mind.

Given the estimate of marginal productivity from equation 7.7, the next step is to calculate its variance. Any discrepancy between the estimated marginal productivity and the existing opportunity cost of the resource may then be tested for statistical significance. Historically, the variance of the marginal productivity has generally been derived by assuming $\hat{Y}$ and X_i to be constants. The variance of $(b_i \hat{Y}/X_i)$ is then given by equation 7.8 where $\text{Var}(b_i)$ is calculated as in equations 4.29 and 4.30.

$$\text{Var}(b_i \hat{Y}/X_i) = (\hat{Y}/X_i)^2 \text{Var}(b_i) \tag{7.8}$$

However, the assumption that $\hat{Y}$ is a constant will seldom hold; $\hat{Y}$ will vary over alternative samples since it is based on b values which are only estimates of true parameters. Nonetheless, for estimates of the marginal productivities with the inputs at their geometric means, equation 7.8 leads to negligible errors in the variance estimate. These errors, however, increase rapidly as marginal productivity estimates are made further away from the geometric mean levels. The implications of nonconstancy of $\hat{Y}$ have been discussed by Carter and Hartley.[25] They derived a more accurate expression of the variance of the marginal productivity estimates. Their formula is given in equation 7.9 where $\text{Var}(\hat{Y})$ is as shown in equation 4.37, m is the vector $b_i (\log X_j - \overline{\log X_j}) + u$ where u is one if i = j and zero otherwise, and C is the matrix of equation 4.24 in terms of log X.

[25] Carter, H. O. and Hartley, H. O. A variance formula for marginal productivity estimates using the Cobb-Douglas function. Econometrica, 26: 306-13.

$$\text{(7.9)} \qquad \mathrm{Var}(b_i\hat{Y}/X_i) = (\hat{Y}/X_i)^2 \;\; (b_i{}^2/n + mCm') \;\; \mathrm{Var}(Y)$$

In equation 7.9, it is assumed that the logarithmic transformation used in the least squares estimation is to the base e. More normally, a transformation to the base 10 will be used, in which case the term (b_i^2 / n) in equation 7.9 must be multiplied by 2.3026^2.

RETURNS TO SCALE

Mention must now be made of scale returns in relation to the general production function depicted by equation 7.10 where it is assumed, for the moment, that no

$$\text{(7.10)} \qquad Y = f(X_1, X_2 \ldots , X_k)$$

relevant inputs have been excluded. Increasing, constant, or decreasing returns to scale prevail at a given point on the surface depicted by equation 7.10 depending on whether a small proportional increase in all inputs leads to a more than proportionate, proportionate, or less than proportionate increase in output.

Physical and Economic Returns to Scale

Practical statements about returns to scale can be made only if the entrepreneur can actually make proportionate changes in all the inputs considered. Where inputs are not under his control, he cannot make a proportionate change in every input factor. There is then little point in telling him that more profit can be attained by increasing or decreasing his scale of operation if uncontrollable factors are included in the recommendation. Needed is an estimate of the returns to scale or production elasticity for inputs under the control of the entrepreneur. The noncontrollable inputs should be held fixed at known levels. Knowledge of a crop farmer that he is operating under constant returns to scale relative to land, labor, capital, moisture, and sunlight has little utility. He cannot make decisions about the amount of weather. But he would have useful knowledge if he knew the range of weather under which decreasing or increasing returns to scale hold relative to land, labor, and various capital items. Distinction therefore may be made between "physical returns to scale," incorporating all inputs, and what might be called "economic returns to scale." The latter would include only those inputs under the control of the entrepreneur. Empirical difficulty arises, however, since exclusion of the uncontrollable inputs from the analysis tend to bias estimates of the economic returns to scale. Such biases will be discussed later.

Euler's Theorem

We now examine how returns to scale at a given point on the general production function indicated by equation 7.10 may be estimated. The procedure is based on the relationship found to prevail between the two sides, LHS and RHS, of equation 7.11 where $\hat{Y}$ is the estimated output with the inputs X_1 to X_k held at the levels used in evaluating the RHS.[26]

$$(7.11) \qquad \hat{Y} \gtrless X_1 \ \delta Y/\delta X_1 + X_2 \ \delta Y/\delta X_2 + \ldots + X_k \ \delta Y/\delta X_k$$

Thus $\hat{Y}$ is the estimated output at the point on the production surface defined by the specified levels of X_1 to X_k. On the RHS, $\delta Y/\delta X_i$ is the partial derivative of the estimate of equation 7.10 with respect to X_i. If, on evaluating equation 7.11, it is found that the LHS equals the RHS, then constant returns to scale prevail. Equivalently, by Euler's Theorem, equation 7.10 is homogeneous of degree one at the specified values of $X_1, X_2, \ldots, X_k$. If the LHS is greater than the RHS, then decreasing returns to scale prevail at the specified point. If the LHS is less than the RHS, increasing returns to scale prevail. The actual size of the returns to scale prevailing at the surface point under consideration is given by dividing the RHS of equation 7.11 by the LHS. Since equation 7.11 can only be evaluated in terms of the estimate of equation 7.10 derived via statistical procedures, statements about returns to scale must be regarded as probability statements and tested accordingly.

As a simple example of the use of Euler's Theorem, consider equation 7.12. It is a two-factor Cobb-Douglas function, $\hat{Y}$, a, b_1, and b_2 being empirical estimates.

$$(7.12) \qquad \hat{Y} = aX_1^{b_1} X_2^{b_2}$$

Corresponding to equation 7.11 we have

$$(7.13) \qquad \hat{Y} \gtrless X_1(b_1 a X_1^{b_1 - 1} X_2^{b_2}) + X_2(b_2 a X_1^{b_1} X_2^{b_2 - 1})$$

which reduces to

$$(7.14) \qquad 1 \lesseqgtr b_1 + b_2.$$

Thus if $(b_1 + b_2)$ is greater than unity, increasing returns to scale prevail, and so on. It is interesting to note that equation 7.14 is independent of X_1 and X_2. In other words, returns to scale in equation 7.12 do not depend on the level of the inputs at the point under examination. By extending equation 7.12 to a model with k input factors, it is easily seen,

[26] For the derivation of equation 7.11 and its justification relative to returns to scale, see Allen, R. G. D. Mathematical analysis for economists. Macmillan and Co., Ltd., London. 1956. Pp. 317-22. For an interesting discussion, see also Wicksell, K. Selected papers on economic theory. Harvard University Press, Cambridge. 1958. Pp. 93-100.

as noted in Chapter 3, that the Cobb-Douglas model implies an unchanging elasticity of production over the whole of the production surface.

Bias in Estimates of Returns to Scale

As mentioned previously, bias will occur in the estimate of physical returns to scale if any input factors are excluded. Moreover, the bias will not be constant over different sets of observations from the same population but will vary as the levels of the excluded inputs vary over the sample of observations to which the function is fitted. It is possible to estimate the bias in scale returns estimates resulting from the exclusion of variable factors which are under the entrepreneur's control but which cannot be included in the analysis. The procedure is outlined in equations 6.36 to 6.44. As an example, suppose equation 7.15 depicts the true production function with all factors included. Alternatively, it may be the production function with all excluded factors held at some constant level throughout the sample observations. Corresponding to equation 7.15, the function actually fitted might be equation 7.16, X_3 having been excluded.

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 \tag{7.15}$$

$$\hat{Y} = b_0 + b_1 X_1 + b_2 X_2 \tag{7.16}$$

Denote by e the true returns to scale, either physical or economic, corresponding to equation 7.15 and by $\hat{e}$ the returns to scale estimated from equation 7.16. By application of equation 7.11 we have:

$$e = (\beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3)/Y \tag{7.17}$$

$$\hat{e} = (b_1 X_1 + b_2 X_2)/\hat{Y} \tag{7.18}$$

The expected bias in $\hat{e}$ is given by equation 7.19.

$$E(\hat{e} - e) = E[(b_1 X_1 + b_2 X_2)/\hat{Y} - (\beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3)/Y] \tag{7.19}$$

Assuming that Y equals $\hat{Y}$ and making use of equation 6.43, equation 7.19 simplifies to equation 7.20 where the p's are the auxiliary regression coefficients of equation 6.42.

$$E(\hat{e} - e) = \beta_3(p_{13} X_1 + p_{23} X_2 - X_3)/Y \tag{7.20}$$

Hence, relative to equation 7.15 we can say that the size of the bias in $\hat{e}$ resulting from the omission of X_3 varies as the level of X_1, X_2, and X_3 vary. Also, for the most likely case with β_3 positive and absolutely large relative to p_{13} and p_{23}, underestimation is the type of bias to be expected.

TECHNOLOGICAL CHANGE

Farm input-output data, either time series or cross-sectional, is generally confounded by technological change. Roughly speaking, the production surface relevant to an individual firm tends to rise over time; it becomes possible to produce the same quantity of a given product with less effort, or to produce a greater quantity of a given product with the same effort. These effects may occur for one or both of two reasons. First, new types of inputs may be added to the input array used in the production process and old types may be discarded. An example would be addition of a welding plant to the machinery services used. More obvious, but rather more subtle in its implications, would be a shift from animal to tractor power. The second source of technological change is improvement in the quality of the inputs used. For instance, the quality of the labor input on farms may improve over time under the influence of higher educational standards and an increasing inventory of experience. Ideally, of course, the use of improved quality inputs might be regarded as the substitution of new inputs for old. However, it is convenient to consider quality changes and distinctive changes in the input array as separate phenomena. The justification is that quality changes are not easily discernible in the short run. The use of inputs of improved quality may often occur without any conscious decision on the part of the entrepreneur. In contrast, changes in the input categories used are quite easily discerned. Moreover, they generally occur as the result of a definite decision by the entrepreneur.

Technological change resulting from both the use of new inputs and improvements in input quality is reflected directly in time series data. Over time, given sufficient data, it is easy to pinpoint the addition of new types of inputs to the production process. To delimit changes in input quality is a more difficult task unless supplementary information is available. Such supplementary information might come from semiexperimental studies of individual inputs. For instance, improvements in particular types of machinery may be gauged from engineering reports on the performance of the machinery. In cross-sectional data, technological advance due to changes in the input array is only reflected indirectly by way of the differential rates of adoption prevailing among farms. Cross-sectional surveys usually disclose some firms operating with an "old" input array, some with a "new" array, and some in a transitional stage. Input quality differences also occur in cross-sectional data. However, such differences largely represent rather fixed discrepancies in quality inherent in the fixed resources of the firms. Probably only to a minor extent do cross-sectional quality differences reflect the positioning of firms at different stages in the sequence of technological advance. Still, exceptions would occur if the cross-sectional sample related to a population whose elements were not relatively contiguous.

So far as the use of new input types is concerned, it would seem possible to take account of such technological change by extending the input

framework of the production function analysis. The problem of technological change due to improvements in input quality is, however, rather intractable in an operational sense. We now consider shifts in the production function for a given product arising from distinctive changes in the set of inputs used.

New Input Types

When new types of inputs are added to the production process, some of the input types previously used may be discarded. Thus, the old and new processes might correspond to equations 7.21 and 7.22, respectively.

$$Y = f(X_1, X_2, \ldots, X_k) \tag{7.21}$$

$$Y = f(X_1, X_2, \ldots, X_{k-m}, X_{k+1}, \ldots, X_{k+h}) \tag{7.22}$$

As depicted in equation 7.22, the new production process involves the exclusion of m inputs, X_{k-m+1} to X_k, and the addition of h new inputs, X_{k+1} to X_{k+h}. However, the old and new production processes corresponding to equations 7.21 and 7.22 may be equally well depicted by the single function shown in equation 7.23. Relative to the old production process, X_{k+1} to X_{k+h} are always at zero level in equation 7.23. Relative to the new production process, X_{k-m+1} to X_k are always at zero level in equation 7.23.

$$Y = f(X_1, X_2, \ldots, X_{k+h}) \tag{7.23}$$

For transitional stages between the old and new techniques of production, most or all of the variables in equation 7.23 would be at nonzero levels. The fact that some inputs are zero at different times does not affect the estimation of equation 7.23 by normal least squares procedures. The only proviso, and it is a major one, is that sufficient data be available for computational purposes. It might be noted that the subsuming of the old and new production functions under the broader framework of equation 7.23 is related to the discussion of hybrid production functions in Chapters 5 and 6. To attempt to fit either equation 7.21 or 7.22 to a set of data encompassing both the old and new processes would result in the fitting of a hybrid function. But to fit equation 7.23 would not lead to hybridity, since equation 7.23 recognizes that X_{k+1} to X_{k+h} are zero in equation 7.21 while X_{k-m+1} to X_k are zero in equation 7.22.

Changes in Input Quality

Reference has already been made in Chapter 6 to the problem of

variation in input quality, at least insofar as it relates to aggregation. Here we are not concerned with quality variations within a given observation but with variations in quality between observations taken over time. These changes in quality may be either positive or negative. For convenience, we speak of the positive case — an improvement in quality. If quality changes are not taken into account, the effects on the derived estimates are equivalent to the biases resulting from errors of observation in the data. The situation is perhaps best illustrated by a simple numerical example.

Suppose that an input is such that of every 10 units used in the production process, only 6 units are actually utilized; the remaining 4 units lost due to frictions beyond the control of the entrepreneur. Further suppose the quality of the input is improved so a 10 per cent increase in utilization results. Then, for every 10 usable units, 6.6 units would now be actually utilized and only 3.4 units lost. For the purpose of estimating a production function, the relevant data for "before" and "after" the quality change are the figures of 6.0 and 6.6 units, respectively. To use the figure of 10 units for both situations would introduce a sizeable error of observation. While the figure of 10 would serve as a satisfactory and legitimate index so long as quality remained unchanged, it becomes an inaccurate index as soon as quality alters significantly. For the two situations discussed, the two readings of 10 units are not comparable. Corresponding to the figures of 6.0 and 6.6, the error-free index values would be 10 and 11, respectively. The practical difficulty is that only the raw data, corresponding to observations of 10 for both cases, are generally available. If data on quality changes were available, it would be possible to index all observations on a particular input in terms of a standardized unit as outlined in relation to equations 6.46 and 6.47. The production function with all inputs evaluated in standardized units might then be depicted as in equation 7.24.

$$Y = F(X_1, X_2, \ldots, X_k; q_1, q_2, \ldots, q_k) \tag{7.24}$$

In equation 7.24, q_{ij} would be a weight used to convert the j-th observation on X_i to units of a standard quality. However, due to lack of knowledge about the q_{ij} values, evaluation of equation 7.24 is generally infeasible in an empirical context. Still, an alternative procedure exists by which approximate account may be taken of such technological change.[27] If quality changes are regarded as equivalent to the introduction of new inputs as they should in a strict theoretical sense, then a new production function, such as equation 7.22 instead of equation 7.21, becomes relevant. The situation is exemplified in Figure 7.2. The

[27] It is worth noting that changes in input quality that occur without any change in the basic input array, imply a movement along the production function estimated in terms of standardized input units. So long as quality changes are accounted for by some standardization procedure, the position of the surface will only be altered when new inputs enter the production process. The relevant definition of a new input is that it be one that cannot be standardized in terms of any of the inputs already in use. A new input must be of a different generic nature to any of the inputs already in use.

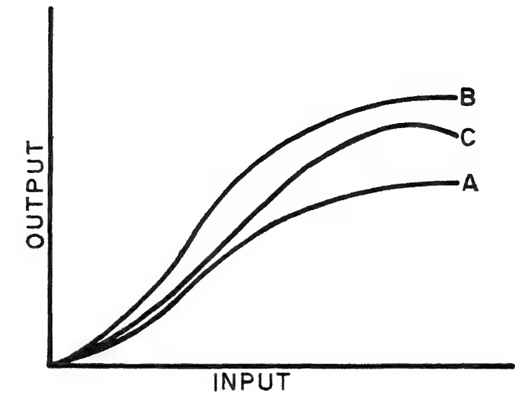

Figure 7.2. Shifts in the production function due to technological change.

curve A corresponds to the production function of equation 7.25 with only X_1 allowed to vary. The B curve also corresponds to equation 7.25, but implies a proportionate increase in the quality of each of the inputs. The input of X_1 is measured along the horizontal axis in unstandardized units, no adjustment being made for the improvement in quality. Strictly, the B curve corresponds to equation 7.26 where the prime on the X's denotes an improvement in quality relative to equation 7.25.

$$Y = f(X_1 \mid X_2, \ldots, X_k) \tag{7.25}$$

$$Y = f(X_{11} \mid X_{21}, \ldots, X_{k1}) \tag{7.26}$$

Because of the assumption of a proportionate quality improvement in each input, the B curve corresponds to the A curve with every ordinate shifted upward by the same proportionate amount. Thus the correspondence between the A and B curves, or between equations 7.25 and 7.26 might be expressed as in equation 7.27 where z is the shift factor. Since curve B lies above A, z is greater than unity. (In drawing the B curve of Figure 7.2, z was taken as equal to 1.5.)

$$f(X_{11} \mid X_{21}, \ldots, X_{k1}) = z f(X_1 \mid X_2, \ldots, X_k) \tag{7.27}$$

As equation 7.27 implies, the shift from function A to function B is such that the marginal rates of substitution between inputs remains unaltered. Technological change of this type has been termed "neutral." It is, of course, a special case. Still, it may serve as a working approximation in empirical studies involving aggregative input categories. If quality improvements occur in only some of the inputs, or if all improve but in different proportions, then a curve such as C might result.

It is also possible to look at distinctive changes in the input array

in terms of Figure 7.2. For instance, the curve A might correspond to equation 7.25 and the curve C to equation 7.28 which involves an additional h input factors over and above those used in equation 7.25.

$$(7.28) \qquad Y = f(X_1 \mid X_2, \ldots, X_{k+h})$$

Quantification of Technological Change

As indicated in discussing Figure 7.2, technological change resulting from either the use of new inputs or quality improvement in old inputs may be regarded — at least approximately — as causing an upward shift in the original production surface. Such a shift may be formally incorporated in the production function. For instance, consider equation 7.29 in which the inputs X_1 to X_k are assumed to be mildly aggregative rather than micro-inputs of highly specialized nature. If the inputs were not aggregated to some extent, a formulation akin to that of equations 7.23 or 7.24 would be pertinent. The variable, t for time, is inserted in equation 7.29 to allow for the influence of technological change.

$$(7.29) \qquad Y = F(X_1, X_2, \ldots, X_k; t)$$

Using such a framework, Solow has suggested (and illustrated empirically) how an approximate assessment of the extent of technical change in time series data may be made.[28] The measure that he suggests is $\Delta F/F$ where this statistic is calculated by way of equation 7.30.

$$(7.30) \qquad \Delta F/F = \Delta Y/Y - \sum_{i=1}^{k} w_i \; \Delta X_i/X_i$$

where

$$(7.31) \qquad \Delta Y = Y_t - Y_{t-1}; \; \Delta X_i = X_{i,t} - X_{i,t-1}$$

and w_i is the relative share of X_i in the total product. The ideal derivation of w_i would be as in equation 7.32 which, however, presupposes knowledge of equation 7.29, the function to be estimated.

$$(7.32) \qquad w_i = \frac{\delta Y}{\delta X_i} \frac{X_i}{Y}$$

An estimate of w_i might be obtained by assuming, perhaps heroically,

[28] Solow, R. M. Technical change and the aggregate production function. Review of Economics and Statistics, 39: 312-20. See also Leser, C. Production functions and British coal mining. Econometrica, 23: 442-6; and Aukrust, O. Investment and economic growth. Productivity Meas. Rev. 16: 35-53.

that the farm-firm is operating at or near its equilibrium position. At equilibrium, the outlay on each factor corresponds to its share in the total product. Hence, denoting the price of Y and X_i by p_y and p_i, respectively, w_i might be approximated via equation 7.33.

$$w_i = p_i X_i / p_y Y \tag{7.33}$$

In the special case of neutral technological change, equation 7.29 takes the form

$$Y = Z(t)\, f(X_1, X_2, \ldots, X_k) \tag{7.34}$$

where the factor Z(t) corresponds to z of equation 7.27. Z(t) denotes the value of z for the set of observations for the t-th time period. $\Delta F/F$ is then equal to $\Delta Z/Z$ and the latter, calculated by way of equation 7.30, may be used to estimate Z(t) as in equation 7.35. For convenience, $Z(t = 1)$ may be set equal to unity.

$$Z(t + 1) = Z(t)\ (1 + \frac{\Delta Z(t)}{Z(t)}) \tag{7.35}$$

Dividing Y_t by Z(t), an adjusted series of output data is obtained. These data may be used to estimate the basic or original production function denoted by equation 7.36.

$$Y = f(X_1, X_2, \ldots, X_k) \tag{7.36}$$

It is this function which prevails at $t = 1$. The shift factor Z(t) specifies the upward shift in the production surface at time t relative to the original position of the surface at $t = 1$. Thus if curve A of Figure 7.2 is specified as the original production surface, curve B corresponds to a neutral shift with Z equal to 1.5.

Whether or not neutral technological change has occurred can be assessed empirically, at least in an approximate way. If the technical change is neutral, the $\Delta F/F$ values will be independent of the X_i values. A plot of $\Delta F/F$ against X_i should suffice to ascertain whether the technical change can be regarded as neutral. Unless a high degree of confidence can be placed in the w_i values used in calculating $\Delta F/F$, the use of more abstruse statistical tests would seem unwarranted.

PROBLEMS IN LIVESTOCK STUDIES

As noted above, there is no dearth of problems in the estimation and interpretation of farm-firm production functions. In large measure, these difficulties arise because the researcher has no *ex ante* control over the generation of the firms' inputs. The farm-firm research worker can only exercise *ex post* discretion over the data to be

analyzed. For instance, purposive sampling procedures may be used. Animal or crop input-output relationships, on the other hand, are usually studied under experimental conditions; the research worker is able to exercise a fair measure of control over the genesis of the data. Still, experimental control with animals is not absolute. They do not behave in a strictly passive way when faced with an array of possible inputs. In consequence, some special difficulties arise in the estimation of livestock production functions. Chief among these are the problems of free versus controlled feeding, physiological restraints on the feed mix vector, changes in the nutritional requirements over time, and forage intake under grazing conditions. Note also needs to be taken of the important distinction between the animal itself as product (as in meat production) and the situation where the product is a flow over time from the animal (as in milk, egg, wool, and power production).

As well, there are some other aspects of livestock input-output analysis that should be mentioned.[29] First, as experimental units, livestock are short-lived and expensive. Research resources being limited, the researcher has to seek a compromise between (a) the number of animals to be penned together as a single experimental unit within a replicate, (b) the size of a replicate in terms of treatments, and (c) the number of replicates to be studied. It is important that sufficient treatments be examined to provide an estimate of the relevant sector of the production surface. Still, excessive experimental units (i.e., pens of animals) should not be used to provide more information than is necessary. These additional experimental units might be better utilized to increase the number of animals within each pen; or to provide a sufficient degree of replication to enable estimation of experimental error. As outlined in Chapter 5, lack of fit analysis may then be carried out. Another general problem is that of autocorrelation in successive observations on the same animal. This difficulty could be circumvented if sufficient research resources were available to enable each treatment to be observed on a different animal or pen of animals; or to space the observations on each animal sufficiently far apart in time to minimize autocorrelation. Even so, such approaches may not be wholly desirable in terms of the finesse of the experiment itself: the effects of variations among animals may be worse than the autocorrelation effects. As noted in Chapter 4, autocorrelation presents problems mostly for probability statements and fiducial limits, rather than in predictions of production and substitution rates. In other words, it mainly raises problems in making tests of significance. Experimental error due to variations between animals may, on the other hand, lead to false estimates of the true production function. Of course, as usually conducted, livestock experiments will contain some errors from both these sources. An indication of how they may be assessed has already been given in chapters 4 and 5.

[29] For a general reference to livestock input-output considerations, see Hoglund, C. R. *et al.* (eds.) Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames. 1959.

Feed Mix Restraints

The treatments applied, in sequence, to an individual animal generally need to be relatively consistent in terms of their feed mix make-up. This point can be illustrated by means of Figure 7.3. It represents a bird's-eye view of the feed plane for the overly simple case of two feed inputs, say harvested forage and grain. The lines R_1, R_2, R_3, and R_4 are ration lines; along each such line the forage-grain ratio remains constant. Suppose the points t_{ij} are taken as defining the desired or observed treatments, the treatment being the pounds of grain and forage corresponding to the point t_{ij}. The points t_{ij} have been located purely for expositional convenience. They are not meant to represent an ideal experimental layout. Each treatment consists of the cumulative total of grain and forage utilized by the animal up until the time of the observation. For the observations to be comparable, each of the animals should be as similar as possible in terms of age, weight, nutritional background, breed, etc., at the start of the experiment.

Since the treatments of Figure 7.3 are cumulative feed totals, there would be no difficulty in observing a sequence such as (t_{31}, t_{32}, t_{33}) on an individual animal. But the sequence (t_{11}, t_{32}, t_{43}) would be clearly impossible on mechanical grounds since t_{32} implies a larger forage intake than t_{43}. More important, the sequence of treatments followed by an individual animal should not diverge too rapidly from a ration line. If the proportions of roughage and concentrate in the ration are suddenly altered markedly, the physiological processes of the animal may be upset; especially if the change is in the direction of more concentrate. While such effects are not likely to be severe in the case of smaller

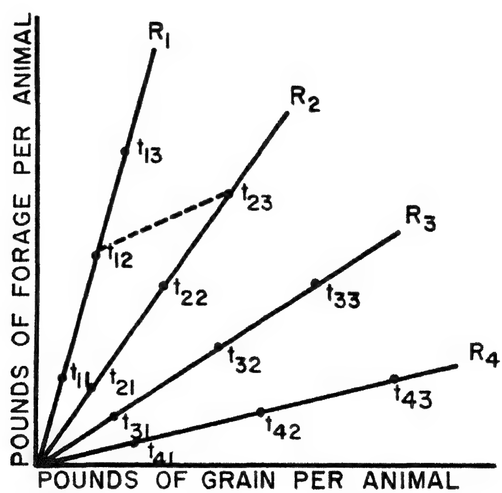

Figure 7.3. Schematic representation of ration lines and treatment locations in the feed plane for a dry-lot experiment.

animals such as poultry, the effects may be severe for larger-bodied animals such as cattle and hogs.[30] With such animals a negative response, even death, may result. Thus, the sequence (t_{11}, t_{12}, t_{23}) may be feasible by following a feed path such as dotted between t_{12} and t_{23}. In contrast, the sequence (t_{21}, t_{43}) is rather infeasible for a large-bodied animal. It would imply immediate change to a ration of nearly 100 per cent grain after reaching t_{21}. While the animal might survive such a swift shift in the feed mix, it would probably lose weight for a period.[31] Output increments would be negative. In contrast, an animal following the path along R_4 to t_{43} would show positive output increments. Total production at t_{43} would most likely be greatest if t_{43} were reached by following the R_4 line rather than by following the path (t_{11}, t_{43}). Hence, the latter is an inefficient feed path; by moving along R_4 a greater output can be obtained from the same quantity of input. Thus the production surface that is probably most efficient in a physical sense and most relevant to farmer decisions is best determined by moving each animal along a ration line. Interpretations of this production surface in terms of optimal movements over the feed plane as feed and product price change should be qualified. We need to remember that recommendations for large-bodied animals, implying a substantial shift from the current feed mix, may lead to a point on another surface; most likely to a point on a lower surface.[32] Recent research has shown, however, that if the adjustment is not too severe in magnitude, the animal may respond after a short period of adjustment, as if it had always been on the same ration. In other words, the effect of the previous ration may not last long.

These points can be illustrated by Figure 7.4 for a two-input situation. Suppose an animal is fed along each of the ration lines R_1 and R_2. Each animal will operate on a production surface relative to the ration line it is following. Because of its physiological adjustment to this ration, sudden and large movements of the animal's feed mix off the ration line may immediately lead to less efficient use of the feed input than by an animal continuously on the new ration line. Hence, product contours indicating the effect of sudden and large movements away from the ration line may be drawn for each animal following a particular ration line. For the animal on ration line R_1, two such isoquants are shown as M_{11} and M_{12}. Corresponding isoquants for an animal on R_2 are M_{21} and M_{22}. M_{11} and M_{21} each refer to the same amount of product. Likewise, M_{12} and M_{22} refer to an identical level of output but one which is

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[30] See Heady, E. O. *et al.* Least-cost rations and optimum marketing weights for turkeys. Iowa Agr. Exp. Sta. Res. Bul. 443. Ames. 1956. P. 870; Jacobson, N. L. Problems in designing feeding experiments from a nutritional standpoint, in Hoglund, C. R. *et al.* (eds.) Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames. 1959. Pp. 206-12.

[31] Relative to cattle, see Southcott, W. H. Drought feeding of beef cattle. Australian Vet. Jour., 35: 126-29. For sheep, see Briggs, P. K. *et al.* The performance of adult metino wethers fed weekly on all-grain rations. Australian Vet. Jour., 32: 299-304.

[32] For an indirect indication of this point, see Heady, Earl O. A production function and marginal rates of substitution in the utilization of feed resources by dairy cows. Jour. Farm Econ., 33: 485-98.

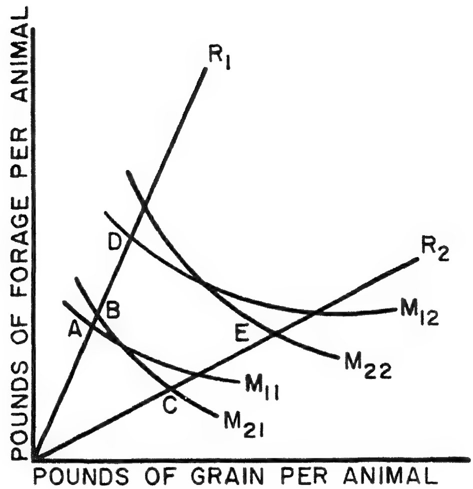

Figure 7.4. Schematic representation in the feed plane of product contours around ration lines.

larger than the level corresponding to M_{11} and M_{21}. Comparison of points such as A and B shows each animal to be more efficient along its current ration line than on another ration line to which it might suddenly be shifted. The same output on the "new" ration line may be obtained with less input from an animal always following the "new" ration line.

In experiments to derive a response surface for production decisions, care must be taken to ensure that there are sufficient treatment responses. In recognition of this fact, livestock production surfaces usually are estimated by studying response along an array of ration lines covering the surface portion of interest. The isoquants derived from most livestock experiments therefore correspond to "isoquants" obtained by connecting up series of points such as A and C, or D and E. The procedure is reasonable for large-bodied animals if it is remembered that such isoquants do not always relate to production possibilities from a single animal once it has started on its feeding program, particularly if wide shifts are to be made from a current ration line.

Often it is necessary to move gradually to a desired ration line, especially by large-bodied ruminants for feed mixes high in proportion of grain or concentrates. Thus in Figure 7.5, responses along the ration line R_4 may be quite different, depending on whether it is approached directly (i.e., always using the feed mix ratio implied by R_4) or as it is reached gradually by, say, the dotted path. The use of such an approach gives an animal the chance to gradually adapt its physiology to the extreme feed mix implied by R_4. Of course, the use of the dotted path implies that the surface area lying between this path and the grain axis

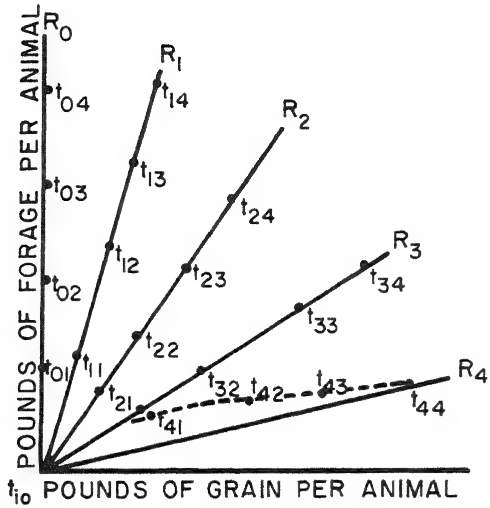

Figure 7.5. Schematic representation in the feed plane of a free feeding dry-lot livestock experiment.

cannot be estimated. So long as this surface section is not relevant to real-world decisions, as might be the case in meat production, the use of the indirect path is reasonable. However, in circumstances where points on the surface below the dotted path are relevant, as may be the case for drought feeding conditions, the use of a gradual approach to R_4 is not justifiable.

Treatment Characterization

Ideally, the treatments used in livestock studies should be specified in terms of (a) the actual feed input, (b) the time over which the input acts, (c) the animal receiving this input, (d) the previous nutritional history of the animal, and (e) the nonnutritional environment of the animal. Obviously, the treatments cannot be specified so exactly. The researcher mainly can attempt to control aspects (c), (d), and (e) so as to enable valid analysis over responses for various feed inputs and time periods. There remains a problem in characterizing the feed inputs used. Often no more can be done than to specify the inputs in such generic forms as alfalfa hay, sorghum silage, corn grain, etc., with an attempt being made to ensure that the feed within each such category is of some specified uniform quality. Such a procedure has an obvious disadvantage from the point of view of making recommendations to real-world decision makers. There are too many different qualities and types of feeds for a production surface to be evaluated for each of the possible combinations among feed types and livestock. It would be far

better if feed inputs could be characterized in terms of a basic standard referring to the chemical and physical character of feed.[33] As yet, such standards are not available. If they were, experiments with a small number of feed types might be used as the basis for developing production surfaces of relevance to a wide range of feeds. The currently available methods of expressing nutritive values (such as TDN, net energy, digestible protein, etc.) are not suited for use as basic standards in the estimation of livestock production surfaces. They generally assume that a given feed quantity has a constant nutritive value independent of both the quantity of feed utilized and of the character of the feed mix, and also of the condition of the animal.[34] In other words, they assume constant marginal rates of substitution between feed types and constant marginal productivities of feed types. Such a linear framework of comparison between and within feeds assumes away the possibility of a production surface showing increasing or diminishing returns.

Free Versus Controlled Feeding

Compared to such production units as machines, soils, and plants, animals are relatively "free" agents. They exercise choice in obvious ways. In consequence, it is more difficult for the researcher to exercise control over the treatments used in livestock experiments. This difficulty is enhanced in production function experiments since they require a large number of distinct treatments.

Two extreme possibilities face the researcher designing a drylot livestock experiment. He may either (a) control the quantity of feed to be supplied in a given time period or (b) he may specify the time interval but not the quantity of feed to be consumed. Intermediate positions between these extremes are possible: Some elements of the ration may be supplied ad libitum while others may be controlled; or, within one experiment, identical feed mixes might be fed under both controlled and free-feeding conditions. Whichever approach is used, certain elementary precautions need to be taken. So far as possible, the various feed types in the over-all ration, or in the segregated parts of the ration under a mixture of controlled and free feeding should be mixed. Otherwise, an animal may exercise its preference strongly, gorging itself on one type of feed and ignoring others. Again, the supply of a given feed type should be as homogeneous as possible in terms of quality and other characteristics during the experiment. In the initial stages of an experiment, allowance should also be made for the adjustment of the

[33] See Lucas, H. L. Experimental designs and analyses for feeding efficiency trials with dairy cattle, in Hoglund, C. R. *et al.* (eds.) Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames. 1959. Pp. 189-92.

[34] Some biological researchers have attempted to remedy these defects. See Blaxter, K. L. and Wainman, F. W. Some observations on the digestibility of food by sheep and on related problems. British Jour. of Nutrition, 10: 69-91; also Brown, W. G. and Arscott, G. H. Animal production functions and optimum ration specifications. Jour. Farm Econ., 42: 69-78.

animals' tastes and physiologies to the feeds and feed mix being studied. Precautions must be taken against (or allowances made for) discrepancies between the quantity of feed placed before an animal and the amount actually consumed. For instance, some of the feed supply may be trampled underfoot or made unusable in other ways. For the biological investigator, the above comments are quite mundane. Still, they are features of the experimental situation that must be recognized by the cooperating economist.

With free or ad libitum feeding, an animal is always faced with a quantity of feed in excess of the amount it can immediately consume. An animal will tend to utilize feed at the limit of its stomach capacity, provided that the ration is not unattractive or distasteful. In either case, we cannot specify in advance the feed quantities that will be utilized in a given time period. We can only specify the proportion of different feeds in the ration placed before the animal and the times at which input and output will be measured. The treatments observed with free feeding are largely determined by the animal. The situation may be illustrated by way of Figure 7.5. Suppose the investigator decides to evaluate the beef production surface by studying the response of cattle along the ration lines R_0, R_1, R_2, R_3, and R_4 with each of these feed mixes being fed ad libitum. (Because of the high grain intake implied by R_4, it may be necessary to approach R_4 indirectly. For instance, R_4 might be eventually attained by following the dotted feed path.) If observations of the response along each ration line are taken at equal time intervals, say weekly or monthly, these observations might correspond to the positions marked t_{ij} ($i = 0, 1, 2, 3, 4$; $j = 0, 1, 2, 3, 4$) in the feed plane. The fact that the observations are not spaced equally along each ration line indicates that the feed consumption varies from one time period to another. Each observed point t_{ij} specifies, through its forage-grain co-ordinates, a treatment in terms of the cumulative quantity of feed fed up to the time of the observation. As a consequence of free feeding, these treatments are only definable *ex post.* Before the event, the researcher could not specify their locations in the feed plane. This situation may now be contrasted with that under controlled feeding conditions, assuming handfeeding is the normal mode of production.

With controlled feeding, the researcher decides how much of a given feed mix is to be made available to the animals in a given time period. Thus, within the specified time periods, it might be decided to feed quantities determined by the intersection of ration lines and a contour defining, for example, total feed poundage or energy intake per animal. With such a system, the observational points or treatments t_{i0}, t_{i1}, ..., t_{i4} of Figure 7.5 could be purposively spaced over the i-th ration line. An approach of this nature would, of course, have to recognize any restraints imposed by the animals' stomach capacity or physiology.

The choice between free and controlled feeding procedures depends in large part on the type of output being studied. If the livestock product is a flow (such as milk, eggs, or power) which is harvested in a semicontinuous manner over time, then controlled feeding must be

strongly recommended. As an important example, consider the case of milk production where daily collection of the product is the general rule.[35] If dairy cows are free fed they will consume feed quantities that are determined by their stomach capacities. Such quantities are depicted by the points $t_1, t_2, \ldots, t_6$ of Figure 7.6. They lie on the stomach line SL. With such an experimental design the observed treatments will not allow estimation of the milk production surface. The treatments would be the same, day by day. Apart from variations over the lactation period and uncontrolled influences, response would only relate to the surface points lying above the points $t_1, t_2, \ldots, t_6$. The only relationships these observations can reveal are (a) the location in the feed plane of the stomach line and (b) maximum milk obtainable from each of the six treatments. The treatments would give no indication of marginal rates of substitution between feed components, nor of marginal yields either along a particular ration line or for a particular type of feed. In short, the experiment of Figure 7.6 does not allow estimation of the production surface. It merely locates a line across the surface. For estimation of the whole surface relevant to either a single day's production or production over the entire lactation period, it would be necessary to have observations scattered purposively throughout the area lying below the stomach line SL of Figure 7.6.

When the livestock product is a stock resource, such as meat harvested upon slaughter of the animal, the analytical limitations imposed

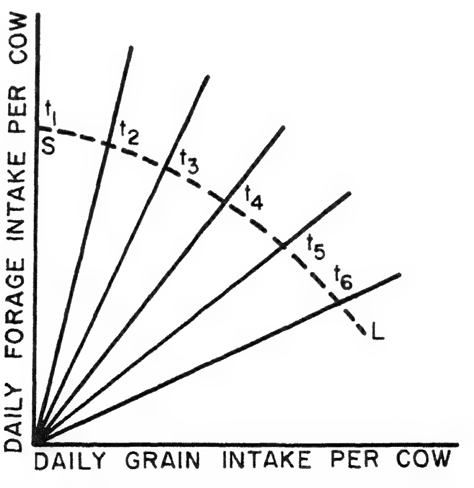

Figure 7.6. Schematic representation in the feed plane of the effect of free feeding in a milk production experiment.

[35] See Heady, E. O. Problems in designing dairy feeding experiments for economic analysis, in Hoglund, C. R. *et al.* Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames. 1959. Pp. 193-205. Also see Orton, F. J. The economy of feed input in milk production. The Farm Economist, 9: 11-25.

by free feeding to stomach capacity are not very severe. They tend to be cancelled out by the longer term nature of the production process. In such instances, the observations required to estimate marginal productivities and marginal rates of substitution relate not so much to daily intake and daily production but to the cumulative quantity of inputs utilized and output produced at each stage of the production process. These major stages in the production process may be periods of weeks or months, depending on the type of animal. In general, for meat production under handfeeding, the farmer's main problem is to know the least-cost ration at different weights of the animal. He may then feed at over-all least-cost to the final weight expected to give the highest market grade and price. Such a procedure is necessary because, in contrast to crop production where the optimal fertilizer input can be applied in one dose, the meat producer has to supply inputs continuously over time to a production unit whose capabilities change over the production period. For instance, changes occur in an animal's nutritional requirements between the growth and fattening stages of the life cycle. As a consequence, the marginal rates of substitution between various feeds are not constant over the production period (a fact which should be taken into account in deciding on the appropriate algebraic form of the production function). Ideally, a least-cost ration should be determined for each weight to marketing. For practical reasons, however, the farmer could not be expected to change rations more than a few times, say once for each major stage in the animals progression to market weight. The researcher, therefore, needs to ascertain rations which average least-cost over selected weight intervals, preferably those commonly used by livestock producers. For instance, with hand-fed hogs, separate production functions might be fitted for the weight intervals: weaning to 75 pounds, 75 to 150 pounds, and 150 to 250 pounds. Rations averaging least-cost over each of these production stages could then be determined. Free feeding experimental procedures would be appropriate for such analyses.

If the date of marketing was an important factor, account would have to be taken of time in the production function, and hence in the experimental design. For instance, a farmer may purchase or breed animals over a range of months, but desire to market them all in a single month. For such situations, experiments would need to be designed with feeding along each ration line controlled at various levels per unit of time. The treatments would be the cumulative weight of each feed mix constituent and the cumulative time from the start of the experiment up to the time of each observation. One set of free feeding treatments would need to be included for completeness. Given a certain amount of time available for attaining the desired market weight, the economically optimum ration for reaching this weight within the available time could be ascertained. As before, recognition would have to be given to restrictions imposed by stomach capacity and physiological factors.

Similar experimental arrangements would be most appropriate for production function studies of semiflow products produced under

handfeeding conditions. Consider the case of wool which is customarily harvested on an annual basis. With only ad libitum feeding treatments, no account could be taken of variations in the extent to which sheep channel various feed types into wool production versus fat production at alternative levels of feed availability. Such considerations are important because optimal handfeeding under certain price regimes might be at a low level of daily feed intake relative to what the animal would consume if free fed.

Other considerations may also necessitate the use of controlled feeding in experiments for semiflow or stock products. Such would be the case were it desired to examine the relationships over time between the various elements of an animal's growth and maintenance processes. For instance, the proportion of the feed mix going to bone, fat, and meat production may vary with the constitution and quantity of the ration received per unit of time.[36] Controlled feeding would be necessary to determine such relationships and their associated marginal quantities. It is also pertinent to note that such experiments would tend to be extremely demanding of research resources. Unless relatively accurate means were available to measure an animal's constituents, without its slaughter, each animal would need to be killed as it reached the desired treatment. Moreover, as noted in Chapter 4, the most appropriate model by which to analyze an animal's growth and maintenance interrelationships may be one using simultaneous equations, also an expensive procedure. Nonetheless, all of these considerations are important; both from the standpoint of basic science in predicting the physiological relationships in livestock nutrition, and for the making of economic decisions.

For flow type products harvested semicontinuously, experiments with some constituents of the ration fed ad libitum and others controlled may suffer from the disadvantage outlined in relation to Figure 7.6. Free feeding will lead to treatment locations along the stomach line. A mixture of free and controlled feed components in an experiment relating to a nonflow product may also tend to be disadvantageous. This is the case where animals gorge themselves on ad libitum feed components and other treatments are not scattered satisfactorily across the feed plane.

Production Functions for Grazing

So far, in discussing livestock problems we have only considered production relations for full handfeeding. But, in actuality, few large-bodied animals are handfed all the year around. This is especially true for the tropical and temperate regions of the globe. The main types of livestock in these regions, cattle and sheep, obtain the bulk of their feed by grazing. And even where climatic conditions necessitate full handfeeding in the fall and winter, grazing is important in the spring and summer.

[36] See Heady, E. O. Economics of agricultural production and resource use. Prentice-Hall, Inc., New York. 1952. Pp. 69-72.

For the production economist, the essential features of pasture are that it is produced (a) at a low cost, (b) in a marked seasonal pattern, and (c) in amounts that are largely beyond the farmer's control in the short run of a single climatic season. In other words, given some prior fertilizer and utilization program, the quantity of pasture feed at a farmer's disposal is largely fixed by climatic factors beyond his control. But, relative to this rather fixed bulk of pasture, there is one important livestock production input that is under the farmer's control. To wit: the animals themselves. It is the intensity of stocking, set by the farmer, that determines the grazing input and the output per animal from a given bulk of pasture.[37] The production function relevant to a farmer's utilization of pasture is therefore broadly of the type depicted in equation 7.37 where, assuming homogeneity within each resource category, Y is the livestock product (e.g., meat, milk or wool); P is the amount of pasture; A is the number of animals; and S is the quantity of supplementary feed used by the animals. For convenience, we first assume equation 7.37 relates to the period from start to finish of a single cycle of the animal in pasture.

$$Y = f(P, A, S) \tag{7.37}$$

As required empirically, each of the input categories of equation 7.37 might be expanded into a number of variables. The quantities of different types of pasture and other grazing crops could be considered as distinct inputs; likewise, different types of animals producing the same product could be segregated. As an example, ewes and wethers could be inserted separately in a wool production function. Again, the function might be expanded to include other inputs; for instance, fertilizer and irrigation water entering the production process via the pasture. Alternatively, the production function for pasture might be substituted in equation 7.37 for the variable P, production of pasture being an intermediate step in the grazing production process. Such a function might be of the general form depicted in equation 7.38 where A and S are as in equation 7.37, L is the area of pasture land, F is the quantity of fertilizer applied to the land area, W is the quantity of irrigation water used, and C is an index of the favorability of the climate to pasture growth during the pasture cycle.

$$Y = f(A, S, L, F, W, C) \tag{7.38}$$

Again, if desired and feasible, a function such as equation 7.38 might be further expanded. Ignoring such possibilities, suppose equation 7.38 is to be fitted. The optimal resource allocation for any particular value of the climatic factor could then be decided by marginal analysis, in the usual way and with all the usual provisos. Similarly, for any fixed area

[37] See Candler, W. V. Wool and wethers. Farm Management Report No. 1. University of New England, Armidale. June, 1958.

of land, the optimum number of animals, and hence the best stocking rate, could be decided. In such calculations the cost of animals must be taken inclusive of their "running" costs involving labor, fencing, and other services that may have to be varied with the stocking rate but are not included in equation 7.38 as direct inputs.

Assuming a satisfactory index of climatic favorability to be available, an experiment to estimate equation 7.38 would have to be conducted over a sufficient number of years to give a satisfactory range of observations on the climatic factor. Alternatively, equation 7.38 might be estimated from an experiment conducted in a year of "average" climate. The fitted function would then correspond to equation 7.39.

$$\hat{Y} = f(A, S, L, F, W, \mid C) \tag{7.39}$$

Normal procedures for response surface designs, as outlined in Chapter 5, should be used for the estimation of grazing input-output functions like equation 7.39. Since the inputs involved are relatively expensive, the rotatable designs are to be strongly recommended, provided that the researcher has some idea of the location of the relevant section of the surface. If he does not, a fractional factorial design may be best. Should the researcher be prepared to assume that constant returns to scale prevail to the controllable factors, the experiment may be simplified, and cheapened, by examining only a single level of the land input. The production surface, inclusive of land as an input, could then be estimated from the constant correspondence between output and proportionate increases in all inputs (including land) as implied by constant returns to scale. Thus, suppose A, S, F, and W of equation 7.39 were observed at various levels with L fixed at some level g. Under constant returns to A, S, L, F, and W, the response at any level of L, say kg, in combination with any observed treatment (A, S, F, W) could be obtained by multiplying the response for the treatment (A, S, F, W) by the factor k. Equation 7.39 might then be fitted to a set of data on Y, A, S, L, F, and W even though L had been held to a single level in the experiment. Alternatively, equation 7.40 might be fitted and equation 7.39 derived by making use of Euler's Theorem as in equation 7.41.

$$Y = f(A, S, F, W, \mid L, C) \tag{7.40}$$

$$\frac{\delta Y}{\delta A}\frac{A}{Y} + \frac{\delta Y}{\delta S}\frac{S}{Y} + \frac{\delta Y}{\delta L}\frac{L}{Y} + \frac{\delta Y}{\delta F}\frac{F}{Y} + \frac{\delta Y}{\delta W}\frac{W}{Y} = 1 \tag{7.41}$$

Functions such as equation 7.37, 7.38, and 7.39 are, however, only broadly applicable to farmers' decisions. They have a number of disadvantages. First, they do not take specific account of seasonal variations in the quality, quantity, and composition of a pasture. Such variations are subsumed within the variable area of pasture. In doing so, the possibility of conserving feed from one stage of the pasture cycle for use at a later stage is ignored. Of course, this criticism could not be

made in situations where there was no marked cycle in pasture growth and composition. The second difficulty with such functions is that their usefulness depends, in large measure, on the extent to which a farmer can vary his stocking rate. On an annual basis, this criticism might not be severe. But it could negate the usefulness of grazing functions that related to each seasonal stage of the pasture cycle; especially if the farmer's livestock breeding program was an important element in his farm operation. The third criticism of output-stocking rate functions is that they make no allowance for risk or uncertainty. Due to the strong dependence of pasture growth on climate, grazing tends to be a very risk-prone operation. As the stocking rate rises, the associated risk increases more than proportionately since drought is not only a function of rainfall but also of stocking rate. The conservation of some pasture growth to serve both as an inter-year drought reserve and as an intra-year supplement therefore becomes an important consideration.[38] As well, so far as the farmer is concerned, an important element of the situation might be the pasture management system to be adopted and its relation to the cropping rotation.[39] To incorporate such factors in the production function analysis would require extremely extensive experiments. Indeed, the most suitable approach might be through the use of real-world data.

The above criticisms basically relate to the fact that such functions as equation 7.39 do not provide a full basis for decisions during the period of pasture availability. Of the resources included, land and animals will generally be fixed in the short run, climate is not controllable, and fertilizer applications will usually have been made in the preceding period. Only the irrigation water and supplementary feed inputs will be currently at the farmer's discretion, and in most instances both these inputs are not used. In contrast, under handfeeding conditions, the farmer can exercise day to day control over the inputs used.

These criticisms lead to the conclusion that the most appropriate approach to the economics of pasture may be a (dynamic) programming one.[40] The mechanics of such procedures do not concern us here. Still,

[38] For a conceptual analysis of the interrelationships between stocking intensities, fodder conservation, and risk see Lloyd, A. G. Fodder conservation in the southern tablelands wool industry. Review of Marketing and Agricultural Economics, 27: 5-50. Insofar as movements in the pasture cycle and drouth occurrences may be characterized in probability terms, the techniques of inventory analysis are also pertinent to the economics of pasture use. See Mauldon, R. G. and Dillon, J. L. Droughts, fodder reserves and stocking rates. Australian Jour. of Agr. Econ., 3(2): 45-57.

[39] See Olson, R. O. and Heady, E. O. Economic use of forages in livestock production on Corn Belt farms. Circular No. 905. USDA, Washington, D.C. 1952; Blaser, R. E. *et al.* Animal performance and yields with methods of utilizing pasturage. Agronomy Jour., 51: 238-42; also Willoughby, W. M. Limitations to animal production imposed by seasonal fluctuations in pasture and by management procedures. Australian Jour. Agr. Res., 10: 248-68.

[40] Basic programming techniques, as related to grazing and storage for later use, have been outlined by Heady, E. O. Economic concepts in directing and designing research for programming of range resources. Jour. Farm Econ., 38: 1604-16. See also Candler, W. V. A new look at budgeting from the viewpoint of linear programming. Australian Jour. Agr. Econ., 3: 46-56; Nelson, M. The production function and linear programming in valuation of intermediate products. Land Econ., 33: 257-61; Nelson, M. and Castle, E. Profitable use of fertilizer on native meadows. Jour. Range Management, 11: 80-3.

it should be noted that estimates of output-stocking rate functions on a seasonal basis would be extremely useful in any programming approach. Too, estimates of such functions are relevant (a) in choosing between grazing and the feeding of harvested forage where the use of grazing is not obviously advantageous and (b) as a guide to public planners in land settlement and the allocation of grazing rights.

A noteworthy feature of the approach of equation 7.37 is that it completely avoids the problem of the amount of pasture actually consumed by an animal. This question is well-nigh irrelevant to the "on the spot" grazing decisions which a farmer must make. Still, specific knowledge of pasture intake is important; for instance, in the consideration of such problems as the physiological effects of different sward types, the efficiency with which different types of animals utilize pasture, and the effects of the grazing animal upon pasture growth and composition. Most such questions are best answered in terms of their relevant response surface since they will rarely depend on a single direct influence. In the past, response surface experiments involving the measurement of grazing intake have not been feasible due to the lack of any reliable measurement technique. Today, however, a sufficient backlog of pasture intake research is available to indicate more reliable measurement procedures that may be followed.[41]

RECOMMENDATIONS FROM FITTED PRODUCTION FUNCTIONS

The estimation of production functions is not an end in itself. They simply provide the framework through which input-output relations may be evaluated in economic terms. Moreover, production phenomena are essentially physical, although they may include some nonphysical elements incapable of direct quantification. From an economics standpoint, the usual criterion of evaluation is the maximization of net revenue from the production process, given the relevant price regime associated with the inputs and outputs. Too, other criteria may sometimes be used by the economist: for instance, the highest monetary return per unit of a given input.[42] Indeed, any production process which is consummated in the market place is most rationally controlled by economic criteria covering both the monetary and other goals of the entrepreneur. Of course, due to a lack of information about the nonmonetary goals, the economist is generally forced to work in monetary terms only. Physical criteria, such as the maximization of physical production, are in general quite irrelevant; except insofar as (a) it is always better to produce

[41]See Reid, J. T. Pasture evaluation in relation to efficiency, in Hoglund, C. R. *et al.* (eds.) Nutritional and economic aspects of feed utilization by dairy cows. Iowa State University Press, Ames. 1959. Pp. 135-51. Also Reid, J. T. *et al.* Symposium on forage evaluation. Agronomy Jour., 51: 212-38.

[42]See Pesek, J. and Heady, E. O. Derivation and application of a method for determining minimum necessary rates of fertilization. Soil Science Society of America Proc., 22: 419-23.

a greater amount of product from a given quantity of input, other factors remaining unchanged, and (b) it is never rational to add inputs when output increments are negative.

The use of estimated production functions as a guide to the economic allocation of resources is fraught with problems. The conditions under which a fitted function will serve as an error-free guide are extremely severe. To be a perfect guide, the estimated function should (a) account for all inputs involved in the production process, (b) cover the relevant range of input and output quantities, and (c) specify the production parameters without error. As well, it is necessary (d) that the function be supplemented by similar information about other production alternatives, (e) that all the relevant prices be known, and (f) that the levels of the inputs not under the entrepreneur's control be known *ex ante*. In addition, even if the fitted function is accurate, it cannot serve as a reliable indicator outside of the statistical population to which it refers.

Obviously, no estimated production function can satisfy all the above conditions. Indeed, this chapter and the three preceding ones have been concerned with methods of accounting for as many of the relevant inputs as possible, and of reducing errors in the estimated parameters. Bluntly, it can be said that no estimated function will ever contain all the relevant inputs or contain error-free estimates of the parameters. Still it is not necessary for such perfection. With reasonable precautions in the collection and analysis of data, complemented by discretion in interpretation, production functions that can serve as reasonable guides in resource allocation may be estimated. This is so even if no information is available about alternative production possibilities; some information of reasonable validity is better than none. But it is certainly better if production functions are available for alternative lines of investment. If resources are limited they may then be allocated among these alternatives in profit maximizing fashion.[43] The last two conditions for perfection noted above, *ex ante* knowledge of future prices and uncontrollable input levels, also are unlikely to be satisfied. In other words, the economic interpretation of a fitted production function will always be shrouded with elements of risk (if objective probability statements can be made about the unknown factors) or of uncertainty (if probability statements cannot be made). It is possible, however, to handle risk and uncertainty in an economically rational fashion.

Another frictional feature in the evaluation of fitted production functions is the time factor. No real-world production process is instantaneous. Some processes will have a natural time length. For others, the length of the production period may be at the discretion of the entrepreneur; if so, time may be included in the function as an input factor. In other cases, where production is relatively continuous over time, the estimated function may refer to some arbitrarily selected

[43] See Henderson, J. M. and Quandt, R. E. Micro-economic theory. McGraw-Hill, New York. 1958. Pp. 72-73. Also Heady, E. O. Economics of agricultural production and resource use. Prentice-Hall, Inc., New York. 1952. Pp. 237-75.

time span; for example, an accounting period. Whatever time period the fitted function refers to, it should take account of all inputs relevant to production during that time period. Thus, some of the inputs may have entered the production process during a prior time period. Still, however time is handled, production functions as estimated empirically are essentially static.[44] Their use as guides for resource allocation can refer only to the length of the production period covered by the function. The resultant resource allocation for profit maximization over this period may be quite different from the allocative pattern necessary to maximize profit over a sequence of such periods. Moreover, as previously noted, the production function for a given process is not invariant over time. It will tend to be affected by technological change, or more accurately, it will tend to be replaced by a new production function.

In providing production function estimates as guides to production decisions, and in making recommendations based on these estimates, the economist or farm adviser assumes that the real-world decision makers desire to maximize profits. This assumption is reasonable even though, in the absence of production function information, entrepreneurs and other planners may follow some simpler motive in their resource allocation. For instance, they may merely search for a pattern of resource use that is satisfactory in terms of their current aspirations; behaving as satisficers rather than maximizers.[45] Given precise recommendations or decision-making guides derived from input-output analyses, however, these real-world decision makers might be expected to shift their aspirations to a level commensurate with profit maximization.

With these broad comments in mind, we may now consider the problems that arise in using farm-firm functions and micro-functions as guides for real-world decisions. Due to the mechanics of data collection, such a division is tantamount to a distinction between functions based on survey data and those from experimental data. To a major degree, the possibility of making useful recommendations depends upon the scope of the fitted function, its reliability in terms of statistical and economic logic, and the quality of the data on which it is based. Some consideration has already been given to these topics in this and preceding chapters. Consequently, these aspects will not be treated in detail here.

Farm-Firm Functions

Typically, farm-firm production functions are estimated from cross-sectional survey data. Estimates so derived are not of extreme usefulness as guides to decision making on individual farms unless the

[44]See Schneider, E. Pricing and equilibrium. William Hodge and Co., London. 1952. Pp. 182-88.

[45]See Simon, H. A. Models of man. John Wiley and Sons, New York. 1957. Pp. 241-60.

functions are relatively disaggregated, both in inputs and outputs. A farmer can be told that he should use more "capital." But the recommendation has little specific meaning where he has dozens of avenues for the use of additional capital, some of which are profitable and others which are not. Likewise if "output" corresponds to a variety of products, the farmer needs specific information about each alternative rather than some broad recommendation in aggregate. Still, a measure of compromise is possible. He might be advised to allocate additional increments of a broad resource category between specific alternatives, in proportions similar to those found on the "average farm" of the sample. Even so, the fitted function must be reliable in terms of the data used and the homogeneity of the production process. A function fitted to bad data can never serve any useful purpose, although it must be recognized that different research workers may, in good faith, have different conclusions about the efficiency of the data. Still, even if there are slight variations among farms in the "true" production function, an average production function of some utility may be calculated.[46]

Other factors also restrict the usefulness of farm-firm functions as guides for individual firms. Satisfactory recommendations may be feasible only for individual farms of an average character. The possible errors associated with recommendations for farms of a sample using atypical input mixes may be too large to warrant recommendations to them. Account also needs to be taken of climatic uncertainty. The resource productivities derived from a cross-sectional farm sample will, in part, depend on the climatic conditions prevailing in the survey year. If this year was extremely atypical, the usefulness of the fitted production function is limited. Many farm resources are relatively fixed in the short run of a few years. The most obvious example is land. To tell a farmer that under current price conditions he should be using another 100 acres of land, when the land may not be available or he may not be able to afford it, is quite meaningless. Such restraints are, of course, best handled by programming procedures. Production function estimates may contribute to these by providing estimates of the input-output coefficients at varying levels of production. Noneconomic constraints may also be important. For instance, a farmer may restrict his input of hired labor because of the psychological burden involved in directing additional labor. Another disadvantage of empirical production functions is that they usually fail to take account of discontinuities that may exist on the input side.[47] It is, for instance, impossible to vary the input of machinery services in the continuous manner possible for fertilizer. Still, machinery services may be hired on a contract basis. Or a three-furrow plow might be used as a compromise between a two-furrow plow and two such plows; alternatively, the two-furrow

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[46]See Konijn, H. S. Estimation of an average production function from surveys. Econ. Record, 35: 118-25.

[47]Functional forms allowing for discontinuities are possible. See Hildreth, C. G. Discrete models with qualitative restrictions, in Baum, E. L. *et al.* (eds.) Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames. 1956. Pp. 62-74.

plow might be used one and a half times as much. Likewise, while half a man cannot be hired, a man might be hired half-time, or a less productive man employed full-time.

Lack of *ex ante* knowledge about future prices also restrains use of farm-firm functions from cross-sectional data. However, it is more of a marginal disadvantage than any of the criticisms noted above. Price risk or uncertainty will always be present regardless of the decision guide used. Programming procedures while eminently suited to individual farm planning, must also face up to the problem of uncertain or unknown future prices.

On all these grounds, cross-sectional production functions must be used warily as guides to individual farmers. In contrast, they may serve more usefully as a guide to problems or policies in broad allocation of resources among regions or between industries, even when based on such aggregates as land, labor, and capital.[48] Although land is immobile, the switching of other inputs such as capital and labor from one region or industry to another is possible. Thus, on the basis of current marginal returns to capital, derived from production functions in two regions A and B, a bank or a government might decide that credit be restricted in region A and increased in B. For capital, such decisions are even feasible in an international context. With labor, however, transfers cannot be made so easily in the short run unless government planning is all supreme and perhaps if little consideration is given to the personal values of the transferred workers.

For production functions to be used as policy guides, discretion must be used in selecting the sample population. If the population is defined with ideal strictness, a cross-sectional sample would never be possible. Variations among individual farms are so many that only individual farms can represent a homogeneous population. Compromise thus has to be made between the degree of homogeneity in the sampled population and the range of applicability of the resultant function. Certainly, given limited research resources, it is desirable that the fitted function be applicable to as large a population as possible.[49]

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[48]For example, see Heady, E. O. and Du Toit, S. Marginal resource productivity for agriculture in selected areas of South Africa and the United States. Jour. Pol. Econ., 62: 494-505; Dillon, J. L. Marginal productivities of resources in two farm areas of N.S.W. Economic Society of Australia and New Zealand. Economic Monograph No. 168. Sydney. 1956; Wang, Y. Resource returns and productivity coefficients for selected crop systems in Taiwan area. Proc. of Agr. Econ. Seminars, Sept. 16-20, 1958. National Taiwan University, Taipei. 1959. Pp. 90-98; Heady, Earl O. Marginal productivity of resources and imputation of shares for rental farms. Iowa Agr. Exp. Sta. Bul. 433. Ames; and Miller, W. G. Comparative efficiency of farm tenure classes in the combination of resources. Agr. Econ. Res., 11: 6-16.

[49]Although farm-firm functions are rarely estimated from time series data, some comments might be made on the use of such functions as guides to resource allocation. If the data refers only to a single farm, its application relative to other farms will be unwarranted unless these other farms are rather similar to the farm studied. This factor will also determine the usefulness of the time series function as a help to interregional resource allocation. Should there have been significant technological changes over the time series period, of course, the fitted function will be of little use on either an inter- or intra-farm basis, or interregionally, unless account is taken of these changes.

Inference From Micro-Analyses

We now outline the difficulties in using production function estimates for individual technologies as guides to real-world decisions. By individual technologies we refer to such phenomena as crop, animal, and machinery input-output relationships. Relative to the over-all farm operation, they are of a micro-nature. In contrast to farm-firm functions which cover a host of micro-production processes within a single aggregative framework, the micro-functions are far more useful as guides for individual farmers. On the other hand, their potential relevance as guides to public planners is not so direct. Still, the foundation of any macro-policy must be in terms of micro-phenomena.[50] For instance, economic growth, in the final analysis, must be based on the development of new micro-production processes and the better utilization of existing processes. Likewise, the use made of micro-processes determines the aggregative supply response. Understanding of supply influences cannot be divorced from micro-phenomena. Insofar, therefore, as micro-functions serve as guides to more efficient resource allocation and hence to welfare gains for the whole of society, they impinge upon public planning. Moreover, micro- and macro-analyses of production are complementary: without some knowledge of both, planning may be inefficient. Micro-recommendations may be valid for individual farmers only as long as masses of farmers do not follow them. Finally, the same basic economic principles apply in analyses relating to micro- and macro-functions and problems. The principles for specifying the optimum resource allocation are essentially the same within a firm, between firms, within industries and regions, and between industries and regions.

A production function based on physical phenomena cannot take direct account of a farmer's nonmaterial goals. Moreover, these farm family goals will vary from farm to farm so that each decision relates unique features. Given knowledge of these nonmonetary aspects, correct recommendation about use of individual technologies might involve resource allocations differing greatly from that implied by profit maximization. It is for this reason that recommendations to individual farmers based on production function analyses should be couched in terms of "do this if you wish to maximize your profits" rather than in dogmatic statements of the type "you should do this."

Also, since the farmer is not an expert in the differential calculus, production function analyses need to be interpreted for him. The researcher's job is not finished when he has estimated a particular production function. He has a responsibility to interpret the analysis in a

[50] For an elaboration of the role of microanalysis in macro-agricultural policy see: Johnson, G. L. Some contributions of microanalysis to agricultural policy, in Baum, E. L. *et al.* (eds.) Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames. 1957. Pp. 362-74; and Heady, E. O. Need for production economics research in solving policy problems, in Baum, E. L. *et al.* (eds.) Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames. 1957. Pp. 348-61.

way the farmer or his go-between can understand and to ensure the channeling of such information to the farmer. Otherwise the research input may be wasted. To this end, such communication devices as price maps, nomograms, and costulators might be used.[51]

Relative to farm-firm analysis, the greater usefulness of micro-functions as farmer guides stems from the fact that (a) they are usually determined under experimental conditions where the researcher can exercise some control and (b) being of a micro-nature, it is far easier to specify the inputs and outputs in a meaningful way. Still, the range of application of these micro-estimates is always limited by the experimental framework used. For instance, a fertilizer-crop production function estimated from experiments on a given soil type cannot be used indiscriminately for other soil types. Likewise, if the level of technical management or other background factors relevant to the response experiment are too different from those prevailing on farms, the usefulness of the fitted function will be curtailed.[52] Indeed, the extent to which a response function for a given technology in a particular environment can be used with reference to the same technology in a different environment is perhaps the major problem in planning and using production function research. It is obvious that research resources are too limited and technological advance too rapid for production function estimates to be derived for all possible production processes in all possible environments. At the same time, many real-world environments are not so dissimilar as to necessitate a separate function for each of them. Moreover, no matter how bad the error may be in extrapolating a production function from one environment to another, such a procedure at least has the salutary effect of forcing those concerned to reconsider their positions. The extent to which such extrapolations can be reasonably made, however, is largely a matter of discretion on the part of those familiar with both the experimental and real-world environments. In this regard, one avenue of worthwhile exploration is the development of adjustment factors. These could be used to adapt results for a particular environment to other environments of real-world significance.[53]

Even within the same basic environment (i.e., soil type, breed of animal, etc.), there are important problems of inference from the experimentally derived function to the real-world situation. These relate, first, to the levels at which the controlled factors were held in conducting the experiment; and, secondly, to the influence of the uncontrolled

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[51] Such devices are illustrated in Profitable use of fertilizer in the Midwest. Wisconsin Agr. Exp. Sta. Bul. 508. Madison. 1954; and Heady, E. O. Integration of physical sciences and agricultural economics. Canadian Jour. Agr. Econ., 4: 1-15.

[52] Such an effect appears to have occurred in some milk production experiments. See Antill, A. G. and Clark, C. The overfeeding of dairy cows. Westminster Bank Review. Feb. 1958. Pp. 13-15.

[53] A start in this direction has been made with respect to initial soil fertility conditions in crop and pasture production by Jensen, D. and Pesek, J. Generalization of yield equations in two or more variables. I. Theoretical considerations. II. Application to yield data. Agronomy Jour., 51: 255-63. Also Rust, R. H. and Odell, R. T. Methods used in evaluating the productivity of some Illinois soils. Proceedings of the Soil Science Society of America, 21: 171-75.

factors operating in the experiment from which the function was estimated.[54] Distinction might also be made between problems of inference arising (a) from inaccurate statistical estimation due to the omission of relevant variables or the inclusion of incorrectly measured variables and (b) from inaccurate economic interpretation due to the omission of input factors which have interaction effects with the included variables. Difficulties from all these sources can be discussed by way of equation 7.42 where X_1 to X_g are the controlled factors that were allowed to vary; X_{g+1} to X_h are the controlled factors held fixed at some level; and X_{h+1} to X_k denote uncontrollable factors that operated at some level (either fixed or variable) over the experiment. As noted in Chapters 5 and 6, all of the unstudied factors X_{g+1} to X_k must be expected to condition the estimated function. For this reason, g should be as large as possible and h - g as small as possible.

$$(7.42)\quad Y = f(X_1, X_2, \ldots, X_g, \mid X_{g+1}, \ldots, X_h, \parallel X_{h+1}, \ldots, X_k)$$

Of course, as estimated empirically, equation 7.42 would only be written in terms of X_1 to X_g, the levels of the other variables being noted in the commentary on the experiment. We will first consider problems of inference relating to the controlled factors.

Influence of controlled factors

The levels at which the variable factors X_1 to X_g are allowed to operate raise no difficulties so long as the use of the fitted function is restricted to the surface section studied experimentally. The estimated function cannot be validly extrapolated outside this region. Recommendations involving levels of X_1 to X_g beyond the studied range must be advanced with extreme caution. This difficulty is best overcome by endeavoring to study as much as possible of the surface that may be of economic significance, given the quantity of research resources available.

Perhaps more important are the problems associated with the factors X_{g+1} to X_h of equation 7.42. If these controllable factors are in fact held fixed over the experiment, the fitted function strictly applies only to real-world situations where these factors operate at the same levels. For instance, in a fertilizer-crop response study, X_{g+1} to X_h might refer to the slope of the experimental plots, the degree of weed infestation, the level of management, the type of cultivation used, and so on. If the experimental level of these factors does not correspond to the farm situation, the fitted function would, in all strictness, not be relevant. The researcher has two alternatives in overcoming this problem, assuming that he has included as many as possible of the controlled factors as variables. The first alternative is to repeat the experiment

[54] See Swanson, E. R. Problems of applying experimental results to commercial practice. Jour. Farm Econ., 39: 382-89.

either in time or space with the fixed controllable factors held at a different level in each experiment. Such a procedure is, of course, equivalent to conducting one large experiment with the "farm" factors (slope, weed infestation, management, cultivation procedure, etc.) treated as variables in the statistical analysis. In general, however, it would seem to be a waste of research resources to carry out such a large experiment or sequence of experiments, even if it were computationally feasible. Research resources might be better used to look at a number of production processes instead of examining a single one in minute detail. The other alternative, advocated by Johnson, is to set up the experiment with X_1 to X_g of equation 7.42 as the treatment combinations but with X_{g+1} to X_h varied throughout the experiment in the fashion prevailing on farms, no attempt being made to incorporate these "farm" factors in the statistical analysis.[55] The fitted function would then be an average function with respect to the factors X_{g+1} to X_h. Such an approach is attractive insofar as these factors are not major determinants in the production process. Moreover, as the array of estimated microfunctions increases, experience can be gained on the degree of importance of these lesser influences. Rules of thumb might then be developed for making inferences from a fitted function to individual farm situations.

Influence of uncontrollable factors

In any experimental situation and its farm counterpart, a host of uncontrollable factors will operate. Chief among these are climatic effects. The researcher, just like the farmer, has to accept these factors as they are stipulated by Nature. For instance, suppose equation 7.42 relates to fertilizer-crop response. The experiment might be carried out, by chance, in a climatically poor crop year. The factors X_{h+1} to X_k, denoting such inputs as rain, heat, wind, etc., might all be at levels inimical to crop production. Obviously, the fitted function would not be relevant to crop production in a good crop year. In situations where the uncontrollable factors were relatively stable, such problems would only be of minor import. Still, most farmers have to face climatic and other risks and uncertainties. But if sufficient research resources are available, variations in the uncontrollable factors can be evaluated in such fashion that reasonable recommendations are possible. Thus, a fertilizer-crop response function might be estimated for a range of climatic situations. For the single variable input case, Figure 7.7 provides a simple illustration. Curves A, B, and C might correspond to the production function for excellent, average, and poor climatic conditions respectively. Alternatively, a single production function might be estimated with climate or other uncontrollable factors included as

[55] Johnson, G. L. Planning agronomic-economic research in view of results to date, in Baum, E. L. *et al.* (eds.) Economic and technical analysis of fertilizer innovations and resource use. Iowa State University Press, Ames. 1957. Pp. 217-25.

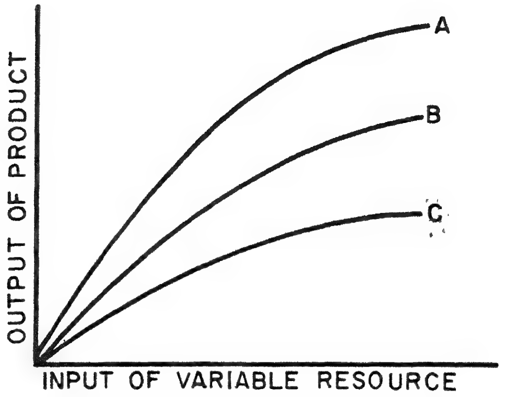

Figure 7.7. Influence on the production function of the level of uncontrollable factors.

variables.[56] How might reasonable recommendations be based on such functions? The answer depends on whether the uncontrollable factors are characterized by risk or by uncertainty. Risk implies that probabilities can be associated with alternative levels of the uncontrolled factors. If probabilities cannot be assigned, then uncertainty prevails.

Risk

Using the situation of Figure 7.7 as our example, suppose the climate can be specified in terms of risk.[57] There might be, say, a probability of 0.6 of average climate, and of 0.3 and 0.1 for poor and excellent climate respectively. Recommendations could then be made in two ways. The farmer might be advised to operate on functions A, B, and C for 1, 6, and 3 years out of 10, respectively. Alternatively, each year he might follow B with 60 per cent of his resources, C with 30 per cent, and A with 10 per cent. Such an approach would allow the farmer to be partly right each year, although he could never be completely correct. The "years out of 10" approach, on the other hand, while it would probably lead to the farmer being right some of the time, could lead to a serious run of inefficient resource allocation patterns from an *ex post* view. To the extent that the making of a whole range of recommendations in probability terms may be too confusing for farmers, an average recommendation might be made with a rider attached giving the probability of the average recommendation being profitable in a given year.[58] Such

[56] Such an approach is to be found in Parks, W. L. and Knetsch, J. L. Corn yields as influenced by nitrogen level and drouth intensity. Agronomy Jour., 51: 363-64. Also Parks, W. L. Methodological problems in agronomic research involving fertilizer and moisture variables, in Baum, E. L. *et al.* (eds.) Methodological procedures in the economic analysis of fertilizer use data. Iowa State University Press, Ames. 1956. Pp. 113-33.

[57] See Heady, E. O. Economics of agricultural production and resource use. Prentice-Hall, Inc., New York. 1952. Pp. 440-53.

[58] Such a procedure has been followed by Hildreth, R. J. Influence of rainfall on ferti-

an approach would be of most value in situations where a fair degree of stability prevailed.

It should be noted that Figure 7.7 reflects a rather simple situation, involving only three possibilities. In real-world situations, there will generally be an infinite array of possibilities involving continuous probability distributions. Moreover, whether the probability distribution is discrete or continuous, its higher moments, especially the variance and skewness, may be extremely important to farmer decisions. For instance, these higher moments are relevant to any consideration of the possibility of extremely large losses or gains resulting from a given course of action within the probabilistic environment. However, without knowledge of the decision maker's preferences about these higher moments of the distribution, their evaluation is impossible.[59]

Uncertainty

If uncertainty prevails with respect to the uncontrollable factors, the appropriate approach would be that of a game against Nature. Thus, even if no probabilities could be assigned to the alternative functions of Figure 7.7, a payoff matrix might be calculated. Suppose equation 7.43 depicts such a matrix where N_a, N_b, and N_c are Nature's possible strategies of excellent, average, and poor weather; and F_a, F_b, and F_c are the farmer's pure alternatives of following functions A, B, or C. As well, the farmer may follow a mixture of these pure alternatives. By definition, no probabilities can be assigned to Nature's possible strategies.

(7.43)

	N_a	N_b	N_c
F_a	7	2	-1
F_b	4	5	0
F_c	2	1	3

The numbers in this payoff matrix are hypothetical examples of the profit the farmer would make if he followed a given strategy when the corresponding state of Nature occurred. Thus, if the farmer selected function B and Nature "chose" N_c, i.e., poor weather, the farmer's profit would be zero. For any such payoff matrix it is possible to decide on courses of action that have strong logical appeal; making use of decision criteria that have been developed for uncertainty situations of the type depicted by Plan 7.43. The best known of such decision-making

lizer profits. Jour. Farm Econ., 39: 522-24. See also Lloyd, A. G. Agricultural experiments and their economic significance. Review of Marketing and Agricultural Economics, 26: 185-209.

[59] In this regard, see Tintner, G. A contribution to the non-static theory of choice. Quarterly Jour. Econ., 56: 274-306; and Dorfman, R. *et al.* Linear programming and economic analysis. McGraw-Hill, New York. 1958. P. 431.

algorithms are those of Wald, Laplace, Hurwicz, and Savage. No attempt will be made to outline these decision criteria here. Their use is rather marginal to the general topic of production function research.[60]

We now pass, in succeeding chapters, to a consideration of actual empirical production function research.

[60] For a detailed outline of these criteria see Luce, R. D. and Raiffa, H. Games and decisions. John Wiley and Sons, New York. 1957. Pp. 278-86. For some examples of their application in a production function context, see: Dillon, J. L. and Heady, E. O. Decision criteria for innovation. Australian Jour. Agr. Econ., 2: 113-20; and Swanson, E. R. Problems of applying experimental results to commercial practice. Jour. Farm Econ., 39: 382-89.

CHAPTER 8

Earl O. Heady
Damon V. Catron
Roger Woodworth
Gordon C. Ashton

Pork Production Functions for Hogs Fed in Drylot

THIS STUDY was conducted jointly by staff members in production economics and swine nutrition. It employs principles from both sciences to obtain more information than conventional research procedures and to provide more complete data for economic analysis of protein levels for hogs. The investigation was designed particularly to provide information allowing specification of optimum protein rations for hogs of different weights. It was made in recognition of the fact that the price of feeds which are high in protein and carbohydrates change relative to each other. Hence, the ration which provides least-cost gains or the most profit per hog is not the same under all price conditions. Hence, experiments were designed which would allow estimation of the pork production surface and the equations of isoquants, isoclines and marginal transformations which relate to them. The chief objective was to determine average rates of substitution over several weight intervals in pork production. The weight interval substitution rates were emphasized because farmers do not change rations each day. As a matter of practicality, they buy or mix and feed the same ration over an interval of time. Thus emphasis was on establishing the average substitution rates between corn and a high protein feed over these weight intervals; rather than to emphasize estimation of functions which allow derivation of substitution ratios at all levels of gain. The power function provides a simple means for estimating average substitution rates over gain intervals and is used accordingly. However, other functional forms also have been derived and are presented in this chapter for other uses, such as estimating optimum marketing weight or other purposes. A few alternative forms of functions were estimated but are not presented (a) because of space limitations and (b) because they provide estimates which appeared no more or less efficient than those presented.

ECONOMY IN RATIONS

Economy in the hog ration depends especially on the manner in which feeds are combined. Many feeds can be used in pork production and they can be combined in many ways. A number of carbohydrate feeds such as corn, barley, or wheat can be substituted for each other

and used in combination. On the other hand, vegetable proteins such as soybean oilmeal, linseed oilmeal, and cottonseed oilmeal, and animal proteins can also be substituted for each other or used in combination. Finally, supplementary proteins and carbohydrate feeds can be substituted for each other and used in many combinations. One of the main economic problems in hog rations is this: What combination of corn and protein should be used in producing 100 pounds of pork? While other carbohydrate feeds sometimes have a more favorable price and supply situation, corn — aside from oats — has had the traditional advantage, both in price and proximity.

Aside from legume pasture, tankage was formerly the important protein source because it supplied vitamin B_{12} and other factors. The absence of the then unknown vitamin B_{12} in vegetable proteins limited their use in pork production up to moderate hog weights. A main economic problem then was deciding (a) the combination of corn and animal proteins to use in producing lightweight hogs and (b) the combination of corn, animal proteins, and vegetable proteins to use in producing heavier hogs. (Minerals, pasture, other feeds, and labor and capital costs, of course, also play a part in profit calculations.)

New technologies in production and recent developments in swine nutrition, however, have caused older notions on hog rations to become obsolete. The discovery that vegetable proteins, balanced or fortified with trace minerals, vitamin B_{12} and antibiotics, can combine with grain as effectively as animal proteins in producing pork changed previous protein level recommendations. Because of the relatively low cost of small amounts of vitamin B_{12}, aureomycin and trace elements, a main problem in the pork ration now appears to be how corn and a vegetable protein such as soybean oilmeal — containing the other feed elements mentioned — can combine with or substitute for each other to give the least-cost ration.[1] Animal proteins are, of course, still an economic consideration in the hog ration.

DESIGN OF EXPERIMENT AND FEEDING METHODS

Data from three hog feeding trials were used in this study. The first protein level experiment — no. 506 — was already completed when the decision was made to analyze the economy of different rations. This experiment included four protein levels of 14, 16, 18, and 20 per cent. These percentages of protein were fed to pigs weighing from 34 to 75 pounds. At weights of 75 pounds, and again at 150 pounds, the protein levels were lowered three percentage points. Under each of the protein levels there were two replicates containing aureomycin and two without. Each replicate had eight pigs per lot. This experiment was not completely suited for the substitution analysis since there were only four protein levels. Also, the shifting of protein levels at different

[1] For purposes of brevity, the term meal is used to replace oilmeal in part of this report.

weights caused difficulty in cumulating weekly gains and feed observations. However, these data are included in part of the analysis although the remaining two experiments provide the main basis for the empirical analysis.

Experiment no. 536 was designed especially for the requirements of this study. It included treatments representing six protein levels and three antibiotic arrangements. The six levels contained 10, 12, 14, 16, 18, and 20 per cent crude protein. There were three replicates without antibiotics, three replicates with aureomycin, and two replicates with terramycin, all using four pigs per lot. A single protein level was fed to each lot throughout the feeding period — because the study was designed for regression analysis, rather than analysis of variance. The replicated lots were retained to fulfill requirements for other studies using analysis of variance. The pigs were weighed each two weeks and feed inputs were accumulated to provide input-output observations at varying points over the growth period. The pigs were taken off the experiment at 200 pounds for use in a study dealing with the effect of protein level on carcass quality. Some of the pigs in this experiment contracted a skin disease. The disease increased the experimental error and reduced feed efficiency.

Experiment no. 554 was the same as experiment no. 536 except that it employed three replicates with aureomycin and three without. In the empirical analysis which follows, the observations for terramycin have not been used in deriving production functions and substitution quantities.

In all three experiments, corn was the main source of carbohydrate and soybean oilmeal was the main source of protein. The soybean oilmeal contained about 45 per cent crude protein. The per cent protein in the corn, however, varied from 8.4 in experiment no. 554 and 8 in experiment no. 506 to 7.2 in experiment no. 536. Each pound of the corn and soybean oilmeal rations was supplemented by .04 pounds of minerals (including steamed bonemeal, iodized calcium carbonate, iodized salt, and trace minerals) and .01 grams of vitamins (including vitamins A, D_2, 5 mcg. of B_{12}, niacin, pyridoxine, riboflavin, and thiamin). The lots with antibiotics received 5 mg. of aureomycin or terramycin per pound of feed. In all experiments the complete ration was ground, mixed, and self-fed. The breed of hogs was not the same in all cases — Duroc in experiment no. 506, Duroc and Poland China x Landrace x Duroc in experiments no. 536 and 554. Pigs were randomly allotted for all three experiments.

STATISTICAL ANALYSIS

Some background information of grain-protein substitution in pork production is already available. Previous experiments suggest that for light hogs, supplementary protein ordinarily is a limitational feed — some amount of it is necessary to attain a given gain. With this information, a particular problem in grain to protein replacement (or protein

levels) for pigs becomes more nearly one of estimating the substitution rates. This experiment was designed accordingly, and the design and analysis allow the testing of alternative hypotheses with respect to the nature of pork isoquants and grain to protein substitution ratios. While an experimental design based on many individual pigs for different treatments scattered over the contour lines and the ration lines would have been better, if each hog had been used as a single observation, funds did not allow this approach. In the analysis which follows, feed and weight were accumulated over the complete feeding period with measurements taken at stated intervals. Each of these measurements then was used as an observation in the regression analysis. This approach gives rise to the problems of autocorrelation discussed in Chapter 6. Mainly the procedure tempers statements which can be made about confidence intervals and does not bias the estimates of mean gain and substitution quantities. The observations between pigs are independent, but the several observations for each pig are not. Thus, given the inclusion of all observations for each pig, the usual coefficients of correlation coefficients and standard errors must be qualified. In general, the regression coefficients used in making estimates of substitution rates and R^2 values were significant even if the number of degrees of freedom were considered to be equal only to the number of hogs, rather than the number of hogs times the number of observations on each. However, the probability levels indicated on pages which follow are based on the total number of observations.

Several types of functions have been fitted to the data. However, most use is made of the power function because, for the purposes mentioned earlier, it allows estimation of the average rate of substitution over a weight interval. The major reason for making this study was to allow determination of minimum cost feed combinations, as an average over the weight intervals stated. Emphasis in United States pork science is now on producing hogs to around 225 pounds of weight, to avoid production of surplus lard and fatty cuts. This is becoming the common practice. Within this framework, the major feeding problem is to determine the least-cost rations in attaining this weight and the emphasis on this study is made accordingly. As mentioned above, average rates of substitution over three weight intervals are estimated by power functions, to conform with certain practical considerations in pork production. In the regression equations which follow, we use the following symbols: Y is gain in pounds per pig after weaning, C is pounds of corn, and P is pounds of soybean meal fed beyond weaning.

Regression Equations

Regression equations were fitted both to gain data over the weight range 34-200 pounds and for each of the weight intervals, 34-75 pounds, 75-150 pounds, and 150-200 pounds. Hereafter, the regressions for individual weight ranges will be referred to as *interval* functions; those

for the 34-200 range will be referred to as *over-all* functions. A different function was fitted for three weight intervals to allow greater flexibility in substitution rates. This statement applies to the logarithmic or Cobb-Douglas function. Although it allows diminishing rates of substitution, and diminishing productivity of both feeds increased in constant proportions or one increased by itself, derived as a single regression equation it does not allow the substitution rate to change with hog weight. In other words for a *ration* or *scale line*, all contour lines will have the same slope or rate of substitution at each point they intersect the ration line. Quadratic, ratio, square root, and other equations allow substitution rates to change along a scale or ration line. However, the power functions for different intervals allow different rates of substitution within each. While a different substitution rate at each pound-change in hog weight is theoretically and logically possible, substitution rates which serve as "averages" over a few weight intervals are sufficient for farmer decisions since the same ration is fed for several days, rather than changed every day. While the Cobb-Douglas functions for the intervals cause the intersection points of all weight contours and ration lines to have the same substitution rate within an interval, it does allow estimation of three different substitution rates, as averages, for the three weight intervals.

The regression equations (production functions) derived from the experimental observations are included below. Regression equations are provided separately for the two major drylot experiments used as a part of this study. The observations from these two experiments have then been pooled as a further basis for estimating interval and over-all functions. It is likely that the pooled results more nearly parallel typical farm conditions than either experiment by itself.

Also, separate regression coefficients have been estimated for the protein level study of experiment 506. The rations and other conditions in this experiment were identical with the others except that each lot was not fed the same ration over the complete weight range; protein levels were lowered as successive weights were reached. Finally, the observations from this study have been pooled with those from the two studies mentioned above for further estimates of the productivity and substitution coefficients. (Regression equations for the single experiment, 506, are recorded elsewhere.)

Many production function equations were derived in the course of this study. Since space prohibits presentation of all these, only enough will be presented to illustrate those which gave great similarity or difference in prediction.

Production Functions[2]

A. Production functions from experiment 536.

I. With aureomycin

(a) Cobb-Douglas

(8.1) 34-75 lbs.: $Y = 1.445C^{.547} P^{.289}$

(8.2) 75-150 lbs.: $Y = .555C^{.795} P^{.151}$

(8.3) 150-200 lbs.: $Y = .467C^{.796} P^{.161}$

(8.4) over-all: $Y = 1.360C^{.630} P^{.201}$

(b) Quadratic

(8.5) over-all: $Y = 2.032 + .324C + .464P - .000129C^2 - .000917P^2 - .000111CP$

II. Without aureomycin

(a) Cobb-Douglas

(8.6) 34-75 lbs.: $Y = 1.658C^{.464} P^{.357}$

(8.7) 75-150 lbs.: $Y = .342C^{.938} P^{.063\dagger}$

(8.8) 150-200 lbs.: $Y = .345C^{.888} P^{.096}$

(8.9) over-all: $Y = 1.174C^{.657} P^{.179}$

(b) Quadratic

(8.10) over-all: $Y = 2.815 + .318C + .348P - .000108C^2 - .000665P^2 - .000118CP^{\dagger}$

B. Production functions from experiment 554.

I. With aureomycin

(a) Cobb-Douglas

(8.11) 34-75 lbs.: $Y = 1.874P^{.323} C^{.493}$

(8.12) 75-150 lbs.: $Y = .815P^{.174} C^{.735}$

(8.13) 150-200 lbs.: $Y = .523P^{.052} C^{.864}$

(8.14) over-all: $Y = 1.412P^{.218} C^{.626}$

[2]In terms of degrees of freedom figured on the basis of all observations, the symbols used have these meanings: ‡ Regression coefficients significant at a probability level of more than 20 per cent; † significant at 10-20 per cent; * significant at 5-10 per cent; no asterisk means a regression coefficient significant at a probability level of 5 per cent or less.

(b) Quadratic

(8.15) over-all: $Y = .125 + .995P + .282C - .000865P^2 + .000046C^2 - .00144PC$

II. Without aureomycin

(a) Cobb-Douglas

(8.16) 34-75 lbs.: $Y = 1.820P^{.277}C^{.529}$

(8.17) 75-150 lbs.: $Y = .814P^{.218}C^{.691}$

(8.18) 150-200 lbs.: $Y = .428P^{.089}C^{.877}$

(8.19) over-all: $Y = 1.577P^{.229}C^{.592}$

(b) Quadratic

(8.20) over-all: $Y = 6.441 + .692P + .258C - .00307P^2 - .000077C^2 + .000615PC$

C. Production functions from experiments 536 and 554 pooled.

I. With aureomycin

(a) Cobb-Douglas

(8.21) 34-75 lbs.: $Y = 1.600P^{.297}C^{.533}$

(8.22) 75-150 lbs.: $Y = .714P^{.142}C^{.767}$

(8.23) 150-200 lbs.: $Y = .439P^{.092}C^{.856}$

(8.24) over-all: $Y = 1.369P^{.200}C^{.636}$

(b) Quadratic

(8.25) over-all: $Y = 3.004 + .582P + .314C - .00144P^2 - .000099C^2 - .000118PC$‡

II. Without aureomycin

(a) Cobb-Douglas

(8.26) 34-75 lbs.: $Y = 1.662P^{.287}C^{.529}$

(8.27) 75-150 lbs.: $Y = .614P^{.163}C^{.770}$

(8.28) 150-200 lbs.: $Y = .343P^{.081}C^{.916}$

(8.29) over-all: $Y = 1.422P^{.208}C^{.616}$

(b) Quadratic

(8.30) $Y = 3.778 + .620P + .288C - .00165P^2 - .000085C^2 - .00009PC$‡

In addition to the regression equations provided above, certain others were also derived. A few of these are given below for experiment 554.

Table 8.1. Number of Observations, Standard Errors, and Values of t for Cobb-Douglas Equations

Treatment	N*	Standard Error (s) and Values of t†			
		$s_{b(p)}$	$s_{b(c)}$	t_p	t_c
A. Experiment 536					
I. With aureomycin					
34-75 lbs.	34	.0366	.0471	7.91	11.61
75-150 lbs.	52	.0232	.0305	6.52	26.05
150-200 lbs.	56	.0264	.0416	6.08	19.15
over-all	142	.0090	.0103	22.46	61.08
II. Without aureomycin					
34-75 lbs.	30	.0407	.0502	8.77	9.23
75-150 lbs.	42	.0409	.0568	1.53	16.52
150-200 lbs.	62	.0316	.0457	3.03	19.41
over-all	119	.0162	.0186	11.06	35.26
B. Experiment 554					
I. With aureomycin					
34-75 lbs.	42	.0184	.0256	17.59	19.26
75-150 lbs.	60	.0150	.0238	11.61	30.89
150-200 lbs.	55	.0196	.0422	2.71	20.48
over-all	161	.0084	.0104	26.03	60.26
II. Without aureomycin					
34-75 lbs.	46	.0216	.0310	12.85	17.06
75-150 lbs.	59	.0164	.0271	13.3	25.5
150-200 lbs.	57	.0247	.0513	3.59	17.1
over-all	161	.0087	.0109	26.38	54.42
C. Experiments 536 and 554 pooled					
I. With aureomycin					
34-75 lbs.	76	.0273	.0279	14.31	19.11
75-150 lbs.	112	.0163	.0244	8.70	31.41
150-200 lbs.	111	.0174	.0318	5.30	26.94
over-all	303	.0130	.0158	15.34	40.35
II. Without aureomycin					
34-75 lbs.	76	.0230	.0311	12.52	16.99
75-150 lbs.	102	.0221	.0343	7.38	22.49
150-200 lbs.	119	.0204	.0342	3.98	26.77
over-all	280	.00917	.0118	21.41	52.32

*Number of observations based on all measurement for all hogs.
†The p subscript refers to soybean meal and the c subscript refers to corn.

D. Other production functions derived for experiment 554.

I. With aureomycin; over-all

(8.31)
$$Y = 12.24C + .372P + .364C - .000977P^2 - .000126C^2 - 1.010C/P$$

(8.32)
$$Y = -8.782 - .794P + .0754C + 4.966\sqrt{P} + 1.196\sqrt{C} + .750\sqrt{PC}$$

II. Without aureomycin; over-all

$$(8.33)\quad Y = 12.659 + .454P + .329C - .00120P^2 - .000107C^2 - .981C/P$$

$$(8.34)\quad Y = -13.302 - .190P + .206C + 8.77\sqrt{P} + .477\sqrt{C}\ddagger + .033\sqrt{PC}\ddagger$$

Standard errors and t values for selected Cobb-Douglas and quadratic equations are provided in tables 8.1 and 8.2. All of the equations for which these equations are presented had R^2 values of .94 or greater. In fact none of the equations from 8.1 through 8.34 had R^2 values smaller than .90.

From these statistics, and considering the relation of measurements to problems of autocorrelation, it was decided that the regression coefficients provided useful estimates for the major objective of this study.

Table 8.2. Standard Errors and t Values for Regression Coefficients and Number of Observations for Quadratic Cross-Product Equations, Pooled Data from Experiments 536 and 554

Standard Errors						
	N	$s_{b(p)}$	$s_{b(c)}$	$s_{b(p)^2}$	$s_{b(c)^2}$	$s_{b(pc)}$
With aureomycin	303	.0461	.0154	.0002	.00003	.00013
Without aureomycin	280	.0565	.0176	.0002	.00027	.00003

t Values					
	t_p	t_c	t_{p^2}	t_{c^2}	t_{pc}
With aureomycin	12.61	20.39	7.30	3.63	0.93
Without aureomycin	10.97	16.34	6.22	2.93	0.59

Product Isoquants

Selected equations of pork isoquants have been derived from the production functions estimated above. These have been derived only for pooled data from experiments 536 and 554. The equations for the interval Cobb-Douglas functions provide isoquant families which all have the same slope within an interval along a ration line intersecting the origin. However, the slopes for the isoquants differ among intervals. The quadratic function provides an isoquant family with each product contour having a different slope along a ration line.

I. With aureomycin
(a) Cobb-Douglas

(8.35) 34-75 lbs.: $C = \left(\frac{Y}{1.605P^{.297}}\right)^{\frac{1}{.533}}$

(8.36) 75-150 lbs.: $C = \left(\frac{Y}{.714P^{.142}}\right)^{\frac{1}{.767}}$

(8.37) 150-200 lbs.: $C = \left(\frac{Y}{.459P^{.092}}\right)^{\frac{1}{.836}}$

(8.38) over-all: $C = \left(\frac{Y}{1.369P^{.200}}\right)^{\frac{1}{.636}}$

(b) Quadratic

(8.39) $C = 1587.384 - .596P$

$\pm 5050.505 \sqrt{.0000005561P^2 - .000304P + .0991 - .000396Y}$

II. Without aureomycin
(a) Cobb-Douglas

(8.40) 34-75 lbs.: $C = \left(\frac{Y}{1.662P^{.287}}\right)^{\frac{1}{.529}}$

(8.41) 75-150 lbs.: $C = \left(\frac{Y}{.614P^{.163}}\right)^{\frac{1}{.770}}$

(8.42) 150-200 lbs.: $C = \left(\frac{Y}{.343P^{.081}}\right)^{\frac{1}{.916}}$

(8.43) over-all: $C = \left(\frac{Y}{1.422P^{.208}}\right)^{\frac{1}{.616}}$

(b) Quadratic

(8.44) $C = 1694.232 - .529P$

$\pm 5882.353 \sqrt{-.000000552P^2 - .000307P + 8.297 - .000340Y}$

The isoquant equations above allow derivation of marginal substitution rates. The Cobb-Douglas equations give rise to substitution

coefficients which are constant, within a weight interval, along a ration line. Substitution ratios for the quadratic function change at each point along a ration line. The derivatives below correspond to the isoquant equations above for pooled data.

I. With aureomycin
(a) Cobb-Douglas

(8.45) 34-75 lbs.: $\frac{\delta C}{\delta P} = .557 \frac{C}{P}$ (8.47) 150-200 lbs.: $\frac{\delta C}{\delta P} = .108 \frac{C}{P}$

(8.46) 75-150 lbs.: $\frac{\delta C}{\delta P} = .185 \frac{C}{P}$ (8.48) over-all: $\frac{\delta C}{\delta P} = .314 \frac{C}{P}$

(b) Quadratic

$$(8.49) \quad \frac{\delta C}{\delta P} = -.596 \pm \frac{-39.419 + .293P}{\sqrt{.000000556P^2 + .9991 - .000304P - .000396Y}}$$

II. Without aureomycin
(a) Cobb-Douglas

(8.50) 34-75 lbs.: $\frac{\delta C}{\delta P} = .543 \frac{C}{P}$ (8.52) 150-200 lbs.: $\frac{\delta C}{\delta P} = .089 \frac{C}{P}$

(8.51) 75-150 lbs.: $\frac{\delta C}{\delta P} = .211 \frac{C}{P}$ (8.53) over-all: $\frac{\delta C}{\delta P} = .338 \frac{C}{P}$

(b) Quadratic

$$(8.54) \quad \frac{\delta C}{\delta P} = -.5294 \pm \frac{-90.465 - .00325P}{\sqrt{-.000000552P^2 - .000307P + 8.297 - .000340Y}}$$

Comparison of Estimates From Different Functions

The main purpose of this study was to derive average rates of substitution of specified weight intervals in pork production. It was not to establish the most efficient equation for estimating all specified quantities which might be derived from a pork production surface. However, the data do provide a basis for some comparisons. A few comparisons of product contours and input-output curves are provided below.

Figures 8.1 and 8.2 show outputs predicted for 16 and 14 per cent protein rations, respectively, by the Cobb-Douglas and quadratic production function equations for experiments 536 and 554. Figures 8.3 and 8.4 allow comparisons between four forms of functions for these two rations under experiment 554. Because of its particular algebraic

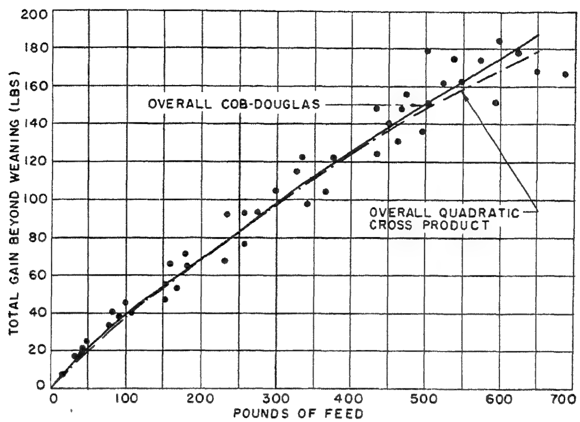

Figure 8.1. Feed-gain relationships for a 16 per cent protein ration derived from experiments 536 and 554 with aureomycin.

restraints, the quadratic cross-product equation tends to underestimate gains at heavy weight intervals. Otherwise, the remaining equations are quite similar. The Cobb-Douglas equation appears to fit the data as well as any of the other forms, probably because marginal feed productivity was relatively constant at heavy weights and the hogs were not carried to levels approaching maximum weights and sharply declining feed productivity.

Table 8.3 provides comparisons of pork gains predicted from Cobb-Douglas and quadratic production functions under experiments 536 and 554 pooled with aureomycin. While they are not presented in the text, interval functions were derived for the quadratic equation as well as

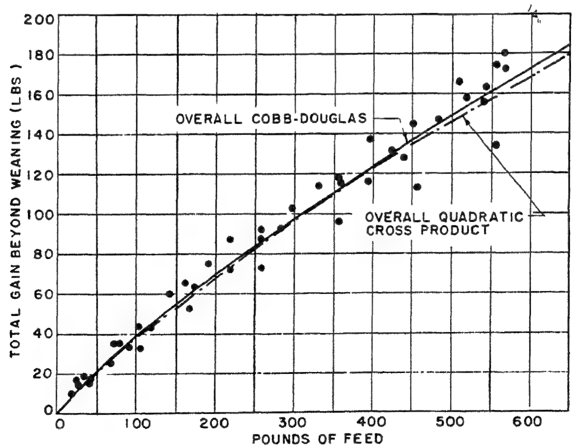

Figure 8.2. Feed-gain relationships for a 14 per cent protein ration derived from experiments 536 and 554 with aureomycin.

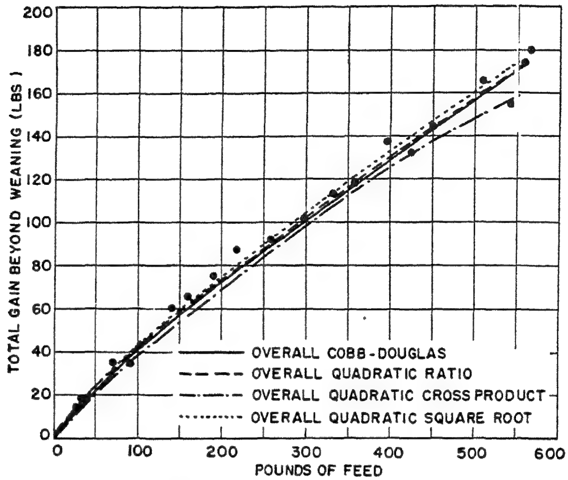

Figure 8.3. Feed-gain relationships for a 14 per cent protein ration derived from experiment 554 with aureomycin.

for the Cobb-Douglas equation. Estimates from both over-all and interval functions in attaining pig gains of 60, 110, and 175 pounds are given in Table 8.3.

As these data indicate, the interval quadratic equations provide estimates which are quite similar to those of the Cobb-Douglas interval function, especially for gain levels of 60 and 175 pounds. The predictions by interval functions were even more similar for the Cobb-Douglas as compared to the ratio and square root equations of 8.31 through 8.34. The Cobb-Douglas and quadratic over-all equations

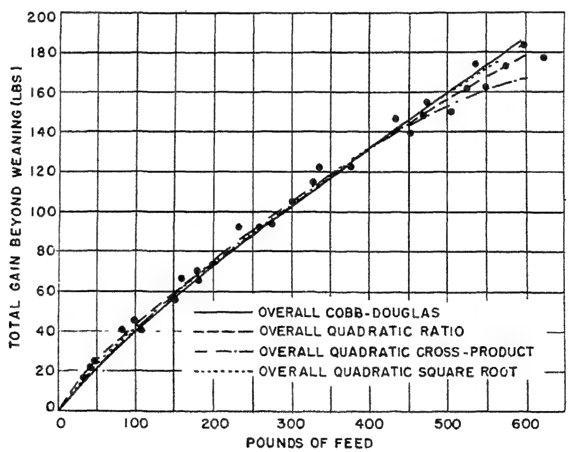

Figure 8.4. Feed-gain relationships for a 16 per cent protein ration derived from experiment 554 with aureomycin.

Table 8.3. Comparisons of Feed Combinations to Produce 100-Pound Gain for 60-, 110-, and 175-Pound Pigs From Four Alternative Functions; Experiments 536 and 554 Pooled and With Aureomycin

Per Cent Protein	Over-all Cobb-Douglas		Interval Cobb-Douglas		Over-all Quadratic		Interval Quadratic	
	Pounds corn	Pounds SBOM	Pounds corn	Pounds SBOM	Pounds corn	Pounds SBOM	Pounds corn	Pounds SBOM
	60-pound pigs							
10	268.9	13.1	326.1	15.9	263.9	12.8	312.7	15.2
11	231.4	21.1	260.4	23.7	246.0	22.4	262.0	23.9
12	218.9	25.2	239.7	27.6	237.2	27.3	242.7	27.9
13	203.4	31.8	214.8	33.5	223.3	34.9	217.8	34.0
14	194.1	36.9	200.3	38.0	213.0	40.5	202.4	38.4
15	182.9	44.6	183.2	44.7	198.7	48.4	183.2	44.7
16	177.8	49.1	175.2	48.4	191.0	52.7	173.7	47.9
17	169.7	56.6	163.8	54.6	181.2	60.4	160.7	53.6
18	165.0	61.9	157.0	58.9	171.1	64.1	152.9	57.3
19	157.7	71.5	146.7	66.5	158.1	71.6	140.8	63.8
20	154.8	75.8	142.7	69.9	152.7	74.8	137.1	67.2
	110-pound pigs							
10	376.7	18.3	331.9	16.1	327.5	15.9	304.2	14.8
11	324.2	29.6	300.9	27.4	304.6	27.8	291.5	26.6
12	306.7	35.3	290.2	33.4	293.7	33.8	284.7	32.7
13	285.0	44.5	276.6	43.2	277.3	43.3	274.1	42.8
14	272.0	51.7	268.3	51.0	265.6	50.4	266.1	50.5
15	256.3	62.5	258.0	62.9	249.7	60.9	254.5	62.0
16	248.8	68.6	253.1	69.8	241.5	66.7	248.1	68.5
17	237.8	79.3	245.7	81.9	228.5	76.2	237.3	79.1
18	231.2	86.7	241.2	90.4	220.3	82.6	230.4	86.4
19	221.0	100.1	234.2	106.1	207.0	93.8	218.4	99.0
20	216.9	106.3	231.3	113.4	201.4	98.7	213.3	104.5
	175-pound pigs							
10	441.0	21.5	364.0	17.7	398.5	19.4	343.1	16.7
11	379.4	34.6	342.5	31.2	368.3	33.6	336.2	30.6
12	359.0	41.3	334.8	38.5	355.5	40.9	332.6	38.2
13	333.6	52.1	325.0	50.8	338.4	52.8	326.6	51.0
14	318.4	60.5	318.9	60.6	327.3	62.2	320.9	61.0
15	299.9	73.1	311.2	75.9	315.2	76.8	313.5	76.4
16	291.2	80.3	307.5	84.9	310.3	85.6	309.1	85.3
17	278.2	92.7	301.9	100.6	304.0	101.3	301.3	100.4
18	270.6	101.4	298.5	111.9	301.6	113.1	296.0	110.9
19	258.7	117.2	293.0	132.8	304.3	137.9	286.2	129.7
20	253.8	124.4	291.0	142.5	307.6	150.7	283.6	139.0

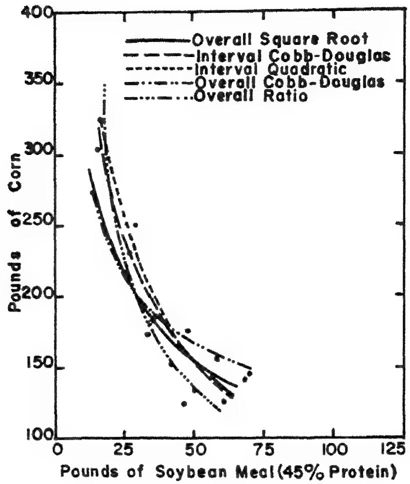

Figure 8.5. Comparison of 60 pound pig contours (showing corn-soybean meal combinations for producing 100 pounds of gain) predicted from six different production functions. Experiment 554 with aureomycin.

provide similar estimates for the 60-pound gain level, but have much greater differences for the 110- and 175-pound gain levels. Again over-all equations by the Cobb-Douglas function gave gain predictions which were much more similar to those of equations 8.31 through 8.34.

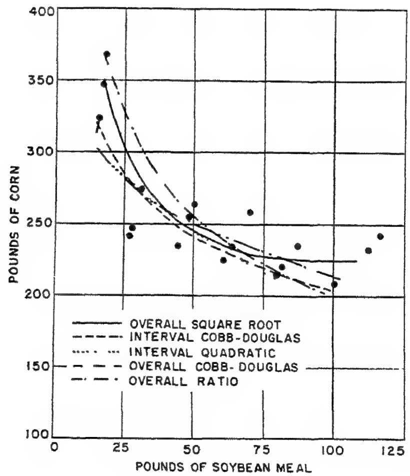

Figure 8.6. Comparison of 110 pound pig contours (showing corn and soybean meal combinations for 100 pounds of gain) predicted from six different production functions. Experiment 554 with aureomycin.

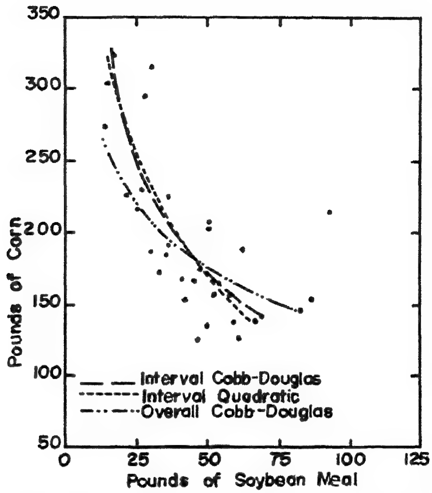

Figure 8.7. Comparison of 60 pound pig contours (showing corn-soybean meal combinations for producing 100 pounds of gain) predicted from four production functions. Experiments 536 and 554 pooled, with aureomycin.

Some comparisons of gain isoquants predicted from different equations are provided in figures 8.5, 8.6, and 8.7. The interval Cobb-Douglas functions provide gain isoquants for all three weight levels which are, on the average, more similar to selected other functions estimated in an over-all manner, than are those from the over-all Cobb-Douglas function. (The data shown in the figures are the means of lot treatments within the earlier stated weight intervals, from which the regression equations were fitted. However, the regression equations were fitted to the individual observations, rather than to the means of these.) The comparisons in these figures are only for experiment 554 with aureomycin. The Cobb-Douglas interval functions and the other over-all functions correspond even more closely for experiments 536 and 544 pooled. Based on these and other comparisons, it was decided that the Cobb-Douglas interval functions might serve best for estimating average marginal rates of substitution over the three intervals. Again, average rates of substitution over intervals were desired because farmers typically feed the same ration over a period of time; rather than changing it with each unit gain of pigs. Hence, the main objectives of this study are continued using the Cobb-Douglas interval functions. Also, certain other comparisons between rations are made using either these functions or the over-all Cobb-Douglas function. This function is used, not because it is most appropriate for all phases of analysis, but because (a) the function does appear to conform with the data about as well as other over-all functions and (b) use of the same type of function throughout provides a "standard" in comparisons.

COMPARISON OF RATIONS

Pigs fed aureomycin evidently had somewhat higher marginal feed gains than those without this additive, on the basis of production functions estimated by Cobb-Douglas and other over-all production functions. Gain levels relative to feed input levels for three levels of protein are indicated in Table 8.4. These comparisons are from the over-all power function when aureomycin rations are compared. Values of t, based on a number of degrees of freedom representing all observations in the study, are included in Table 8.5 for comparisons of regression coefficients between interval production functions. Even considering a much smaller number of degrees of freedom, these differ significantly. However, based on the number of degrees of freedom representing all observations in the study, it is not possible to say that regression coefficients for the production functions differ significantly. (See Table 8.6.) Since most protein feed supplements for hogs now contain antibiotics, the functions with aureomycin will be emphasized in certain of the analyses and comparisons which follow. At the time of the study, the cost of the antibiotic added was about 60 cents per 1,000 pounds of feed.

Table 8.4. Comparison of Total Product From Weaning From 50-Pound Feed Increments With and Without Aureomycin, Derived From Over-all Cobb-Douglas Equations for Experiments 536 and 554 Pooled*

Pounds Feed	12 Per Cent Protein		14 Per Cent Protein		16 Per Cent Protein	
	W	WO	W	WO	W	WO
50	20.8	20.3	21.8	21.3	22.1	21.8
100	37.1	35.9	38.8	37.8	39.5	38.5
150	52.0	50.1	54.5	52.7	55.4	53.8
200	66.1	63.7	69.3	66.8	70.4	68.2
250	79.7	76.2	83.5	80.3	84.9	81.9
300	92.8	88.6	97.2	93.3	98.8	95.2
350	105.5	100.6	110.6	105.9	112.4	108.1
400	118.0	112.3	123.7	118.2	125.7	120.6
450	130.2	123.7	136.4	130.3	138.7	132.9
500	142.2	134.9	149.0	142.1	151.4	145.0
550	154.0	145.9	161.3	153.7	164.0	156.8
600	165.6	156.8	173.5	165.1	176.3	168.4
650	177.0	167.4	185.5	176.3	188.5	179.9
700	188.3	178.0	197.3	187.4	200.6	191.2
750	199.5	188.4	209.0	198.4	212.5	202.4
800	210.5	198.7	220.6	202.2	224.3	213.4

*W = with aureomycin; WO = without aureomycin.

Table 8.5. Values of t for Difference between Over-all Cobb-Douglas Regression Coefficients With and Without Aureomycin, Experiments 536 and 554.

Weight Range	Protein Coefficient	Corn Coefficient
34-75	.37	.13
75-150	1.08‡	.13
150-200	.59	1.82*
over-all	.71	1.43†

*5 < p < 10;
†10 < p < 20;
‡p > 40; based on all observations in study.

Marginal Feed Productivity

The equations of marginal feed productivity for the Cobb-Douglas over-all function applied to rations including aureomycin are:

$$\text{(8.55)} \qquad \text{Corn: } \frac{\delta Y}{\delta C} = .871P^{.200}C^{-.364}$$

$$\text{(8.56)} \qquad \text{Soybean meal: } \frac{\delta Y}{\delta P} = .274C^{.636}P^{-.800}$$

Based on the Cobb-Douglas over-all function, the total and marginal gains for aureomycin are presented in Table 8.7. In these data the ration has been held constant in the sense that corn and soybean meal are always fed in the same proportions. Only data for aureomycin are shown in the text figures since it appears that this antibiotic is economical to include in the ration (in the quantities of this study) at any prospective price level. Even with no increase in an "average year," this small investment is likely good insurance against certain diseases.

The data show that the marginal productivity of feed declines as more of it is fed, in a constant-proportion ration. For example, the

Table 8.6. Values of t for Testing Interval Regression Coefficients Against Each Other — Experiments 536 and 554 Pooled — Cobb-Douglas Functions*

Experiment and Comparison	With Aureomycin		Without Aureomycin	
	Protein	Corn	Protein	Corn
34-75 lbs. vs. 75-150 lbs. . .	6.67	9.09	5.56	7.14
34-75 lbs. vs. 150-200 lbs.. .	9.09	11.11	10.00	11.11
75-150 lbs. vs. 150-200 lbs. .	2.94	2.78	3.85	5.88

*Significance level for all figures is 0 < p < 5, based on all observations in study.

Table 8.7. Total Gain Beyond Weaning and Marginal Gain for Different Feed Levels per Pig With Rations in Fixed Proportions; (Gain Over Beginning Weight of 34 Pounds) Experiments 536 and 554 Pooled With Aureomycin

Pounds of Feed After Weaning	Total Gain						Marginal or Additional Gain per Pound Added Feed					
	Per cent protein in ration						Per cent protein in ration					
	10	12	14	16	18	20	10	12	14	16	18	20
50	18.41	20.77	21.77	22.12	22.12	21.83	.3682	.4154	.4353	.4425	.4423	.4367
100	32.85	37.06	38.84	39.47	39.46	38.96	.2888	.3258	.3414	.3470	.3469	.3425
150	46.10	52.00	54.49	55.39	55.37	54.66	.2649	.2988	.3131	.3182	.3182	.3141
200	58.62	66.13	69.30	70.43	70.41	69.51	.2504	.2825	.2961	.3009	.3008	.2970
250	70.63	70.68	83.50	84.87	84.84	83.76	.2402	.2710	.2840	.2886	.2886	.2849
300	82.25	92.79	97.23	98.83	98.80	97.54	.2324	.2622	.2748	.2792	.2792	.2753
350	93.55	105.54	110.59	112.41	112.38	110.94	.2261	.2551	.2672	.2716	.2716	.2680
400	104.59	117.99	123.65	125.68	125.64	124.04	.2207	.2490	.2612	.2654	.2652	.2620
450	115.41	130.19	136.43	138.67	138.63	136.86	.2164	.2440	.2556	.2598	.2598	.2564
500	126.03	142.17	148.98	151.43	151.38	149.45	.2124	.2396	.2510	.2552	.2550	.2518
550	136.47	153.95	161.33	163.98	163.93	161.84	.2088	.2356	.2470	.2510	.2510	.2478
600	146.76	165.56	173.49	176.34	176.29	174.04	.2058	.2322	.2432	.2472	.2472	.2440
650	156.91	177.01	185.49	188.54	188.48	186.08	.2030	.2290	.2400	.2440	.2438	.2408
700	166.93	188.31	197.34	200.58	200.52	197.96	.2004	.2260	.2370	.2400	.2408	.2376
750	176.84	199.48	209.04	212.48	212.41	209.70	.1982	.2234	.2340	.2380	.2378	.2348
800	186.63	210.53	220.62	224.25	224.18	221.32	.1958	.2210	.2316	.2354	.2354	.2324

"50th pound" of feed under a 10 per cent ration adds .368 pound to hog weight; at 150 pounds of total feed, "one more pound" adds .265 pound to hog weight; at 400 pounds the added gain is only .221; while at 800 pounds it is only .196 pound. With a ration of 14 per cent protein the added or marginal gain per added pound of feed is .435, .313, .261, and .232 pound for total feeding levels of 50, 150, 400, and 800 pounds, respectively.

The same relationship is also shown in the total gain figures. The total weight increases by smaller and smaller increments as feed intake per hog is increased. Of particular importance, however, is the difference between protein rations in total gain for a given feed intake. With a feed input of 400 pounds, total gain is 105 pounds under a 10 per cent ration but 124 pounds under a 14 per cent ration and 126 pounds under a 16 per cent ration. Total gain for a given feed input under an 18 per cent ration is practically the same, or slightly lower, than for a 16 per cent ration. The predicted total gain is less for a 20 per cent ration than for a 16 per cent ration. Since soybean meal costs more than corn, it cannot be fed profitably in quantities giving 18 or 20 per cent protein in the ration: for any given total feed intake per hog, total gain and value of hog decreases, and total cost of feed increases. These problems can be better solved, however, from the substitution data of later sections. How far the ration should be extended between 14 and 16 per cent protein depends on the substitution and price ratios for the two feeds.

Average Elasticities

Diminishing marginal transformation of feed into pork is apparent in the exponents or regression coefficients in the Cobb-Douglas production functions. The exponent for either corn or protein in these equations (i.e., the regression coefficients for the data in logarithmic form) can be termed the production elasticities.

They are average elasticities since they indicate, as constants over the surface when estimated by a Cobb-Douglas function, the percentage increase in weight for each 1 per cent increase in feed intake per pig. If these quantities are 1.0, they indicate that a 1 per cent increase in the amount of feed increases hog weight by 1 per cent — feed has a constant added productivity. In all cases, however, these quantities are less than 1.0 for either corn or protein denoting (1) that if each feed is increased alone by 1 per cent, weight will increase by less than 1 per cent, and hence (2) the marginal production (the amount *added* to total weight or production from an *added* pound of feed) becomes smaller and smaller as feed intake per hog increases.

The 536 and 554 pooled production functions show that for 34-75 pound pigs with aureomycin, corn has an elasticity of .533 — a 1 per cent increase in the amount of corn fed will increase pig weight by only .533 per cent. Similarly, the elasticity for soybean oilmeal is .297, indicating that this percentage increase in weight can be expected from a 1 per cent feed increase. For all of the Cobb-Douglas regression coefficients presented, the quantity is less than 1.0 for either corn or soybean meal alone.

The sums of the exponents for corn and soybean meal suggest the general nature of the input-output or feed transformation ratio as they are fed in a ration of fixed proportions (i.e., a ration of say 16 per cent protein fed from one weight to another). A sum of 1.0 indicates that a 1 per cent increase in the two feeds, always held in constant proportions, will result in a 1 per cent increase in weight or pork output. For all of the logarithmic functions derived, however, the sum of the exponents is less than 1.0. This indicates that on the average, as more feed is fed, always with corn and protein in the same proportion, the hog will gain weight — but at a diminishing marginal rate. In the over-all function for 536-554 pooled with aureomycin, the sum of the elasticities is .836 (.200 + .636) denoting that a 1 per cent increase in feed intake increases hog weight only .836 per cent.

In the Cobb-Douglas equation, and given the average nature of the elasticities so defined, three tendencies are evident and were expected from previous nutritional research:

1. The elasticity coefficient for protein declines with higher weight intervals. This phenomenon is expected since a small pig is more dependent on protein for growth than a heavier one.
2. The elasticity for corn increases between weight intervals; a phenomenon also expected since a mature hog needs more carbohydrate feeds in fattening.

Table 8.8. Summary of Elasticity Coefficients From Pooled Data for Logarithmic Functions*

Interval and Experiment	Corn Elasticity	Protein Elasticity	Coefficient of Substitution
1. 34-75 pounds			
536-554 pooled WA	.533	.297	.557
536-554 pooled WOA.	.529	.287	.543
536-554-506 pooled WA.	.530	.303	.572
536-554-506 pooled WOA.	.514	.289	.562
Average of 4 for 34-75 pounds . .	.523	.294	.557
2. 75-150 pounds			
536-554 pooled WA.	.767	.142	.185
536-554 pooled WOA.	.770	.163	.211
536-554-506 pooled WA.	.762	.136	.178
536-554-506 pooled WOA.	.805	.158	.197
Average of 4 for 75-150 pounds. .	.776	.150	.193
3. 150-200 pounds			
536-554 pooled WA.	.856	.092	.108
536-554 pooled WOA.	.916	.081	.089
536-554-506 pooled WA.	.849	.110	.130
536-554-506 pooled WOA.	.907	.090	.099
Average of 4 for 150-200 pounds.	.882	.093	.107

*WA indicates "with aureomycin" and WOA means "without aureomycin."

3. The sum of the elasticities increases between weight intervals. A suggestion of declining feed productivity is the decline in the production function constants between weight intervals.

The clear tendency for gain elasticities of oilmeal to decline with weight are shown in Table 8.8. For example, over the weights 34-75, 75-150, and 150-200 pounds, the average of all elasticities computed were, respectively, .294, .150, and .093. In contrast, the corresponding elasticities for corn were .523, .776, and .882. (See Table 8.8.) Similarly, the coefficients of substitution of soybean meal for corn decline consistently between weight intervals, as estimated by the interval functions. As an average for all interval functions estimated, these coefficients (Table 8.8) are .557, .193, and .107 for the weight ranges 34-75, 75-150, and 150-200 pounds. (The coefficients in column 3 of Table 8.8 are not the complete derivatives of the isoquant equations, but are only the multipliers of the feed ratios as indicated in equations 8.45 through 8.53.) Similar changes in elasticity and substitution coefficients are apparent from other algebraic forms of over-all equations. However, those for the Cobb-Douglas interval equations provide quantities which might be looked upon as averages for these weight ranges.

Previous nutritional research has indicated that the rate at which protein replaces corn decreases as the weight of the hog increases. This condition is, of course, inherent in (a) the coefficients of the logarithmic functions presented above and (b) the signs of the coefficients in the quadratic equations for the pooled data. That protein and corn substitute for each other at a diminishing marginal rate, with hog weight at a fixed level, is inherent in the derived equations for the same reason; it is obvious in the Cobb-Douglas equations, equations 8.35 through 8.43, since a constant value of Y is divided by an increasingly larger quantity relating to protein (as protein is raised to the relevant power). These

same conditions are reflected in the coefficients for the equations of marginal substitution rates. The decline in the substitution rate (of soybean meal for corn) is greater, of course, with higher weight intervals than these interval coefficients suggest.

The reason is this. As the hog grows to heavier weights, a greater amount of feed is required to produce a given gain; for any one level of protein feeding, more corn is required to produce the given gain. Hence the C/P ratio, as well as the substitution coefficient itself, declines for heavier hogs in the equations of marginal substitution rates.

CORN AND SOYBEAN MEAL SUBSTITUTION RATES

We now turn to the major objective of this study, namely, the estimation and use of average rates of feed substitution over selected weight intervals. The equations of isoquants and marginal substitution rates, based on the interval Cobb-Douglas functions, have been used to derive the relevant quantities, expressed per 100 pounds of gain, in Table 8.9. The first two columns under each weight show the various combinations of soybean oilmeal and corn which will produce 100 pounds of pork.[3] They may be looked upon as providing the substitution possibilities over the three weight ranges. More specifically they are calculated as the substitution possibilities within the ranges, but at exactly 60, 110, and 175 pounds.

The figures again show that the amount of corn required to produce 100 pounds of gain decreases as more protein is fed and vice versa. In other words, they show the many possible combinations of feed which will produce a given amount of pork for pigs of three different weights.

The marginal substitution rates show the amount of corn replaced by 1 pound of soybean oilmeal *at* each of the combinations shown in the first two columns. These substitution rates are of a diminishing marginal nature; each added pound of protein substitutes for less corn than the previous pound in producing a given amount of gain. For 60-pound pigs (pigs in the 34-75 pound range) 1 pound of soybean oilmeal substitutes for 12.5 pounds of corn when the ration includes 15 pounds of soybean oilmeal and 337 pounds of corn; it substitutes for only 4.3 pounds of corn when the feed combination is 30 pounds of soybean oilmeal and 229 of corn; with 75 pounds of soybean oilmeal and 137 pounds of corn, 1 pound of protein replaces only a pound of corn.

Diminishing substitution rates also hold true for weights of 110 and 175 pounds. Thus the rate of replacement of corn by protein declines as the growing stage merges into the fattening stage. In Table 8.9 with 30 pounds of soybean oilmeal in the ration, 1 pound of soybean oilmeal for a 60-pound pig replaces 4.3 pounds of corn; it replaces only 1.8 pounds of corn for a 110-pound pig, and 1.2 pounds for a 175-pound pig.

[3] The feed quantities were first derived for the gains necessary to take a pig from the beginning of the specific interval to the weight indicated. Next, these were transformed to the equivalent of a 100-pound gain since this is the manner in which many persons consider rations.

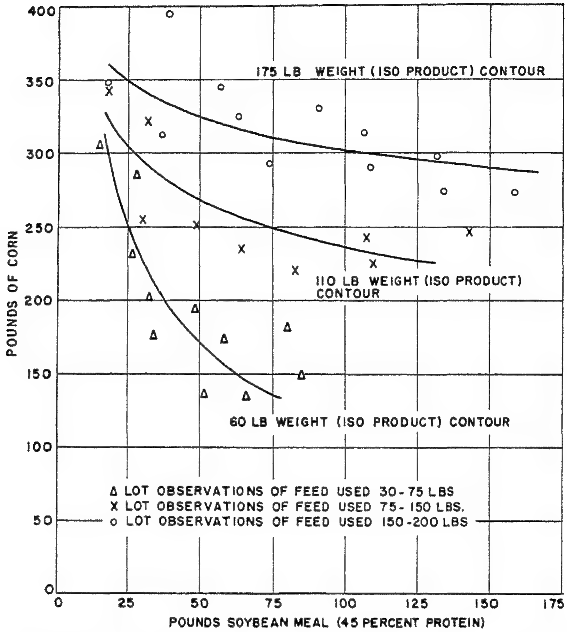

Figure 8.8. Pork isoquants showing the combination of corn and soybean meal which will produce 100 pounds of gain for pigs of 60, 110 and 175 pounds. Derived from interval Cobb-Douglas equations. (The lot observations are used only to suggest the nature of the fit for average gain data over each interval. The functions were not fitted to these single lot observations but to many more observations scattered over each specific interval.)

These differences in substitution rates are also reflected in the pork isoquants shown in Figure 8.8. These curves, drawn from the data in Table 8.9, show that, for a given gain, as the rate at which soybean oilmeal substitutes for corn increases, the slope of the isoproduct curve also increases, and vice versa (i.e., the slope of the curve declines as more protein and less corn is fed to get a given gain). The slopes of the curves also decline as the pig becomes heavier. This indicates that the rate at which soybean oilmeal substitutes for corn decreases as the pig increases in weight.

The isoquants in Figure 8.8 are not all measured from the 34-pound weight at the beginning of the experiment. Only the lower contour is so measured and represented in the figure. The other two are measured relative to the beginning of their respective weight intervals, with feed inputs representing those beyond this level. In this sense, the three isoquants do not express a production function represented by a single "continuous" family of product contours.

Table 8.9. Feed Combinations to Produce 100 Pounds of Gain and Feed Replacement Rates With Aureomycin for Hogs of 60 Pounds, 110 Pounds, and 175 Pounds. Experiments 536 and 554 Pooled

60-Pound Pigs				110-Pound Pigs				175-Pound Pigs			
Feed quantities to produce 100 pounds gain		Marginal substitution rate of soybean meal for corn $\frac{\delta C}{\delta P}$; pounds corn replaced by 1-pound SBOM	Protein in total ration (per cent)	Feed quantities to produce 100 pounds gain		Marginal substitution rate of soybean meal for corn $\frac{\delta C}{\delta P}$; pounds corn replaced by 1-pound SBOM	Protein in total ration (per cent)	Feed quantities to produce 100 pounds gain		Marginal substitution rate of soybean meal for corn $\frac{\delta C}{\delta P}$; pounds corn replaced by 1-pound SBOM	Protein in total ration (per cent)
pounds SBOM	pounds corn			pounds SBOM	pounds corn			pounds SBOM	pounds corn		
10	421.7	23.51	9.1	10	356.8	6.61	9.3	10	387.0	4.17	9.2
15	336.5	12.50	9.8	15	336.3	4.15	9.8	15	370.5	2.66	9.7
20	286.7	7.99	10.6	20	319.0	2.95	10.4	20	359.3	1.94	10.2
25	253.1	5.64	11.5	25	306.0	2.27	11.0	25	350.7	1.51	10.7
30	228.7	4.25	12.4	30	295.0	1.83	11.6	30	342.0	1.24	11.2
35	209.8	3.34	13.4	35	297.5	1.52	12.2	35	338.2	1.04	11.6
40	194.8	2.71	14.4	40	280.6	1.30	12.7	40	333.3	.90	12.1
45	182.4	2.26	15.3	45	274.6	1.13	13.3	45	329.1	.79	12.6
50	172.0	1.92	16.3	50	269.2	1.00	13.9	50	325.4	.62	13.0
55	163.1	1.65	17.3	55	264.4	.89	14.4	55	322.2	.59	13.5
60	155.4	1.44	18.2	60	260.2	.80	15.0	60	319.0	.57	13.9
65	148.6	1.27	18.5	65	256.4	.73	15.5	65	316.4	.52	14.4
70	142.6	1.14	20.0	70	252.9	.67	16.0				
75	137.2	1.02	20.9								

Minimum Cost Rations

The substitution data from Table 8.9 provide the necessary information for specifying the ration which will give the lowest cost for any given gain. Since the substitution or replacement rates are of a declining rather than a constant nature, a different ration is required with each change in the corn and soybean meal price ratio to give the least-cost gains. Feed costs for a given gain are lowest when the substitution ratio equals the price ratio. Hence with a soybean oilmeal price of 4.52 cents per pound and a corn price of 2.0 cents, a price ratio of 4.52/2.0 or 2.26, specifies that a combination of 45 pounds of soybean meal and 182 pounds of corn gives the lowest feed cost for 60-pound pigs; the marginal rate of substitution in Table 8.9 is also 2.26 for this feed combination. With a soybean meal price of 3.34 cents per pound and a corn price of 1.0 cent, the price ratio of 3.34/1.0 or 3.34 specifies that 35 pounds of soybean oilmeal and 210 pounds of corn gives the lowest feed cost for pigs of this weight. At this point the marginal rate of substitution of soybean oilmeal for corn is also 3.34.

Table 8.10 has been provided to simplify computation of the combination which will give the lowest feed cost for a given gain. In effect, it specifies the feed combination at which the marginal rate of substitution equals the price ratio being examined. One need only divide the soybean oilmeal price or cost per pound (including vitamins, antibiotics, and other ingredients) by the price of corn per pound. This ratio (or the ratio nearest to it) can be located in the first column of Table 8.10. Then the least-cost ration can be found along the same line, under the appropriate weight. Suppose that soybean oilmeal costs 4.4 cents per pound and corn is 2 cents per pound. Since the ratio is 2.2, a ration with 45.8 pounds of soybean oilmeal and 180.7 pounds of corn or 15.5 per cent protein will give the lowest feed cost for 60-pound pigs; an 11.1 per cent protein ration — for this same price ratio — will give lowest feed costs for 110-pound pigs and a 10.0 per cent protein ration will give lowest costs for 175-pound pigs. (Actually, the rations can be applied throughout the weight ranges.)

From 1936-52, Iowa prices averaged $1.74 and $3.40 per 100 pounds of corn and soybean meal, respectively. Under this average yearly price ratio of 1.96 the rations which give lowest feed costs for 60-, 110-, and 175-pound hogs have 16.0, 11.3, and 10.1 per cent protein, respectively. If a single ration were fed from weaning to marketing weight (34-225 pounds) under this price ratio, one having 14 per cent protein would give lowest feed costs per 100 pounds of gain. The price ratio fluctuates between years, of course. In the 15 years between 1938 and 1953, the ratio was as high as 2.8 and as low as 1.3 in Iowa.

TIME OF GAINS

Minimum feed cost, while an important consideration in maximizing pork profits, is only one facet of the economic problem. Also to be

Table 8.10. Least-Cost Ration and Feed Quantities for Stated Price Ratios; Based on Cobb-Douglas Interval Functions With Aureomycin — Pounds Feed per 100 Pounds Gain

Price ratio: Price per Pound SBOM / Price per Pound Corn	60-Pound Pig (34 to 75 Pounds)			110-Pound Pig (75 to 150 Pounds)			175-Pound Pig (150 to 200 Pounds)		
	Corn	SBOM	Per Cent Protein*	Corn	SBOM	Per Cent Protein*	Corn	SBOM	Per Cent Protein*
1.0	136.3	75.9	21.0	269.3	49.9	13.9	336.9	36.3	11.8
1.1	141.0	71.4	20.3	273.3	46.0	13.4	340.1	33.3	11.5
1.2	145.5	67.5	19.6	277.1	42.8	13.1	342.9	30.8	11.2
1.3	149.7	64.2	19.0	280.5	40.0	12.7	345.6	28.6	11.0
1.4	153.7	61.2	18.4	283.9	37.6	12.5	348.1	26.8	10.8
1.5	157.5	58.5	17.9	296.9	35.4	12.2	350.5	25.2	10.7
1.6	161.3	56.2	17.5	289.9	33.6	12.0	352.7	23.7	10.6
1.7	164.8	54.0	17.1	292.6	31.9	11.8	354.7	22.5	10.4
1.8	168.1	52.0	16.7	295.1	30.4	11.6	356.7	21.3	10.3
1.9	171.4	50.3	16.4	297.7	29.0	11.5	358.6	20.3	10.2
2.0	174.7	48.7	16.0	300.1	27.8	11.3	360.4	19.4	10.1
2.1	177.7	47.2	15.8	302.4	26.7	11.2	362.1	19.6	10.0
2.2	180.7	45.8	15.5	304.6	25.7	11.1	363.7	17.8	10.0
2.3	183.6	44.5	15.2	306.8	24.7	11.0	365.3	17.1	9.9
2.4	186.4	43.3	15.0	308.8	23.8	10.9	366.8	16.5	9.8
2.5	189.2	42.2	14.8	310.8	23.0	10.8	368.3	15.9	9.8
2.6	191.9	41.1	14.6	312.7	22.3	10.7	369.7	15.3	9.7
2.7	194.4	40.1	14.4	314.6	21.6	10.6	371.1	14.8	9.7
2.8	197.0	39.2	14.2	316.3	20.9	10.5	372.3	14.3	9.6
2.9	199.5	38.3	14.0	318.1	20.3	10.4	373.7	13.9	9.6
3.0	201.9	37.5	13.9	319.8	19.7	10.4	374.9	13.5	9.5
3.1	204.3	36.7	13.7	321.4	19.2	10.3	376.1	13.1	9.5
3.2	206.6	36.0	13.6	323.0	18.7	10.3	377.3	12.7	9.5
3.5	213.4	34.0	13.2	327.5	17.3	10.1	380.6	11.7	9.4
4.0	223.8	31.2	12.7	334.5	15.5	10.0	385.5	10.4	9.3

*To determine per cent protein in the total ration, soybean oilmeal of 45 per cent protein and corn of 8.4 per cent protein have been used.

considered are labor and other capital costs. However, as has been pointed out elsewhere, feed represents such a major portion of total pork production costs that any reduction in feed costs usually lowers total expenses. An additional part of the feeding problem is that of time. Gain comes slowly for small pigs which receive only small amounts or no supplemental protein. Because of low daily gains, the production period is extended accordingly and the selling price may be increased or decreased from different rations depending on whether faster gains put the pigs on the market in a higher or lower price period.

Hence, the farmer must consider the price to be received for his hogs as well as the feed cost per pound of gain in deciding which ration is most profitable. If a higher percentage of protein in the ration increases cost per 100 pounds of gain but allows an even greater increase in the price received for pork, the most profitable ration will not be the one which gives the minimum feed cost.

Time Functions and Daily Gains

Since time and rates of gain are necessary along with substitution rates and feed costs — to specify the most profitable ration, the three total time functions shown below have been derived.

$$(8.57)\quad T = 6.958 + .038P\ddagger + .291C + .00197P^2 - .000093C^2 - .001073PC$$

$$(8.58)\quad T = 3.969 + .083P\ddagger + .246C + .000078P^2\ddagger - .000123C^2 + .735C/P$$

$$(8.59)\quad T = -9.837 + .690P + .167C - .432\sqrt{P} + 4.264\sqrt{C} - .590\sqrt{PC}$$

In these equations, total time (T) to consume a specified quantity of feed is considered a function of the amount of corn and protein fed.[4] Total time rather than daily rate of gain has been predicted since we are interested in estimating the time required to reach marketing weights under different rations.

With knowledge of the weights attained from different rations and amounts of feeds, information on the total time required to attain these weights and consume the corresponding amount of feed makes possible the determination of the effect of different rations in putting hog marketings into periods of high or low prices. An equation for predicting the change in the daily rate of gain in relation to the amount of protein fed was computed as follows where D refers to daily rate of gain in pounds:

$$(8.60)\quad D = \frac{1.369P^{.200}C^{.636}}{-9.837 + .690P + .167C - .432\sqrt{P} + 4.264\sqrt{C} - .590\sqrt{PC}}$$

Using these two over-all functions, the total gains in Table 8.7 divided by the total time in Table 8.11 gives the daily gain rates in Table 8.12. Where gains over different intervals are being analyzed, the production function for the particular interval, rather than for the over-all function, is divided by the corresponding time function. Total time as a function of feeds fed, rather than daily rate of gain, was predicted in the equations since our main interest is in predicting the time to reach market weights under different rations. From the daily gain equation it is possible to specify the change in the daily gain (the marginal daily gain) associated with change in the amount of protein fed — with total corn and total protein inputs at specified levels. The equation is shown below where the derivative specifies the change in the daily gain for each 1 pound change in the amount of protein fed:

$$(8.61)\quad \frac{\delta D}{\delta P} =$$

$$\frac{1.369C^{.636}(.0334C - 1.967 + .853C^{.5})P^{-.8} + .138P^{-.7} + .13P^{-.3} + .177C^{.5}P^{-.3} - .69P^{.2}}{(.69P + .167C - .432\sqrt{P} + 4.264\sqrt{C} - .59\sqrt{PC} - .9837)^2}$$

After examining the regression coefficients and their standard errors and after plotting the functions against the observations, it was

[4] In the equations the coefficients indicated by ‡ are not significant at the 20 per cent level of probability; those without asterisks are significant at the 5 or 1 per cent probability levels on the basis of all weight observations included in the study. The coefficients of determination for the three equations are .971, .966, and .973, respectively.

Table 8.11. Total Time to Consume Various Quantities of Feed for Rations of 10, 12, 14, 16, 18, 20 Per Cent Protein (Experiment 554 With Aureomycin and Equation 8.59 With Square Root Terms)

Pounds of Feed*	Total Days (To Feed Out Different Quantities of Feed) With Protein Levels of†					
	10 per cent	12 per cent	14 per cent	16 per cent	18 per cent	20 per cent
50	22.4	19.3	17.9	16.9	16.2	15.8
100	36.8	32.6	30.3	29.0	28.3	28.2
150	49.0	43.2	40.3	38.7	38.2	38.4
200	59.9	52.5	48.9	47.2	46.8	47.5
250	69.8	60.9	56.7	54.9	54.7	55.8
300	79.1	68.7	63.9	62.0	62.1	63.7
350	87.9	76.1	70.7	68.7	69.0	71.1
400	96.4	83.0	77.1	75.0	75.6	78.2
450	104.5	89.7	83.3	81.1	82.0	85.1
500	112.3	96.1	89.1	86.9	88.1	91.8
550	120.0	102.3	94.8	92.6	94.1	98.3
600	127.4	108.3	100.3	98.1	99.8	104.7
650	134.6	114.2	105.7	103.4	105.5	110.9
700	141.8	119.9	110.9	108.6	111.0	117.0
750	148.7	125.5	116.0	113.7	116.4	122.9
800	155.6	130.9	120.9	118.6	121.7	128.8

*Pounds of feed fed beyond weaning.
†Days beyond weaning.

decided that the square root function, equation 8.59, fits the data best. This function allows prediction of the amount of time associated with consumption of different quantities of feed under the various corn-protein rations. The figures in Table 8.11 indicate that time is decreased in the ration up to slightly beyond 16 per cent protein; further increases in protein lengthen the time period because a smaller proportion of carbohydrates is available for the fattening stage of the production process.

Using the total gain and the time functions from above, the figures of Table 8.12 can be computed. These show the average daily rate of gain for 50-pound feed intervals. These figures show greater daily rates of gain for small pigs with rations containing as much as 18 per cent protein. For pigs weighing 85 pounds, the higher protein percentage lowers the average daily rate of gain; a 16 per cent protein ration gives a greater average daily rate of gain beyond this weight. In other words, pigs receiving 18 per cent protein are predicted to have the greatest daily rate of gain at the outset but eventually those with a

Table 8.12. Average Daily Rates of Gain at Various Total Levels of Feed Intake for Different Protein Rations* (With Aureomycin)

Pounds of Feed	Average Daily Gain up to Time of Feed Level Shown for Protein Levels of					
	10 per cent	12 per cent	14 per cent	16 per cent	18 per cent	20 per cent
100	.894	1.137	1.281	1.362	1.392	1.382
150	.940	1.203	1.357	1.431	1.450	1.424
200	.979	1.259	1.417	1.493	1.504	1.465
250	1.012	1.308	1.473	1.547	1.551	1.501
300	1.040	1.350	1.521	1.595	1.592	1.532
350	1.064	1.387	1.564	1.637	1.629	1.560
400	1.085	1.421	1.603	1.675	1.661	1.585
450	1.105	1.451	1.639	1.710	1.691	1.608
500	1.122	1.479	1.671	1.742	1.718	1.628
550	1.138	1.505	1.701	1.771	1.743	1.646
600	1.152	1.528	1.729	1.798	1.766	1.663
650	1.165	1.550	1.755	1.824	1.787	1.678
700	1.178	1.571	1.780	1.847	1.807	1.693

*Daily gains are averages from weaning to the time necessary to consume the feed shown. They are not daily rates of gain at exactly the feed levels indicated.

16 per cent ration are expected to catch up — ending the production period with a higher average daily rate of gain.[5]

Table 8.13 shows the predicted daily rates of gain between 50-pound feed intervals. In contrast to Table 8.12 which showed the cumulative average, these data show the average gain rate only within each feed interval. In this form, daily gain rates for a 16 per cent ration surpass those of an 18 per cent ration except for the first 100 pounds of feed. Those for a 14 per cent protein ration are only slightly below those for a 16 per cent ration as feed intake and hog weights reach higher levels.

Of more direct importance to the farmer is the total time required to add different amounts of gain on hogs of specified weights. These figures are presented in Table 8.14 and are rounded to the nearest day, since the farmer does not refine his feeding and marketing decisions to fractions of days. For 34-75 pound pigs a gain from the beginning to the end of the weight interval is attained with minimum time when the ration reaches 18 per cent protein; over the weight interval 75-150

[5]An earlier experiment at Iowa State University (Catron, D. V. *et al.* Re-evaluation of protein requirements. Animal Science, 2(2): 221-32. 1952.) even showed that pigs on a 12 per cent protein ration, while lagging behind those with higher protein rations at the outset, eventually increased their daily gain to a point where it surpassed the rate for pigs receiving more protein. Pigs receiving 14 per cent protein reached a weight of 200 pounds as quickly as those getting 16 per cent protein.

Table 8.13. Average Daily Gain Within Feed Intervals for Different Percentages of Protein (With Aureomycin)

Pounds of Feed	Daily Gain Computed as an Average Between 50-Pound Feed Increments With Protein Levels of					
	10 per cent	12 per cent	14 per cent	16 per cent	18 per cent	20 per cent
100	1.003	1.229	1.371	1.432	1.431	1.383
150	1.080	1.404	1.575	1.638	1.618	1.540
200	1.154	1.522	1.713	1.775	1.740	1.637
250	1.208	1.610	1.818	1.879	1.829	1.706
300	1.249	1.678	1.902	1.962	1.899	1.758
350	1.282	1.740	1.972	2.030	1.956	1.799
400	1.309	1.787	2.034	2.089	2.003	1.834
450	1.333	1.830	2.084	2.138	2.044	1.861
500	1.353	1.868	2.131	2.183	2.079	1.885
550	1.369	1.900	2.173	2.222	2.111	1.906
600	1.385	1.931	2.209	2.258	2.138	1.923
650	1.398	1.958	2.244	2.291	2.162	1.939
700	1.409	1.981	2.275	2.311	2.185	1.951

pounds, minimum time is attained when the ration reaches 13 per cent protein; the corresponding figure is 13 per cent for the weight interval 150-225 pounds. For a single ration over the entire weight interval, minimum time is reached with a 15 per cent ration.[6]

Returns Considering Feed Costs and Time of Marketing

Maximum daily gain or minimum time to attain a given marketing weight is not a sufficient criterion for the most profitable hog ration. A shorter production period which throws marketings into a higher price period will lower profits if it increases the cost of feed more than the increased profit from a higher selling price. On the other hand, a quicker gain will lower profits if (1) it increases costs by more than price or (2) it moves hog marketings back into a period of lower prices while leaving costs unchanged (or higher).

[6]While other rations give time as low as some of those mentioned, interest generally is in the lowest protein ration which gets hogs to market in the shortest time period. The number of days for a specified gain are rounded in Table 8.14, to the nearest day. The following rations give a shorter time period than any other ration:

34-75 lbs.	27.8 days	20.3 per cent protein
75-150 lbs.	41.4 days	14.8 per cent protein
150-225 lbs.	35.4 days	14.4 per cent protein

Table 8.14. Total Days to Attain Specified Gains (With Aureomycin)

Per Cent Protein In Ration	Days Required Beyond Weaning To Take Pigs Between Weights of			
	34-225 lbs. (191 lbs. gain)	34-75 lbs. (41 lbs. gain)	75-150 lbs. (75 lbs. gain)	150-225 lbs. (75 lbs. gain)
10	151	51	53	46
11	127	41	46	40
12	120	38	44	38
13	113	34	42	36
14	111	33	42	36
15	108	31	42	36
16	108	30	42	36
17	110	29	43	38
18	112	28	44	40
19	118	28	47	44
20	122	28	48	46

Since seasonal prices are affected by movements in the general price level, the farmer should use price outlook material and balance the economy of timeliness against the cost of the ration. If national economic outlook suggests a break in prices, he may want to feed more protein and rush his pigs to market before the price decline, even if feed costs per 100 pounds of gain are increased. If continued inflation and rising prices are in prospect, he will wish to hold down daily rate of gain, particularly if a lower protein content ration results in lower feed costs.

Using the average prices for pork, corn and soybean oilmeal over the period 1937-52 and without considering expectations of falling or increasing prices due to national inflation or deflation, the data of Table 8.15 have been computed to show the relationship of prices received, time of gain, and feed cost to net return per hog. These data are for fall pigs weaned on November 1 and November 15.

The following figures result for a farmer who makes his decision for future marketings and the most profitable ration on the basis of corn and protein prices at weaning time; the average SBOM to corn price ratio on November 15 (for the period 1937-52) was 2.1. Pigs weaned on November 15 receiving a 16 per cent protein ration require the least time to attain a marketing weight of 225 pounds. The earlier marketing date of March 3 would not, however, have given the highest price. A 14 per cent ration with a marketing date of March 5 would have given the highest price per hundredweight. A 12 per cent protein would have given the highest net return above the cost of feed after weaning.

For pigs weaned on November 1, a 16 per cent protein ration again would have given the most rapid gains and the earliest marketing dates; a 12 per cent ration would have given the lowest cost for the gain after weaning, and a 20 per cent ration would have given the highest price for pork. The 12 per cent ration would have given the greatest return above feed costs after weaning. Hence, it is apparent that whether

Table 8.15. Days to Attain Market Weight of 225 Pounds, Price per Hundredweight and Costs and Returns From Different Protein Rations. Based on 1937-52 Average Prices for Pork at Marketing Date and Feed at Weaning Dates (With Aureomycin)

Per Cent Protein in Ration	Pounds of Feed To Gain 191 Pounds*		Days to Gain 191 Pounds Over 34-225 Pound Weight Interval*	Marketing Date	Average Price Per Hundred-weight	Gross Income Per Hog	Feed Cost Per Hog*	Net Income Over Feed Costs Per Hog After Weaning*
	Corn	Protein						
	pigs weaned November 1							
10	705	34	150	Mar. 31	14.53	32.68	13.44	19.24
12	610	70	120	Mar. 1	14.57	32.79	12.99	19.80
14	561	107	110	Feb. 19	14.52	32.67	13.39	19.28
16	529	146	108	Feb. 17	14.37	32.33	14.16	18.17
18	504	189	112	Feb. 21	14.50	32.63	15.18	17.45
20	483	237	122	Mar. 3	14.71	33.11	16.46	16.65
	pigs weaned November 15							
10	705	34	150	Apr. 14	14.26	32.08	13.44	18.64
12	610	70	120	Mar. 15	14.61	32.86	12.99	19.87
14	561	107	110	Mar. 5	14.74	33.17	13.39	19.78
16	529	146	108	Mar. 3	14.71	33.11	14.16	18.95
18	504	189	112	Mar. 7	14.73	33.15	15.18	19.97
20	483	237	122	Mar. 17	14.61	32.86	16.46	16.40

These data are based on a single low-cost ration over the entire weight range.

priority should be given to rations which maximize daily gains or to those which minimize feed costs depends upon the weaning date of the pigs, and the feed price ratio in relation to likely pork prices at marketing. Quite a range of rations can be used without changing feed costs per 100 pounds of gain.

Usually the farmer will want to consider different rations for hogs of different weights as suggested by the data in tables 8.9 and 8.10. These rations then can be related to weaning date and time in the manner indicated in Table 8.14. (Table 8.15 uses the same ration throughout the production period.) The farmer also has the opportunity to adjust his weaning and marketing dates by selecting different dates for breeding and farrowing.

The calculations in Table 8.15 are for average selling prices over the period 1937-52. Table 8.16 not only emphasizes that in different years pigs weaned at the same time may need different rations, but is also of some interest in appraising the risks involved in different rations. Since low protein rations require a longer period for growing and fattening a market hog, it might be said that more risk and uncertainty is involved. The longer period gives more time for prices to change and the further into the future the farmer must extend his price expectations, the less accurate he may be in gauging the likelihood of different price outcomes. An examination of Table 8.16 suggests that this situation does hold true for hogs weaned on November 15 and fed a 10 per cent protein ration; this ration gives a marketing date with the lowest price in 11 out of 14 years and the highest price in 2 years. In this same sense there is a lower probability of obtaining the lowest price of the six marketing dates when higher protein rations are fed, and a slightly higher probability of obtaining the highest price. For the

Table 8.16. Number of Years in Which Prices Were Highest or Lowest for Six Marketing Dates for Pigs Weaned November 15 and Fed Different Rations, 1937-52*

Per Cent Protein In Ration	Days to Reach 225 Pounds	Marketing Date	Number of Years Highest Price Paid At This Time†	Number of Years Lowest Price Paid At This Time†
November 1 weaning				
10	159	March 31	6	5
12	121	March 1	1	4
14	108	February 19	5	4
16	105	February 17	5	6
18	107	February 21	2	4
20	113	March 3	3	2
November 15 weaning				
10	159	April 14	2	11
12	121	March 15	4	0
14	108	March 5	5	1
16	105	March 3	4	3
18	107	March 7	5	1
20	113	March 17	4	0

*The war years 1945 and 1946 were omitted because the price paid was the same for all marketing dates included here.
†Computed relative to the six marketing dates shown in column 3 and not in terms of prices over the entire year.

earlier weaning date of November 1, the marketing date for a 10 per cent protein ration is 15 days earlier and the "price risk" is less pronounced.

However, in 14 out of 16 years for pigs weaned November 15 the least-cost ration would have resulted in greater profits. In other words the price gain was greater than cost increase for the least-time ration in only two years. The ration which gave the lowest cost of gain would, of course, have differed between years.

LEAST-COST RATIONS AND OPTIMUM MARKETING WEIGHTS

In the end the farmer must make two decisions in livestock feeding: (1) the ration to be fed for any one level of production or output (the feed substitution problem) and (2) the amount of the best ration to feed the animal and the marketing weight or level of production to be attained. The appropriate procedure is to decide first the least-cost ration and then the amount to be fed (i.e., the marketing weight or production level).

Usually the farmer knows the current feed prices but must estimate the future livestock price. In order to facilitate these decisions in pork

production, tables have been computed for these purposes. They then show the price per 100 pounds which must be received to "break even" on feed costs if hogs are sold at various weights under different feed prices.[7]

These procedures can be used for determining either least-cost or least-time rations. First, the farmer may decide on the amount of protein and corn which will allow the lowest feed costs per pound of gain. He then can compute the cost per 100 pounds of this ration and use it along with the hog price data to estimate the necessary selling price to "break even." Or, he can select a least-time ration and apply the same procedure.

PRACTICAL APPLICATIONS

Previous sections have been concerned largely with estimation of substitution coefficients which might have use in practical application. A few tables have been presented which summarize the data and application of principles to simple forms. For example, Table 8.10 could be understood and used by most high school and college students and by most commerical hog producers. However, since some readers may complain that the algebra and calculus presented is too complex and confusing to be used by farmers, we illustrate one further step in simplification of findings and in application of economic principles.

Communication of Findings

The communication of results stemming from complex research should not be confused with the research effort itself. Research and communication are two entirely different steps in scientific and educational activity. In most cases, the two can be separated, although it is certainly true that the research worker does not always excel in communicating results to the person who will use them. Often the research worker has no specialist in education or communication methods at hand and must complete the "translation step" by himself, if the findings are to be put to use. But even in this case, he should consider the two steps to be more or less separable. He should employ the research techniques which are most efficient for the predictions at hand. Then, he should use the communication media which are most effective in allowing operational use of basic data and principles.

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[7] See tables 14 and 15 in the publication: Heady, Earl O. *et al.* Replacement rates of corn and soybean oilmeal in fortified rations for growing-fattening swine. Iowa Agr. Exp. Sta. Bul. 409. Ames. 1954.

Pork Costulator

It is known, of course, that few hog farmers understand calculus or the application of basic economic principles. They cannot go out to the pig pen and equate the appropriate partial derivatives and price ratios, to specify the optimum ration or marketing weight. However, it is possible to develop mechanical aids to let a farmer make calculations and decisions as if he understood the basic mathematical and economic principles. To illustrate this point, the pork costulator developed by the authors is illustrated in Figure 8.9.

The costulator is a mechanical device constructed from three sheets of heavy metal. The center or square sheet is about 12 inches square. Disks of a slightly smaller diameter were cut from the other two sheets, with one riveted to each side of the square piece. Prices of corn were printed on the center square, near the perimeter of each disk. Prices of protein supplement were printed at the edge of each disk. A slit, through which is read the least-cost and least-time rations for particular price ratios, was cut in the disk on one side for 35-75 pound and 75-150 pound hogs. A similar slit was provided for 150-225 pound hogs on the disk on the other side. The ration data is printed on the center square. By matching the current protein price printed on the edge of the disk with the current corn price printed on the center square, the appropriate price ratio is specified. By glancing through the slit for hogs of a particular weight, the producer can read off the ration which minimizes feed costs per pound of gain, or the ration which minimizes time required for marketing at 225 pounds. Certain auxiliary information also is indicated and includes the components of these rations, the range price ratios over which it would be profitable to provide self-feeding and free choice of rations by hogs, rates of gain and time to marketing. In effect, the farmer can use the costulator as his "mathematical brain." In matching up the feed prices and reading off the appropriate information, he is accomplishing the equivalent of (a) equating partial derivatives with price ratios and (b) simultaneously solving a set of equations for magnitude of feed variables for attaining least-cost or least-time rations. He need not know mathematics to apply and use the principle.

The costulator in Figure 8.9 represents one practical mechanism for encouraging use of appropriate principle and data. Others could be suggested. For example, the price maps illustrated in Chapter 9 represent an alternative.

Figure 8.9. Pork Costulator derived from production functions for farmer use of economic principles.

CHAPTER 9

Damon V. Catron
Earl O. Heady
Dean E. McKee
Gordon C. Ashton
Vaughn C. Speer

Pork Production Functions and Substitution Coefficients for Hogs on Pasture

THIS CHAPTER reports a study similar to the drylot experiments explained in Chapter 8, except that it refers to hogs produced on pasture. Like the drylot study, the objectives of the pasture experiment were to estimate (1) the production function, (2) the substitution rate between corn and soybean oilmeal at different points on the production surface, (3) the least-cost ration for different soybean oilmeal to corn price ratios, (4) the relationship between the rate of hog gains and the input of corn and soybean oilmeal, and (5) the proportion of the years in which a least-cost feeding system results in greater profits than a least-time feeding system. Substitution between major classes of feed such as corn and soybean oilmeal is possible mainly where the rations are fortified with appropriate quantities of trace minerals (as well as antibiotics in the case of drylot feeding). These fortifying elements have been included in the rations of this study.

DESCRIPTION OF THE EXPERIMENTS

Two experiments were conducted co-operatively by research personnel in swine nutrition and production economics to obtain the necessary data. The first experiment, A. H. 597, was conducted during the summer of 1953. The second experiment, A. H. 597A, was conducted during the summer of 1954. Both experiments were conducted on an alfalfa pasture. The data from the two experiments were combined for the purposes of this study.

Both experiments were randomized complete block designs and included 12 treatment combinations with three replications each. Treatment combinations consisted of six rations, an antibiotic treatment and an antibiotic check. The rations were: 8 per cent, 10 per cent, 12 per cent, 14 per cent, 16 per cent, and 18 per cent protein. The antibiotic treatment consisted of crystalline chlorotetracyline (aureomycin) fed at the rate of 5 mg. per pound of ration. The rations were composed of ground yellow corn and solvent-extracted soybean oilmeal fortified with dicalcium phosphate, calcium carbonate, salt, trace minerals, and vitamins. Aside from corn and soybean oilmeal, other feed ingredients were retained as a constant amount per 100 pounds of feed. The experimental

unit was an individual hog, and each hog received the same ration throughout the entire experiment. Forage intake was not measured, although it is a resource in pork production just as are corn and soybean oilmeal. However, farmers generally put their hogs on pasture and then decide on the best ration when hogs already have access to free choice of forage. Hence, the functions are estimated within this framework. They do not allow evaluation directly of the quantitative effects of forage but allow computation of least-cost rations for hogs with free choice of pasture.

The hogs were fed individually in portable field units. Each field unit consisted of three pens equipped with individual self-feeders and waterers. The units were aligned side by side on pasture. They were moved each Monday, Wednesday, and Friday during the experiment. The original order of the units on the field was maintained at all times.

The pasture sward for experiment 597 was composed of a mixture of alfalfa and bromegrass. Mower clipping was used to maintain a maximum herbage height of about 8 inches; the pasture area was clipped several times during the trial to prevent excessive growth. Moisture was sufficient in both seasons so that the herbage remained of good quality over the experimental period.

Treatment combinations and the pigs were randomly assigned to pens within a block. Of the three replications, one included females, and the other two included males. The hogs were weighed every second week while they were on the experiment and were removed from the experiment as each hog reached 200 pounds.

The breeding of the hogs used in experiment A. H. 597 was Duroc x Poland China x Landrace x Duroc and Poland China x Landrace x Duroc. A Poland China x Landrace x Duroc cross was used in experiment A. H. 597A. Thirty-six hogs were required for each experiment, a total of 72 hogs for both experiments.

ESTIMATION OF THE PRODUCTION FUNCTION

Two steps have been followed in estimating the production function. First, three alternative types of functions have been fitted to all observations of the two experiments. These functions are denoted as *over-all functions*. Second, each of the three types of functions have been fitted to the observations on each of the six rations separately. The latter functions are called *individual ration functions*. Interest is mainly in the over-all functions; they express the relationship between hog gains and the input of any one of many combinations of the feeds. Individual ration functions express the relationship between hog gains and feed input when feeds are held in fixed proportions. The input-output curves for different rations varied in fixed proportions can be readily obtained from the over-all function. Therefore, comparisons of the feed-gain relationship estimated by the over-all function with that estimated from the individual ration function provides a simple means of checking the reliability of the over-all function.

The production functions express total gain beyond weaning as a function of total feed consumption beyond weaning. Experimental observations were taken on the consumption of feed and the amount of gain over 2-week intervals. The interval observations were progressively totaled over the entire feeding period to obtain a series of cumulative summations of gain, corn consumption, and soybean oilmeal consumption beyond weaning for each hog. The over-all production functions were then fitted to the 72 series of observations. Each individual ration function is fitted to 12 such series of observations. Analyses of variance were made for both experiments and for two weight intervals. They indicated no significant antibiotic effects for daily gain or feed consumption under conditions given. For feed consumption per pound of gain, antibiotic effects were significant only for the initial-to-200-pound weight interval in experiment 597. Antibiotic effects upon feed per 100 pounds gain are significant at the 5 per cent probability level for only the linear term for the initial-to-the-200-pound weight interval. In general, the analyses of variance do not support the hypothesis that the antibiotic treatment and the check lots constitute separate populations. Consequently, data from treatment and check lots are pooled for estimation of the production function and each protein level includes observations from 12 hogs.

Autocorrelation

Fitting of the functions for the cumulative series again introduces a problem of autocorrelation. The different observations for each hog are not independent since (a) the second observation taken on a hog is the sum of the feed consumption and gain over the first and second 2-week intervals, (b) the third observation is the sum of the feed consumption and gains in each of three 2-week periods, etc. Although the series of observations taken on a hog is itself autocorrelated, it is independent of the series of observations taken on other hogs. Since the over-all production function is fitted to all observations in each series, the autocorrelation coefficient for the entire collection of data is likely to have a value greater than zero.

The presence of autocorrelation in the observations does not present problems in predicting the relationship between the dependent and the independent variables, but it does introduce problems in making tests of significance. The effect of autocorrelation is to reduce the number of effective observations to which the function is fitted. In other words, the number of degrees of freedom used for tests of significance of uncorrelated series is reduced when autocorrelation is present.

Procedures are available for approximating the effective number of degrees of freedom in autocorrelated series. However, the necessity of calculating the autocorrelation coefficient and approximating the effective number of observations may be avoided by basing the tests of significance on a minimum number of effective observations to which the series would be reduced by autocorrelation.

Since the observations taken on different animals are independent, the minimum number of effective observations may be regarded as equal to the number of hogs from which observations were taken. The minimum number of effective observations is 72 for the over-all function and 12 for the individual ration functions. If the tests are significant on the basis of the minimum number of effective observations, the null hypothesis may be rejected. If the tests are not significant, the null hypothesis cannot be accepted without further testing. In the latter case, the test must be conducted on the basis of the actual number of observations used, disregarding autocorrelation for the moment. If the test still is not significant at an acceptable probability level with the greater number of degrees of freedom, the null hypothesis may then be accepted.[1]

Over-all Functions

In the functions which follow, C refers to pounds of corn, P refers to pounds of soybean oilmeal and Y refers to pounds of gain, all measured beyond weaning.

Quadratic:

$$Y = -1.7536 + .2988C + .9828P - .00003012C^2 - .003880P^2 - .0001684CP \tag{9.1}$$

Square root:

$$Y = -17.4939 + .2472C + .03568P + 1.4249\sqrt{C} + 6.6133\sqrt{P} - .08138\sqrt{C}\sqrt{P} \tag{9.2}$$

Cobb-Douglas:

$$Y = .5493C^{.8426}P^{.1604} \tag{9.3}$$

The quadratic function equation 9.1 explains 98.3 per cent of the variance in hog gains (Table 9.1), the square root function explains 98.1 per cent, while the power function explains only 94.2 per cent of gain variance. Using the minimum number of degrees of freedom, the linear and squared terms of equation 9.1 are significant at the .01 and .05 probability levels. The cross-product term is acceptable at a probability level between .10 and .15. The linear term for P in equation 9.2 is significant only at a probability level greater than .30. Both terms for the Cobb-Douglas function, equation 9.3, are significant at a .01 probability level.

[1]The effective number of observations need to be approximated only if the tests are not significant on the basis of the minimum effective number of observations but are significant on the basis of the actual number of observations taken. For example, if the calculated "t" for a regression coefficient in the over-all function were 2.616, the regression coefficient would be significant at the .01 level of probability on the basis of 500 degrees of freedom. On the basis of the minimum number of effective observations for the over-all function, 72, the regression coefficient would not be significant. If the autocorrelation reduces the number of effective observations to less than 125, the null hypothesis would be accepted at the .01 level of probability. If the effective number of observations is 126 or greater, the null hypothesis would be accepted.

Table 9.1. Correlation Coefficients, Standard Errors and t Values for Over-All Functions. Standard Errors and t Values in Order Given in Equations

Equation	n	R^2	Standard Errors					Value of t				
			s_{b1}	s_{b2}	s_{b3}	s_{b4}	s_{b5}	t_{b1}	t_{b2}	t_{b3}	t_{b4}	t_{b5}
1	521	.983	.0091	.0348	.00001	.00026	.00011	32.89*	28.25*	3.00*	15.06*	1.55‡
2	521	.981	.0146	.0353	.3760	.5107	.0339	16.89*	1.01§	3.79*	12.95*	2.40†
3	521	.942	.0213	.0072	-----	-----	-----	68.78*	22.41*	----	-----	----

*$p < .01$
†$.05 > p > .01$
‡$.15 > p > .10$
§$p > .30$

The sum of the elasticities for corn and soybean oilmeal in the power function is equal to 1.003, indicating slightly increasing returns to proportional increases in the input of the two feeds. This relationship appears unlikely in pork production, and, for the pasture data, the function appears to overestimate gains for higher levels of feed inputs. (This characteristic holds true only for the particular observations of this study and is not a characteristic of the same function fitted to other data.) The quadratic and the quadratic root functions express decreasing returns to proportional increases in both feeds.

Comparisons Between Over-all Functions

Relationships among predictions from the three functions are shown in Figure 9.1 for a 12 per cent protein ration. Similar estimates are obtained from all three functions up to a feed input of about 250 pounds. Beyond 250 pounds of feed, the curve estimated by the Cobb-Douglas function rises above the curves estimated by the other functions. The quadratic and quadratic root functions give very similar results throughout the entire range of the curves. Below feed inputs of 350 pounds, the curve for the quadratic function lies below the curve for the square root function. Beyond feed inputs of 350 pounds, the positions of the two curves are reversed.

The relationships between the three functions at other protein levels are similar. At lower protein levels, the curves from the two quadratic-type functions are more nearly linear and correspond more closely to the estimates obtained from the Cobb-Douglas function. At higher protein levels, the curves for the quadratic-type functions have greater curvature and fall away from the Cobb-Douglas curve more rapidly. The quadratic functions produce curves that are most consistent with the scatter diagrams at all protein levels.

Individual Ration Functions

Since the over-all functions have been fitted to all observations on

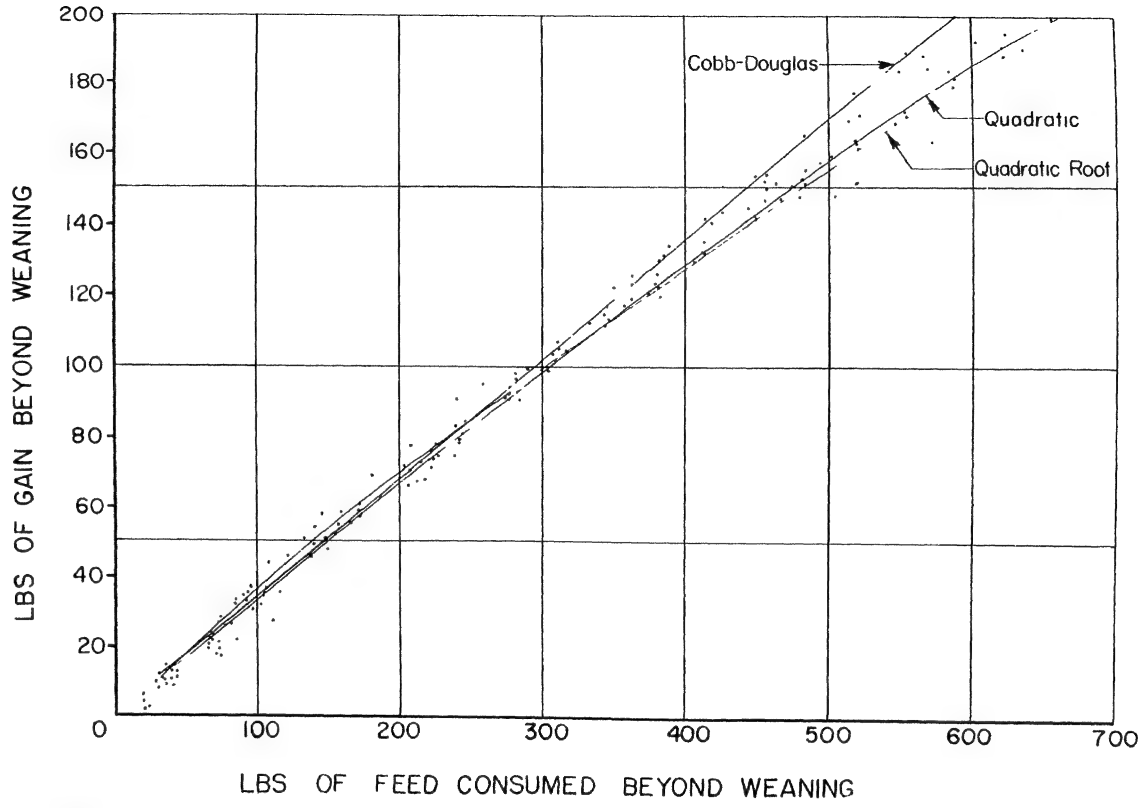


Figure 9.1. Growth curve for 12 per cent protein ration as estimated by the three over-all functions.

Table 9.2. Correlation Coefficients, Standard Errors of Regression Coefficients and t Values for Individual Ration Functions

Equation	R^2	Standard Errors		t Values	
		s_{b_1}	s_{b_2}	t_{b_1}	t_{b_2}
	$Y = a + b_1C + b_2C^2$				
8 per cent ration	.965	.0288	.000048	10.06	.16
10 per cent ration	.983	.0187	.000031	17.94	1.30
12 per cent ration	.992	.0136	.000025	31.95	5.60
14 per cent ration	.991	.0160	.000032	31.27	7.43
16 per cent ration	.976	.0997	.000210	5.72	1.87
18 per cent ration	.983	.0232	.000052	25.28	8.08
	$Y = a + b_1C + b_2\sqrt{C}$				
8 per cent ration	.965	.0412	1.2257	7.77	.64
10 per cent ration	.982	.0268	.8159	10.35	1.26
12 per cent ration	.992	.0199	.5691	13.04	5.06
14 per cent ration	.991	.0242	.6614	9.20	6.78
16 per cent ration	.975	.0406	1.1010	3.13	6.54
18 per cent ration	.987	.0298	.7772	3.24	10.47
	$Y = aC^{b_1}$				
8 per cent ration	.932	.0424	--	27.24	--
10 per cent ration	.978	.0165	--	62.22	--
12 per cent ration	.947	.0256	--	41.63	--
14 per cent ration	.987	.0122	--	79.58	--
16 per cent ration	.939	.0252	--	37.73	--
18 per cent ration	.971	.0154	--	56.19	--

the production surface, they might result in "abnormal" predictions for individual rations. Single-variable functions express the result of a single ration without encountering some of the "joint relationships" inherent in the over-all functions, and are, therefore, compared with the over-all functions. These comparisons show that "spurious" predictions do not arise from the over-all functions. In these graphic comparisons, corn alone is the independent variable. In any one ration, the ratio of corn to soybean oilmeal is fixed: An increase in the consumption of corn must be accompanied by a constant proportion of soybean oilmeal. There is no necessity for measurements to include both feeds in the individual ration function.

The individual ration functions, paralleling the three over-all functions, with gain as the dependent variable and corn as the independent variable, are as follows:

Quadratic functions:

(9.4) 8 per cent protein ration $Y = -5.102 + .290C + .000009C^2$

(9.5) 10 per cent protein ration $Y = -1.200 + .335C - .00004C^2$

(9.6) 12 per cent protein ration $Y = -3.062 + .433C - .0001C^2$

(9.7) 14 per cent protein ration $Y = -1.982 + .500C - .0002C^2$

(9.8) 16 per cent protein ration $Y = -1.664 + .570C - .0004C^2$

(9.9) 18 per cent protein ration $Y = 1.260 + .586C - .0004C^2$

Square root functions:

(9.10) 8 per cent protein ration $Y = -\ .189 + .320C - \ .784\sqrt{C}$

(9.11) 10 per cent protein ration $Y = -\ 5.854 + .278C + 1.028\sqrt{C}$

(9.12) 12 per cent protein ration $Y = -14.831 + .260C + 2.880\sqrt{C}$

(9.13) 14 per cent protein ration $Y = -20.064 + .223C + 4.484\sqrt{C}$

(9.14) 16 per cent protein ration $Y = -31.031 + .127C + 7.200\sqrt{C}$

(9.15) 18 per cent protein ration $Y = -32.685 + .096C + 8.137\sqrt{C}$

Cobb-Douglas functions:

(9.16) 8 per cent protein ration $Y = .111C^{1.156}$

(9.17) 10 per cent protein ration $Y = .272C^{1.026}$

(9.18) 12 per cent protein ration $Y = .258C^{1.067}$

(9.19) 14 per cent protein ration $Y = .505C^{.967}$

(9.20) 16 per cent protein ration $Y = .598C^{.948}$

(9.21) 18 per cent protein ration $Y = 1.000C^{.865}$

For all three equations, estimates for the 8 per cent ration show an increasing marginal productivity of feed. The quadratic and square root functions show a decreasing marginal productivity for rations with 10 per cent or more of protein. The Cobb-Douglas function shows increasing marginal productivity through the 12 per cent protein ration.[2] Increasing marginal feed productivity for low protein rations may be an effect of pasture. Young pigs consume very little forage, but, as they mature, they consume increasingly greater amounts. Hence, with the low palatability of a low-protein ration, small pigs may obtain insufficient amounts of protein from forage. As they grow, however, forage intake and hence gain per pound of concentrates may increase sharply, even for low-protein rations. Forage then becomes a substitute source of protein for hogs obtaining a small proportion of soybean oilmeal in the concentrate ration. However, this substitution is possible mainly as the hog grows. The tendency to substitute forage protein for concentrate protein is less with rations high in protein because of their greater palatability and nutritional "completeness." This phenomena would not have been expressed if feed value of forages could have been measured and used in predictions.

Comparison of Over-all and Single-Variable Estimates

After examination of the various statistics for the three over-all

[2] Only equation 9.17 of the single ration power functions has an elasticity which does not differ significantly from one at a probability level of 5 per cent.

functions, the quadratic equation, equation 9.1, was selected as the best estimator for the production surface. The Cobb-Douglas over-all function, equation 9.3, was eliminated because of the smaller proportion of the gain variance explained and the greater algebraic restrictions imposed by its logarithmic form. The square root over-all function, equation 9.2, provides estimates highly similar to the quadratic function. However, since it explains a slightly lower portion of variance in gains and has a relatively greater standard error for the P terms, it was rejected in favor of the quadratic function.[3] Hence, the text comparisons which follow compare estimates of single-line, input-output curves derived from over-all and single-variable equations for the latter functions.

"Growth curves" for six rations estimated by the single-variable and the over-all quadratic functions are shown in Figures 9.2 to 9.7. Similar curves are obtained from the estimates of the two types of quadratic functions. At the 8 per cent, 10 per cent, and 12 per cent protein levels, the curve estimated from the over-all quadratic function is almost identical to the curve estimated from the function fitted to each ration separately. The curves for the 14 per cent, 16 per cent, and 18 per cent protein levels are similar up to feed inputs of about 500 pounds. Beyond this, the curve estimated by the over-all function has slightly greater slope than the curve estimated by the individual ration function for 14 per cent and 16 per cent protein levels; the reverse is true for the 18 per cent protein level.

Interval Estimates from Cobb-Douglas

Since the over-all Cobb-Douglas equation appeared less satisfactory than other functions for the over-all surface, an attempt was made to predict three interval functions; and hence, to eliminate "overestimates" of gain at high feed inputs. As mentioned in Chapter 8, a reason for this attempt was to provide "average rations" for three gain intervals and to conform with the normal practice of changing rations two or three times during the growing-fattening period. The estimated relations are:

(9.22) Weaning to 75 pounds: $Y = .3350C^{.9087}P^{.2704}$

(9.23) 75 pounds to 150 pounds: $Y = .6543C^{.8072}P^{.1408}$

(9.24) 150 pounds to 200 pounds: $Y = .3127C^{.9875}P^{.0270}$

The three equations have R^2 values of .82, .89, and .73, respectively. All regression coefficients are significant at a 1 per cent level of probability, except for the coefficient of soybean oilmeal in the last equation. It is significant at a 20 per cent level. The elasticity of production for

[3]While the regression coefficient for the cross-product term was significant at a probability level greater than .10 but less than .15, it has been retained in the over-all quadratic function (9.1) since it adds some precision to estimates.

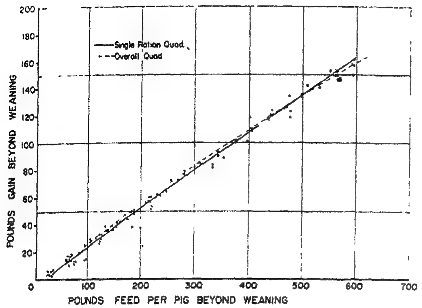


Figure 9.2. Growth curves for the 8 per cent protein ration estimated from individual ration and the several quadratic functions.

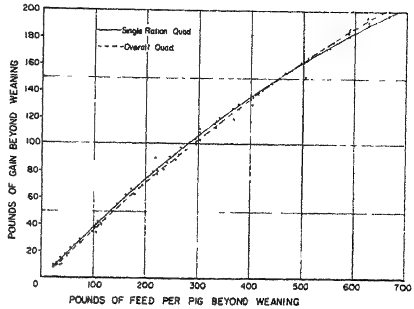


Figure 9.5. Growth curves for the 14 per cent protein ration estimated from the individual ration and over-all quadratic functions.

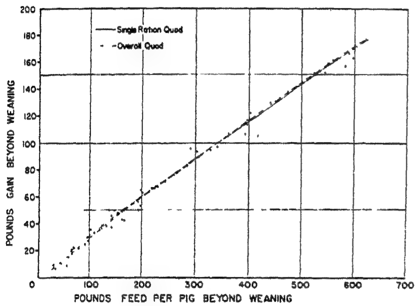


Figure 9.3. Growth curves for the 10 per cent protein ration estimated from the individual ration and over-all quadratic functions.

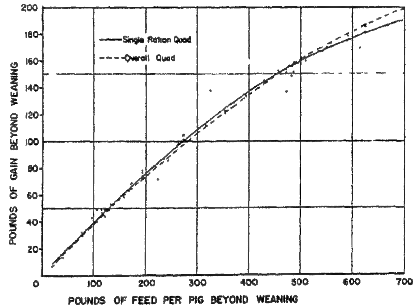


Figure 9.6. Growth curves for the 16 per cent protein ration estimated from the individual ration and over-all quadratic functions.

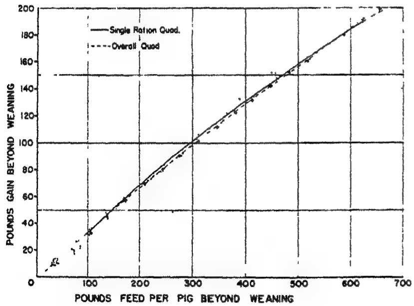


Figure 9.4. Growth curves for the 12 per cent protein ration estimated from the individual ration and over-all quadratic functions.

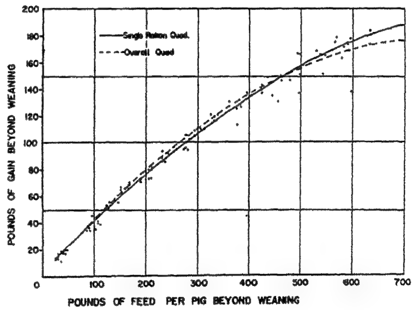


Figure 9.7. Growth curves for the 18 per cent protein ration estimated from the individual ration and over-all quadratic functions.

soybean oilmeal declines, from lower weight to higher weights, as expected. However, the elasticity of production for corn falls and then rises, instead of consistently rising from low weights to high weights as expected. The sum of the elasticities of production for the two feeds are 1.17 for the first interval, .95 for the second, and 1.01 for the third. This relationship, increasing feed productivity followed by decreasing feed productivity and then increasing feed productivity, is inconsistent with known biological conditions.

While the interval Cobb-Douglas approach gave satisfactory results in estimating "average rations" to be fed over a gain interval in the drylot study, it does not appear to be appropriate for the pasture data. The quadratic over-all function again appears to be the best choice among the various alternative functions examined, although modifications must be made in its use for determining rations to be used as "averages over gain intervals."

PRODUCTION SURFACE ESTIMATES

The pork production surface for corn and soybean oilmeal, based on equation 9.1, is shown in Figure 9.8. Consumption of corn and soybean oilmeal is measured by the vertical distance of the surface. The gains in hog weight, between weaning and market weight, follow a path over the face of the surface. The location of the path upon the surface is determined by the ration fed. A ration consists of a fixed combination of corn and soybean oilmeal and represents a vertical slice of the surface through the origin. The ration is represented by a straight line drawn in the horizontal or feed plane of the surface passing through the origin of the graph.

Marginal Productivity of Feeds in Fixed Proportions

Predicted total gains beyond weaning and the marginal productivity of feed at several levels of total feed consumption are shown in Table 9.3 for six rations. Marginal productivities are calculated on the basis of a proportional increase in the consumption of both corn and soybean oilmeal. In other words, the marginal productivities show the increase in hog weight resulting from a 1-pound increase in the quantity of the ration consumed.

The change in nutrient requirements as the hog approaches maturity is partly reflected in the total gains for each ration (Table 9.3). Fifty pounds of an 8 per cent protein ration produce 12.5 pounds of gain; 50 pounds of an 18 per cent protein ration produce 20.6 pounds of gain. The higher protein ration supplies more of the protein necessary for tissue building and growth at low weights. However, 700 pounds of the 8 per cent protein ration produce 185.8 pounds of gain while the same amount of 18 per cent protein ration produces only 176 pounds of gain.

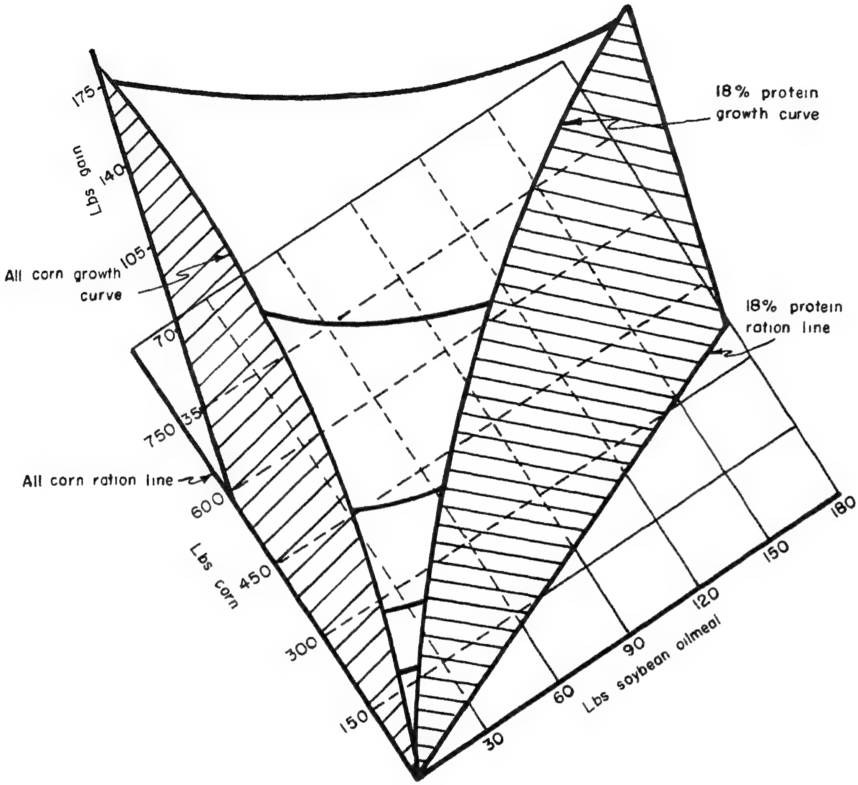


Figure 9.8. Production surface for hogs on pasture (lines or contours on surface are gain isoquants).

The high protein ration does not supply a sufficient amount of carbohydrate for production of fat at later stages of growth. These differences are brought out even more clearly by the marginal productivity figures. Up to a total feed intake of 250 pounds, marginal feed productivity is highest with an 18 per cent ration. At the 300-pound feed level, an additional pound of the 16 per cent ration has the same marginal productivity as an 18 per cent ration. The marginal productivity of the 18 per cent protein ration declines from .61 at the 50-pound feed level to .06 at the 700-pound feed level because of the decline in the protein requirements as the hog matures.

Iso-Product Contours

Figure 9.9, a contour or pork isoquant map corresponding to the production surface in Figure 9.8, is estimated from the quadratic

Table 9.3. Total Gain Beyond Weaning and Marginal Productivities of a Pound of Ration for Various Protein Rations

Pounds of Feed Consumed Beyond Weaning	Total Gain Beyond Weaning						Marginal Productivity of Feed*					
	Per cent protein in the ration						Per cent protein in the ration					
	8	10	12	14	16	18	8	10	12	14	16	18
50	12.54	13.67	15.76	17.30	18.93	20.64	.296	.334	.399	.461	.529	.611
100	26.70	28.90	32.91	35.72	38.64	41.55	.293	.330	.390	.447	.503	.568
150	40.72	43.96	49.70	53.54	57.36	60.96	.290	.326	.382	.433	.477	.526
200	54.60	58.84	66.13	70.74	75.10	78.88	.287	.322	.374	.418	.452	.484
250	68.34	73.54	82.20	87.32	91.87	95.31	.284	.318	.365	.405	.426	.422
300	81.95	88.06	97.90	103.30	107.65	110.24	.282	.314	.357	.390	.400	.400
350	95.41	102.39	113.25	118.65	122.45	123.68	.279	.311	.349	.376	.375	.358
400	108.74	116.55	128.24	133.38	136.27	135.63	.276	.309	.340	.362	.349	.316
450	121.92	130.52	142.86	147.50	149.11	146.09	.273	.303	.332	.348	.323	.274
500	134.97	144.32	157.13	161.01	160.97	155.06	.270	.299	.324	.333	.298	.232
550	147.88	157.93	171.03	173.90	171.85	162.53	.267	.295	.316	.319	.272	.190
600	160.65	171.36	184.57	186.18	181.75	168.51	.264	.291	.307	.305	.246	.148
650	173.29	184.61	197.76	197.84	190.67	173.00	.261	.287	.299	.291	.221	.105
700	185.78	197.68	210.58	208.88	198.61	176.00	.258	.283	.291	.276	.195	.063

*Added gain resulting from an added pound of ration. All figures predicted as derivatives from equation 9.1.

equation 9.1, for 26, 76, and 141 pounds of gain beyond weaning. The isoquant equation is:

$$(9.25) \qquad C = 4960.36 - 2.7961P \pm (-16{,}600.2656)\,[-.00000044P^2 + .00001774P + .08907733 - .00012048Y]^{.5}.$$

The quantity of corn and soybean oilmeal required to produce 26, 76, and 141 pounds of gain beyond weaning also has been determined from the single-variable equations 9.4 through 9.9 for each of the six rations. These feed quantities have been plotted in Figure 9.9. (In every instance, the quantities estimated from the individual ration functions fall very close to the contours estimated from the over-all equation.) The close agreement between the estimates from the individual ration functions and the over-all function is further proof that the over-all equation provides reliable predictions of the relationship expressed within the experimental data.

The experiments conducted did not include rations beyond the 18 per cent protein level. Therefore, the portion of the iso-product contours lying below the 18 per cent protein ration line is an extrapolation beyond the range of the data. The present study provides no information on the shape of the contours in that section of the production surface. However, with increasing levels of protein, the contours should flatten out and eventually approach a zero slope.

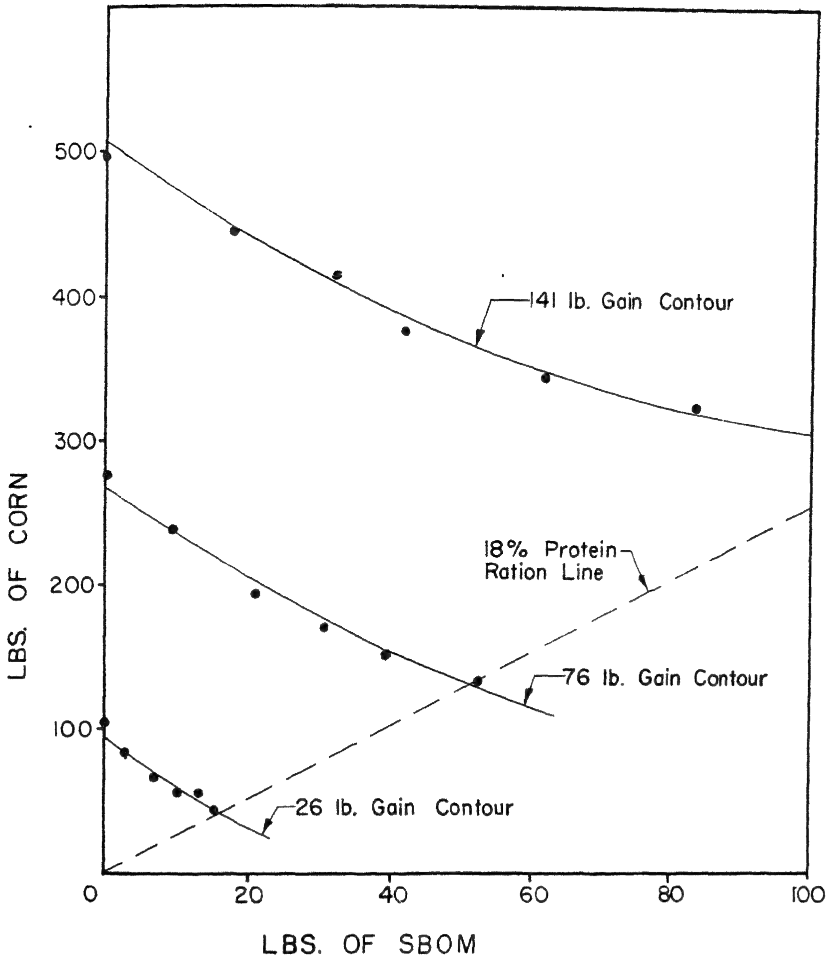


Figure 9.9. Iso-product contours for pasture fed swine.

Rates of Substitution

The rate at which soybean oilmeal substitutes for corn in the hog ration at a given level of output is indicated by the slope at a particular point on the iso-product contour. The iso-product contours in Figure 9.9 are curved. Consequently, the rate of substitution of soybean oilmeal for corn declines as the ration includes a greater percentage of protein.

Marginal rates of substitution of soybean oilmeal for corn can be computed from equation 9.26.

$$\frac{\delta C}{\delta P} = \frac{-.982769 + .007760P + .000168C}{-.298812 + .000060C + .000168P} \tag{9.26}$$

Table 9.4. Corn and Soybean Oilmeal Quantities and Substitution Rates Along the 26-, 76-, and 141-Pound Gain Isoquants Derived from Equations 9.25 and 9.26

Proportion of SBOM in Ration*	Per Cent Protein†	26 Pounds of Gain			76 Pounds of Gain			141 Pounds of Gain		
		Feed required		Marginal rate of substitution‡	Feed required		Marginal rate of substitution‡	Feed required		Marginal rate of substitution‡
		corn (lbs.)	SBOM (lbs.)		corn (lbs.)	SBOM (lbs.)		corn (lbs.)	SBOM (lbs.)	
.02	8.7	88.0	1.76	3.25	251.1	5.02	3.19	472.8	9.46	3.09
.04	9.4	82.9	3.32	3.22	237.1	9.48	3.07	447.6	17.90	2.86
.06	10.0	78.5	4.71	3.18	225.0	13.50	2.97	426.5	25.59	2.65
.08	10.6	74.5	5.96	3.15	214.3	17.15	2.87	408.4	32.67	2.46
.10	11.2	71.0	7.10	3.12	204.9	20.49	2.79	392.6	39.28	2.28
.12	11.8	67.8	8.13	3.10	196.4	23.57	2.71	379.1	45.49	2.11
.14	12.4	64.8	9.08	3.07	188.7	26.42	2.64	367.1	51.40	1.95
.16	12.9	62.2	9.95	3.05	181.8	29.09	2.57	356.6	57.05	1.79
.18	13.5	59.7	10.75	3.03	175.5	31.59	2.50	347.2	62.50	1.64
.20	14.0	57.5	11.50	3.01	169.7	33.94	2.44	338.9	67.78	1.50
.22	14.5	55.4	12.19	2.99	164.3	36.15	2.38	331.6	72.94	1.35
.24	14.9	53.5	12.83	2.98	159.4	38.25	2.33	325.0	78.01	1.21
.26	15.4	51.7	13.44	2.96	154.8	40.24	2.28	319.3	83.02	1.07
.28	15.8	50.0	14.00	2.95	150.5	42.14	2.23	314.3	88.01	.93
.30	16.3	48.4	14.53	2.94	146.5	43.95	2.18	310.0	93.01	.79
.32	16.7	47.0	15.04	2.92	142.8	45.69	2.14	306.4	98.05	.65
.34	17.1	45.6	15.51	2.91	139.3	47.35	2.10	303.5	103.18	.50
.36	17.5	44.3	15.95	2.90	135.9	48.94	2.05	301.3	108.45	.35
.38	17.9	43.1	16.37	2.89	132.8	50.47	2.01	299.8	113.93	.18
.40	18.2	41.9	16.77	2.88	129.9	51.95	1.98	299.2	119.68	.01

*The figures show the pounds of soybean oilmeal for each pound of corn. Hence, the figure .20 refers to .2 pounds of soybean oilmeal for each 1 pound of corn.
†Based upon a protein content of 45 per cent for soybean oilmeal and 8.2 per cent for corn.
‡The negative signs have been omitted from the substitution rates. The substitution ratios are the derivatives in equation 9.26. The feed combinations for specified gains have been derived from equation 9.25.

Table 9.4 includes prediction of the pork isoquants and the marginal rates of substitution associated with them. With a corn to SBOM ratio of .20, a 14 per cent protein ration, 57.5 pounds of corn, and 11.5 pounds of soybean oilmeal are required to produce 26 pounds of gain. The rate of substitution on the 26-pound gain contour with a 14 per cent protein ration is 3.01 (i.e., a pound of soybean oilmeal replaces 3.01 pounds of corn at the particular point on the 26-pound contour). For the same ration, 169.7 pounds of corn and 33.94 pounds of soybean oilmeal are required to produce a 76-pound gain. However, the quantity of corn replaced by a 1-pound increase in soybean oilmeal drops to 2.44 pounds for this level of gain. For a gain of 141 pounds, 338.9 pounds of corn and 67.78 pounds of soybean oilmeal are required and the rate of substitution drops to 1.50. Hence, the substitution rate and relative feed value of soybean oilmeal declines as the hog increases in weight. While this point has been illustrated for a 14 per cent ration only, it also holds true for rations containing other percentages of protein.

Changes in Substitution Rates for a Given Gain

Not only do the substitution rates for protein decline as the hog attains greater weight, but also they decline as the proportion of protein

increases for growth to a given weight. For example, substitution rates vary from 3.25 to 2.88 over the range for which the 26-pound gain contour has been predicted. In a ration containing .02 pounds of soybean oilmeal per pound of corn, 1 pound of soybean oilmeal replaces 3.25 pounds of corn. A pound of soybean oilmeal replaces only 2.88 pounds of corn at the same gain level in a ration with .40 pounds of soybean oilmeal per pound of corn. One pound of the former replaces only 2.99 pounds of the latter when the ratio of soybean oilmeal is .22.

For the 76-pound gain contour, the marginal rate of substitution varies from 3.19 with soybean oilmeal to corn ratio of .02 to 1.98 with a soybean oilmeal to corn ratio of .40. The rate of substitution declines much more rapidly along the 76-pound gain contour than along the 26-pound gain contour because of the greater hog weight. The range in magnitude of substitution rates is even greater along the 141-pound gain contour. A pound of soybean oilmeal replaces 3.09 pounds of corn with a soybean oilmeal to corn ratio of .20, but only .01 pounds of corn with a ratio of .40.

The production function, equation 9.1, predicts the gain resulting from various total amounts of feed consumed beyond weaning. Hence, each contour, such as in Figure 9.9 and Table 9.4, is derived with reference to the origin or weaning weight. The predictions suppose that the quantities of corn and soybean oilmeal specified by the co-ordinates of a point on a contour are fed in that proportion from weaning to the level of gain represented by the contour.

LEAST-COST RATIONS

The least-cost ration can be defined by equation 9.26 where the derivative or marginal rate of substitution of corn for soybean oilmeal is equated to the price ratio of soybean oilmeal and corn. However, least-cost rations determined by use of substitution rates estimated from the over-all production function must be carefully interpreted. The hogs in these experiments were each fed a constant ration throughout the entire course of the experiments. The production function therefore expresses, under a constant ration system of feeding, the relationship of total weight gain to total consumption of corn and soybean oilmeal from weaning weight. The iso-product contours (Figure 9.10) derived from the over-all production function, therefore, show the possible combinations of corn and soybean oilmeal to produce various levels of gain under a single ration technique of feeding. The ration which is determined by equating the substitution rate on these contours to the feed price ratio gives the total quantity of corn and soybean oilmeal which will produce the amount of gain *represented by the respective contour,* and not over-all gain contours, with the lowest outlay for the feed under a system of feeding a single ration throughout the feeding period. For example, point G on the 225-pound contour in Figure 9.10 is the locus, on that contour, where soybean oilmeal substitutes for corn

at the rate of 2.5. Therefore, the co-ordinates of point G are the quantities of corn and soybean oilmeal which will produce 194 pounds of gain beyond weaning under a constant ration system of feeding. With a soybean oilmeal price of 2.5 times the price of corn, corn and soybean oilmeal would be fed, from weaning to the 225-weight level, in the proportions represented by the line OG.

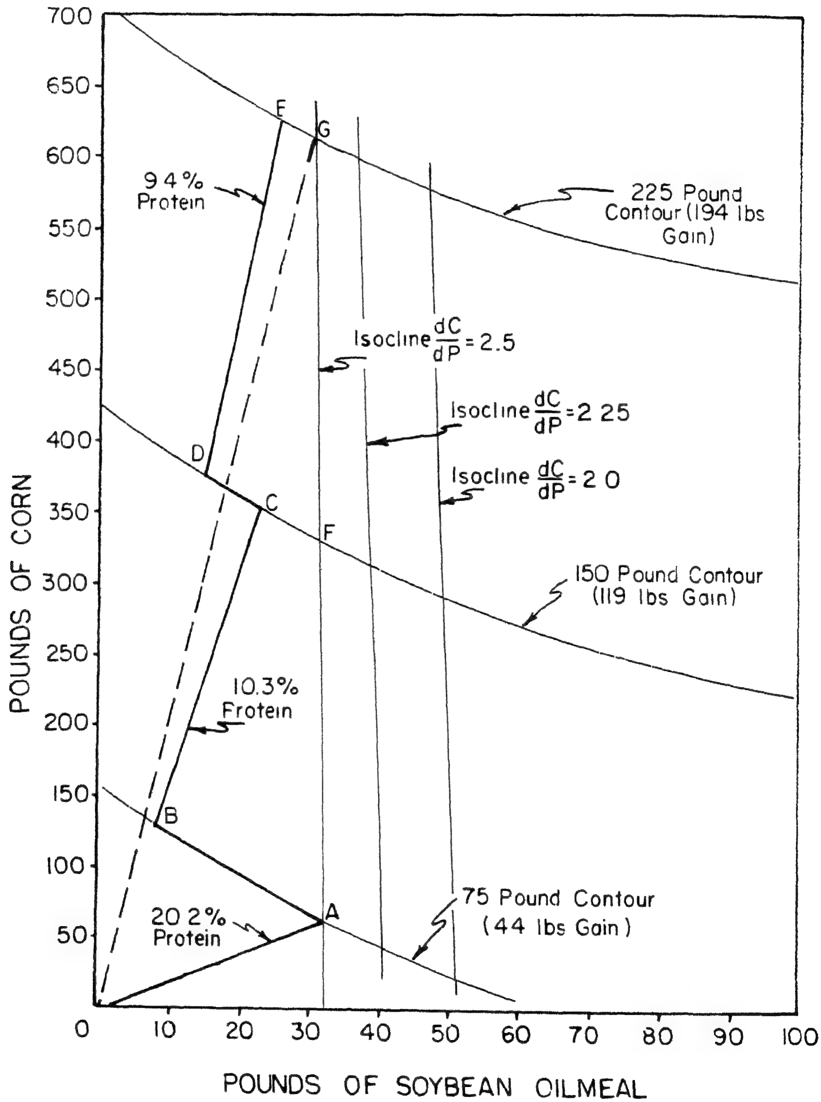


Figure 9.10. Expansion path of the least cost ration with a price ratio of 2.5, 2.25, and 2.0.

A single ration fed throughout the entire feeding period is one possible system to follow; but it is obvious from the relationships shown in Figure 9.10 that this system does not result in the lowest possible feed cost. Feed costs can be further reduced by adjusting the proportion of corn and soybean oilmeal fed at intermediate points throughout the feeding period. The line AFG in Figure 9.10 is an isocline joining all points on successive contours having a slope of 2.5 (i.e., a substitution rate of 2.5). Comparing line AFG to line OG, it is obvious that below 194 pounds least-cost gains are not attained if the same ration is fed throughout the entire feeding period. One hundred and nineteen pounds of gain can be produced at lower cost by feeding corn and soybean oilmeal in the proportions represented by the feed quantity at point F. This ration has a higher proportion of soybean oilmeal than the ration represented by the line OG. Similarly, the ration for producing 44 pounds of gain has a higher proportion of soybean oilmeal than the ration for producing 119 pounds of gain under a single ration feeding system. The rate of substitution between corn and soybean oilmeal is changing continuously as the hog gains in weight. Consequently, if feed costs are to be minimized, a different ration should be fed for each successive pound of gain produced.

Particular Interpretation of Isoclines

The isocline equation is:

$$C = \frac{.299K + .0077P - .00016PK - .98}{.0006K - .00017} \quad . \tag{9.27}$$

The isoclines in Figure 9.10 might appear to be interpreted as indicating that the least-cost ration to reach 75 pounds includes the amount of protein shown at F, with some of this taken away from the pig as he attained 119 and then 225 pounds. However, this is not the case. Point A shows only the feed combination, fed as a single ration, which would be used to attain minimum cost if the pig were to be taken only to 75 pounds; point F shows the least-cost feed combination if the pig were to be fed to exactly 150 pounds; point G shows the feed combination for lowest cost if the hog were fed a single ration and taken to exactly 225 pounds. These isoclines do not directly show how the corn and soybean oilmeal proportion are to be adjusted throughout the period from weaning to market weight in order to minimize feed cost per each successive pound of gain produced. The isoclines refer only to a constant ration feeding system, to the gain level indicated. They are, however, indicative of the necessity for feeding a lower protein ration for each successive pound of gain produced.

Generally, isoclines have a positive slope. In Figure 9.10 they appear to be vertical, or to have a slight negative slope. This phenomenon arises mainly because of the nature of the experiment and measurements. The quantities measured in the feed plane are corn and soybean

oilmeal. The third feed, forage, is not measured. If the protein in forage were added to that in soybean oilmeal, the isoclines would have a positive slope. Small pigs eat very little forage because their digestive organs cannot handle it. However, as hogs progress in weight, they can and do consume much more forage relative to concentrates. Thus for a 225-pound hog, the feed equivalent of soybean oilmeal in forage would, if added to the soybean oilmeal measured in the study, fall at a point to the right of G in Figure 9.10.

In a study designed to relate the gain surface physically with feed input, forage should be measured and introduced into the production function. The current study did not, however, have this objective. It was designed to allow specification of least-cost rations under conditions representing the environment in which most farmers make their decisions. Most farmers turn their pigs on pasture as a disease control precaution, as well as to obtain some feed advantage. Yet hog pasture usually includes an abundance of forage and no attempt is made to fully utilize it in matching costs of forage against concentrate feeds. The farmer is concerned, given an ample supply of forage and the quantity of it consumed when different concentrate rations are fed, of *balancing* corn and protein supplement feeds in a manner to minimize concentrate costs.

Interval Rations

A fixed ratio of corn and soybean oilmeal fed over an interval of gains does not result in the lowest possible feed costs for the entire gain interval because substitution rates change continuously as the hog increases in weight. Therefore, if feed costs are to be minimized, the proportions in which the corn and soybean oilmeal are fed should be changed for each unit of gain produced (i.e., should follow an isocline). In practice, it is impossible to make such extremely small changes in the ration. To adjust the ration for gains even as small as a pound, the hogs would have to be fed individually and the rations changed daily. Farmers are concerned with the least-cost ration for rather wide intervals of gain. From a practical standpoint, they may consider changing the ration only two or three times in the course of the entire feeding period from weaning to market weight.

Hence, in providing practical figures for farmer recommendations, the production surface has been divided into three weight intervals. The three weight intervals are: weaning to 75 pounds liveweight, 75 to 150 pounds liveweight and 150 to 225 pounds liveweight (i.e., the total weight contours shown in Figure 9.10). For a given ratio of the feed prices, a constant ration is selected for each interval. The ration selected should produce the gain at a lower feed cost than any other constant ration fed over the same interval.[4]

[4]Greater refinement in the least-cost feeding system can be achieved by dividing the surface into a greater number of weight intervals. The ration then can be altered more frequently over the total production period.

Table 9.5. Least-Cost Rations on Alfalfa Pasture for Various Soybean Oilmeal to Corn Price Ratios

Soybean Oilmeal to Corn Price Ratio	Weaning to 75 Pounds				75 to 150 Pounds				150 to 225 Pounds			
	Feed required		Per cent protein	Days to feed over interval	Feed required		Per cent protein	Days to feed over interval	Feed required		Per cent protein	Days to feed over interval
	corn	SBOM			corn	SBOM			corn	SBOM		
1.1					174	42	14.9	43	232	23	11.2	32
1.2					175	41	14.7	43	235	22	11.0	32
1.3					177	39	14.5	43	236	20	10.8	32
1.4					178	38	14.3	43	238	19	10.6	32
1.5					181	37	14.1	43	239	18	10.5	32
1.6					182	36	13.9	43	241	16	10.2	33
1.7					186	34	13.4	43	243	15	10.1	33
1.8					190	30	12.9	43	245	14	9.9	33
1.9					196	27	12.4	43	248	13	9.8	33
2.0					204	23	11.7	43	249	12	9.6	34
2.1					209	20	11.0	44	249	12	9.6	34
2.2					213	19	10.9	44	249	12	9.6	34
2.3					216	17	10.6	44	251	11	9.5	34
2.4					218	16	10.5	44	251	11	9.5	34
2.5					221	15	10.3	45	254	10	9.4	34
2.6					223	14	10.1	46	254	10	9.4	34
2.7	85	24	15.8	33	230	12	9.7	47	256	9	9.2	34
2.8	93	20	14.5	36	234	10	9.5	47	256	9	9.2	34
2.9	104	17	12.9	40	236	9	9.3	48	256	9	9.2	34
3.0	118	12	11.2	46	242	8	9.1	49	259	8	9.1	34
3.1	130	8	10.0	52	242	8	9.1	49	259	8	9.1	34

The computed least-cost ration for each interval is given in Table 9.5 for a series of soybean oilmeal to corn price ratios. These rations were approximated from the over-all quadratic production function in this manner: A series of ration lines was projected through the surface from weaning to a liveweight of 225 pounds. The total feed requirements beyond weaning then were computed for producing a 75-, 150-, and 225-pound hog along each of the ration lines. In other words, the various quantities of corn and soybean oilmeal which can be used for producing 44, 119, and 194 pounds of gain beyond weaning were determined. The difference for a particular ration in respect to protein percentage, between the total feed requirements at the beginning and the end of each weight interval was used as the feed requirements for the particular interval. The least-cost ration for the interval is then determined by summing the value of corn and soybean oilmeal, for the numerous rations. While the procedure is an approximation (in contrast to the equation of the derivative of the isoquant equation with the price ratio), it gives estimates of the least-cost ration, accurate within a few tenths of a per cent in protein and sufficiently accurate for practical uses.

Indication of Rations

In Table 9.5, least-cost rations can be determined as follows: With a price of 4.05 cents per pound for soybean oilmeal and 1.5 cents per pound for corn, the price ratio is 2.7. At this price ratio, the least-cost ration over the weaning to the 75-pound interval is 15.8 per cent protein. The ration will include 85 pounds of corn and 24 pounds of soybean oilmeal for this amount of gain, plus the 2.5 pounds of the minerals indicated on page 302 for each 100 pounds of feed. For gains in the 75-150 pound interval for a price of $1.12 per bushel for corn and $4.00 per hundredweight for soybean oilmeal, a price ratio of 2.0 on a per-pound basis, the least-cost ration includes 204 pounds of corn and 23 pounds of soybean oilmeal, plus 2.5 pounds of minerals per 100 pounds of feed. With 11.7 per cent protein, growth over this gain interval requires 43 days

The described above procedure for finding the least-cost ration can be contrasted to other concepts of feeding in Figure 9.10. With a price ratio of 2.5, the rations for each of the three weight intervals would be represented by line OABCDE. From weaning to 75 pounds liveweight, corn and soybean oilmeal would be fed in the proportion represented by the line OA. At the 75-pound contour, a shift is made along the contour to point B. From 75 to 150 pounds, corn and soybean oilmeal would be fed in the proportion represented by BC. The ration is shifted again at the 150-pound contour to the proportions represented by DE. The line OABCDE might be called the "practical expansion path" of rations for a price ratio of 2.5. It represents a compromise between feeding of a different ration for each pound of gain (line AG) and feeding the same ration over the entire feeding period (line OG). The first alternative is impractical while the second does not minimize feed costs.

It can be seen by comparing the line OG with line OABCDE that the constant ration over the entire production period results in underfeeding of soybean oilmeal throughout the first and second weight intervals and overfeeding of soybean oilmeal throughout the third weight interval. Cost of feed is not minimized in any of the three weight intervals by feeding the constant ration. The same condition may hold true for OABCDE within each weight interval.

Least-Cost Graph

Figure 9.11, based on the rations in Table 9.5, provides a graph for

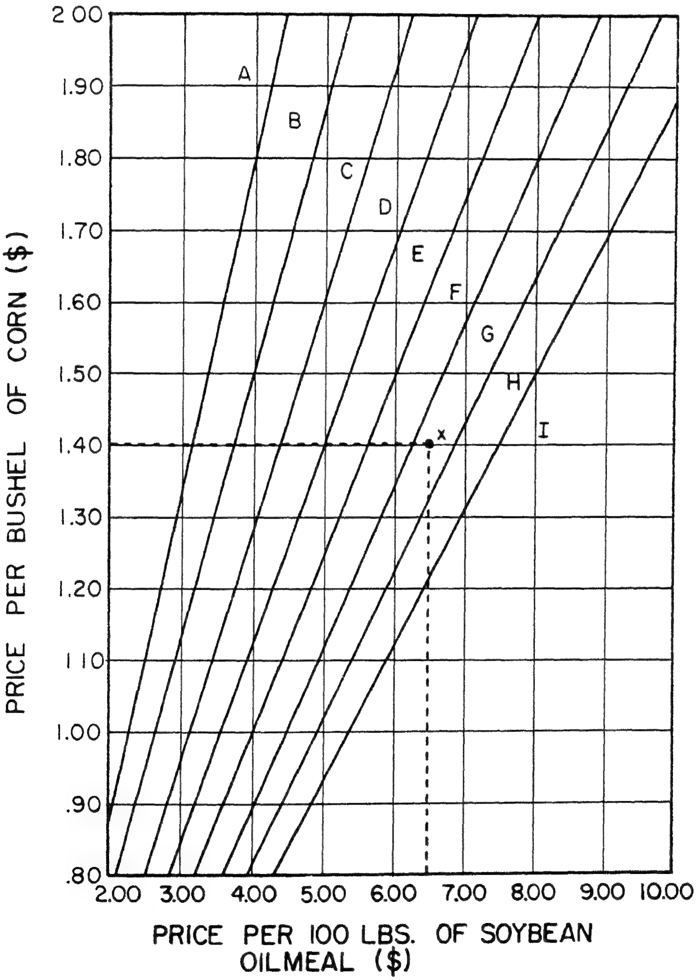


Figure 9.11. Least-cost ration graph.

calculation of least-cost rations under different prices for corn and soybean oilmeal. The series of iso-price ratio lines in Figure 9.11 show all combinations of corn and soybean oilmeal prices giving the same price ratio. Rather than to consider an infinite series of price ratio lines with minute changes in the proportions of corn and soybean oilmeal in the ration, only eight lines have been drawn. This procedure amounts to assuming that the gain isoquant is made up of a series of linear segments (rather than of continuous points on a smooth curve).

The least-cost ration for a given set of prices for corn and soybean oilmeal is found by reading up the corn axis of Figure 9.11 to the given price of corn, and then reading across in a horizontal direction until a point is reached directly above the given price of soybean oilmeal. The area of the graph in which the point lies determines which of the rations in Table 9.6 is the least-cost ration to be fed for the gain intervals of 75-150 and 150-225 for growing-fattening hogs on alfalfa pasture. For example, if the price of corn is $1.40 per bushel and the price of soybean oilmeal is $6.50 per hundredweight, these prices form the coordinates of point X on the graph. Point X falls in area G. The least-cost ration for hogs between 75 and 150 pounds liveweight is given opposite line G in section I of Table 9.6. For hogs between 150 and 225 pounds liveweight, the least-cost ration is given opposite line G in section II. The rations in Table 9.6 are in terms of feed requirements per hundred pounds of gain rather than in terms of the quantity of feed required for a single hog to produce the amount of gain for each weight interval.

Table 9.6. Least-Cost Rations for Pasture Fed Hogs in Terms of Feed Per 100 Pounds of Gain

	Corn	SBOM	Per cent protein	Average daily gain	Number of days
			I. 75 to 150 pounds		
A	232	56	14.9	1.74	43
B	237	51	14.3	1.75	43
C	243	48	13.9	1.76	43
D	261	36	12.4	1.75	43
E	279	27	11.0	1.71	44
F	291	21	10.5	1.68	44
G	297	19	10.1	1.64	46
H	315	12	9.3	1.54	48
I	323	11	9.1	1.52	49
			II. 150 to 225 pounds		
A	309	31	11.2	2.33	32
B	317	25	10.6	2.33	32
C	321	21	10.2	2.31	33
D	331	17	9.8	2.29	33
E	332	16	9.6	2.27	34
F	335	15	9.5	2.25	34
G	339	13	9.4	2.23	34
H	341	12	9.2	2.20	34
I	345	11	9.0	2.18	34

RATE OF GAIN

To allow prediction of the effect of rations on rate of gain, two types of functions are examined as alternatives for expressing the relationship between the inputs of corn (C) and soybean oilmeal (P) and the number of days (T) required to consume various quantities of the two feeds. The two functions are the quadratic, equation 9.28, and the square root, equation 9.29. Each function has been fitted over the observations from all six rations in both experiments.

$$T = 4.2477 + .4414C - .3673P - .0003C^2 + .0047P^2 - .0010CP \tag{9.28}$$

$$T = -23.0421 - .0064C + .5304P + 7.9090\sqrt{C} - 4.0347\sqrt{P} - .2120\sqrt{C}\sqrt{P} \tag{9.29}$$

The statistics for the two functions are presented in Table 9.7. Equation 9.29, the quadratic root, explains only a slightly greater proportion of the variation in the dependent variable, (T), than does equation 9.28. Equation 9.28 gives a relation showing time as a maximum at a level of feed consumption within the range of experimental observations. For total time to reach a maximum would mean that the hog would have to "die off" and cease feed intake, an unrealistic situation. It appears more logical that the slope of the total time function should fall off rapidly at low total feed input and approach linearity as total feed consumption reaches a high level and the hog approaches maturity. A mature hog has nearly a constant daily feed intake upon reaching maturity. Conceivably, the hog could live several years, with total time continuing to increase with age and continued feed consumption.

Table 9.7. Correlation Coefficients, Standard Errors, and t Values for Time Functions in Order of Regression Coefficients in Equations 9.28 and 9.29

Equation	R^2	Standard Errors					t Values*				
		s_{b_1}	s_{b_2}	s_{b_3}	s_{b_4}	s_{b_5}	t_{b_1}	t_{b_2}	t_{b_3}	t_{b_4}	t_{b_5}
9.28	.912	.0141	.0539	.000027	.00040	.00018	31.35	6.81	12.04	11.83	5.60
9.29	.918	.0021	.0503	.5358	.7283	.0483	3.05	10.55	14.76	5.54	4.39

*All t values exceed those for a probability level of .05 or less.

Equation 9.29, the square root function, more nearly allows the latter conditions. The relations obtained from the two types of equations are plotted in Figures 9.12 and 9.13 for two rations. In terms of comparisons, equation 9.29 has been used as the basis for estimating rate of gain.

Estimates of the total time required to consume various quantities of feed for six different rations are shown in Table 9.8. At low levels of feed input, the least time required to consume a given quantity of

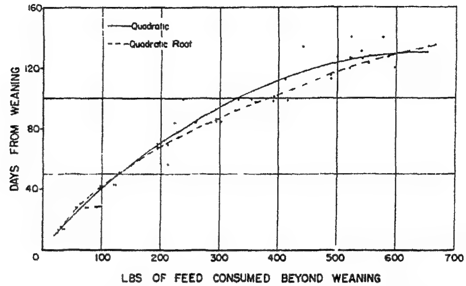


Figure 9.12. Time relation for 10 per cent protein ration.

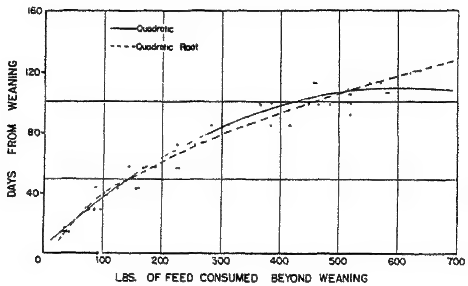


Figure 9.13. Time relation for 12 per cent protein ration.

feed is obtained with the 18 per cent protein ration. As the pigs become older and heavier, the advantage of the extremely high protein ration becomes increasingly smaller and finally gives slower gains than a ration with somewhat less protein.

An equation of daily rates of gain can be expressed as in equation 9.30 where D is gain per day, T (the equation in the denominator) is total time to consume a given amount of feed, Y (the equation in the numerator) is gain forthcoming from the same feed, and C and P refer to corn and soybean oilmeal consumption per pig.

$$(9.30) \quad D = \frac{-1.75 + .2988C + .9828P - .00003C^2 - .0039P^2 - .00017CP}{-23.0421 - .0064C + .5304P + 7.909\sqrt{C} - 4.035\sqrt{P} - .212\sqrt{C}\sqrt{P}}$$

The average daily rate of gain, between 50-pound feed increments, of the 18 per cent protein ration rises up to a total feed input of 250 pounds (Table 9.9). Between inputs of 200 and 250 pounds, a maximum average daily rate of gain of 1.685 pounds is reached. Beyond the 250-pound level of feed consumption, each additional 50-pound increment of the ration results in lower daily gains. The 16 per cent protein ration produces gains at a slower rate than the 18 per cent protein ration up to

Table 9.8. Total Time Required to Consume a Given Amount of Feed in Rations of Various Protein Levels

Pounds of Feed	Total Days To Consume Specified Feed Quantity for Protein Percentages of					
	8	10	12	14	16	18
50	31.5	23.8	19.5	16.5	14.0	11.7
100	53.9	42.4	36.6	32.8	29.9	27.2
150	71.0	56.5	49.6	45.3	42.3	39.7
200	85.4	68.1	60.4	55.9	52.9	50.5
250	98.0	78.2	69.8	65.2	62.3	60.2
300	109.4	87.2	78.3	73.6	70.9	69.2
350	119.9	95.4	86.0	81.3	78.9	77.7
400	129.6	102.9	93.1	88.5	86.4	85.7
450	138.7	109.9	99.7	95.2	93.5	93.3
500	147.3	116.4	106.0	101.6	100.3	100.6
550	155.4	122.6	111.9	107.7	106.7	107.7
600	163.2	128.4	117.4	113.4	113.0	114.6
650	170.7	133.9	122.8	119.0	119.0	121.2
700	177.8	139.2	127.9	124.3	124.8	127.7
750	184.7	144.2	132.8	129.5	130.5	134.0
800	191.4	149.1	137.5	134.5	135.9	140.2
850	197.8	153.7	142.0	139.3	141.3	146.3
900	204.0	158.2	146.4	144.0	146.5	152.2

an input of 200 pounds of feed. The same comparison holds true between the 14 per cent and the 16 per cent protein rations. As the hog consumes more feed and increases in weight, the rate of gain falls off with the higher protein ration. It again is evident from the data in Table 9.9 that the protein content of the ration must be decreased over the production period if the rates of gain are to be kept at a maximum.

LEAST-COST VERSUS LEAST-TIME RATIONS

Rations for least-cost and least-time rations are much more similar on pasture than on drylot (see Chapter 8). This condition holds true on pasture because of the availability of protein in the forage. If the price ratio is one favoring the use of a small percentage of protein in the concentrate mix, the hog can supplement the protein intake by consuming more forage. Accordingly, the rate of gain is not decreased much when the concentrate ration is adjusted to include less protein, for hogs on pasture. In drylot, a shift in ration to meet a higher protein to corn price ratio cannot be offset by a greater intake of protein from forage.

Table 9.9. Average Daily Gain Between Feed Intervals for Various Protein Levels

Pounds of Feed	Daily Gain Computed as an Average Between 50-Pound Feed Increments With Protein Percentages of					
	8	10	12	14	16	18
100	.632	.817	1.000	1.128	1.241	1.343
150	.819	1.074	1.294	1.422	1.511	1.563
200	.965	1.280	1.520	1.630	1.676	1.661
250	1.088	1.457	1.704	1.784	1.777	1.685
300	1.194	1.613	1.860	1.901	1.833	1.658
350	1.288	1.754	1.992	1.988	1.852	1.591
400	1.373	1.882	2.106	2.051	1.843	1.493
450	1.449	2.000	2.205	2.094	1.809	1.369
500	1.519	2.110	2.289	2.118	1.754	1.222
550	1.583	2.211	2.362	2.126	1.680	1.056
600	1.642	2.305	2.423	2.119	1.590	.873
650	1.696	2.394	2.474	2.099	1.484	.675
700	1.746	2.476	2.516	2.067	1.365	.463

Historic Outcomes

The effect of the seasonal fluctuation in hog prices on returns over feed cost has been examined over the 20-year period from 1935 through 1954 for spring hogs farrowed at four different dates. These figures indicate the number of years in which either the least-time or least-cost rations would have been most profitable. The market price at which the hogs would have been sold under each system has been determined by taking into account the time required to produce a 225-pound hog with least-cost and least-time rations. The feed prices used were the average annual price of soybean oilmeal in each of the years and the price of corn in the month at which the hogs reach weaning weight, 75 pounds liveweight, and 150 pounds liveweight. The price of corn was assumed constant for the duration of each weight interval. It was further assumed that 6 weeks would be required to raise pigs from farrowing to weaning weight. The farrowing dates considered were February 1, March 1, April 1, and May 1.

In the 20 years, with hogs farrowed on February 1, the least-cost rations would have resulted in the greater return over feed costs in 15 of the years. The least-time ration would have given the greater returns over feed cost in only 5 years. With hogs farrowed on March 1, the least-cost ration would have given the greatest return over feed cost in 19 years; the least-time ration would have given greater returns in only 1 out of the 20 years. For hogs farrowed on April 1, the least-cost

rations would have been more profitable in 13 years, while the least-time ration would have been more profitable in 7 out of the 20 years. For a May 1 farrowing date, the least-cost ration was more profitable in 9 years, while the least-time ration was more profitable in 11 of the 20 years.

Rate of gain is of lesser importance with hogs farrowed early in the season. Hogs farrowed in February, March, and April can be produced more profitably on the least-cost ration a greater proportion of the time than on the least-time ration. Rate of gain is of much greater importance for hogs farrowed late in the season, because of sharp seasonal price declines in October. The lower market price often more than offsets the feed economies obtained by feeding the least-cost ration.

The average feed costs for the 20-year period in producing a 225-pound market hog on pasture differ only slightly for least-cost and least-time rations. The small difference is due to the fact that the hogs are on pasture, with no costs figured for the latter. Protein from legumes replaces some of that which would otherwise be obtained at a cost from soybean oilmeal in drylot. The rates at which soybean oilmeal substitutes for corn in the hog ration under a pasture feeding system are such that the least-cost ration deviates only slightly from the rations which maximize the rate of gain. The modal gain for the least-cost ration was less than 50 cents over the 20 years examined. The largest difference was for the February 1 farrowing date. In 2 of the 20 years, the returns for the least-cost ration exceed the returns for the least-time ration by more than $2.50. In 10 of the years, the returns with the least-cost ration were within $1 of the returns with the least-time ration. The returns from the least-cost and least-time rations differed by $1 or less for all 20 of the years under a March 1 farrowing date. On the average, over a period of years, little gain is forthcoming from feeding the least-cost, as compared to the least-time, ration on pasture. However, in a few years the economic advantage of the least-cost ration with a pasture feeding system is quite large.

Grinding and Mixing Versus Free Choice

Feeding either least-time or least-cost rations required grinding and mixing of concentrate feeds. These extra steps add to the cost of the rations. Hence, an additional study is needed for comparison between (a) the costs of rations fed free choice and (b) the costs of rations plus the costs of grinding and mixing, for least-time and least-cost rations. These comparisons are not possible from the data of this study.

Earl O. Heady
Stanley Balloun
Robert McAlexander

CHAPTER 10

Production Functions, Least-cost Rations, and Optimum Marketing Weights for Broilers

Feed is the major cost in broiler production. Hence, one of the main opportunities for increasing profit is to lower feed costs per unit of gain. Great progress has been made in recent years in developing high energy feeds which lessen feed inputs for producing birds of a given weight. However, the types of cost-minimizing and profit-maximizing principles explained elsewhere in this book have not been widely applied. This void has existed because of lack of data in appropriate forms. This study was designed to provide data adapted for these purposes.

Ordinarily, broiler feeds are made up of two major categories of feeds, along with the proper vitamins and minerals. These two categories include feeds high in carbohydrate such as corn and feeds high in protein such as soybean oilmeal. If prices of these feeds did not change, the least-cost ration determined at one point in time also would be the least-cost ration at all later points in time. However, the prices of these major feed sources do change. In recent years the price of corn has been as low as 1.8 cents per pound with soybean oilmeal as high as 4.5 cents per pound, a SBOM to corn price ratio of 2.5; in other years the price of corn has been as high as 4.5 cents per pound with soybean oilmeal as low as 3.5 cents per pound, a SBOM to corn price ratio of .8. The ration or combination of these two feeds which minimizes costs of gains under one of these price ratios will not also minimize costs under the other ratio. The least-cost ration can be determined by relating the prices of the feed sources to the rates of substitution of the feeds.

OBJECTIVES

The specific purposes of this study are: To predict the broiler production surface (function) of gains in relation to two feed categories; to predict input-output relationships of gain in terms of a fixed combination of the two feeds; to predict gain or growth isoquants indicating the possible combinations of two feeds which will result in a fixed gain level; to predict the marginal rates at which high-carbohydrate and high-protein feeds substitute for each other in producing a particular level

of gain; to predict isoclines indicating feed combinations for particular gain levels which have the same rate of substitution; to indicate rations which will minimize costs of gains under various price relationships; to predict rations which minimize the time required for attaining a particular marketing weight and to indicate marketing weights which will maximize profits above feed costs for various price relationships.

However, the study is partly of a methodological nature. Since broiler production functions have not been estimated previously, several types of equations have been fitted to the data. For the practical use of the data in specifying rations which average least-cost over two weight ranges, a power function is employed although it does not serve most efficiently in predicting the entire surface. However, as mentioned in Chapter 9 for hogs, broiler producers are interested in rations applicable for a feeding period rather than in daily recalculating the least-cost ration. Finally, comparisons are made in prediction of rations and marketing weights when several types of equations are used.

DESIGN OF EXPERIMENT AND FEEDING METHODS

Data for this study were obtained from an experiment conducted by the Department of Poultry Husbandry at Iowa State University. Six hundred New Hampshire chicks were used in the experiment. These chicks were randomly assigned to 30 pens (batteries) with a restriction of having 10 cockerels and 10 pullets per pen. The broilers were self-fed on six different rations consisting of 16, 18, 20, 22, 24, and 26 per cent protein levels. The experiment was designed so that there were at least two replicates on each ration. Twelve groups of broilers were fed rations with fixed proportions of corn to protein for the entire period. In other words, two pens each of the birds were fed the entire period on the 16, 18, 20, 22, 24, and 26 per cent rations. The other 18 pens of birds were fed up to a weight of approximately 1.32 pounds per bird, then changed to lower protein rations for the remainder of the feeding experiment as shown in Table 10.1. The birds were weighed each week and corresponding feed inputs were determined to provide observations for regression analysis. The birds were taken off the experiment at the end of 11 weeks. The experimental unit was a pen, with each weighing becoming an observation.

Corn was the main source of carbohydrates and soybean oilmeal was the main source of protein. The soybean oilmeal contained approximately 45 per cent crude protein while the corn contained approximately 8.4 per cent. (See Table 10.2.)

ESTIMATION OF PRODUCTION FUNCTIONS

Four over-all equations were fitted to the experimental observations as one step in the objectives of this study. They are quadratic equation

Table 10.1. Design of Experiment for Broiler Study

Pen-Numbers Replicates I	II	Per Cent Protein Rations Fed Broilers From Weight of .09 to 1.32 Pounds	Per Cent Protein Rations Fed Broilers From Weight of 1.32 Pounds to End of Feeding Period
22	25	16	16
27	20	18	18
16	29	18	16
2	10	20	20
3	8	20	18
28	19	20	16
16	24	22	22
17	5	22	18
6	13	22	16
18	14	24	24
11	9	24	20
7	30	24	16
1	4	26	26
12	23	26	22
21	15	26	18

10.1, a square root, equation 10.2, a Cobb-Douglas, equation 10.3, a resistance, equation 10.4, and a Spillman-type, equation 10.5, functions. These functions are based on the 12 pens (two pens per ration) of rations which were continued from initiation of the experiment up to an average liveweight of 3.13 pounds, with 11 to 13 weighings per pen for a total of 146 observations.[1] The functions for over-all production surface estimates are as follows:

Table 10.2. Pounds of Ingredients Used per 100 Pounds of Feed in Broiler Experiment

Ingredients	Per Cent Protein in Ration					
	16	18	20	22	24	26
Ground yellow corn	71.0	65.5	59.6	53.9	48.2	42.5
Wheat middlings	5.0	5.0	5.0	5.0	5.0	5.0
Dehydrated alfalfa meal (17%)	2.5	2.5	2.5	2.5	2.5	2.5
Soybean oilmeal	15.0	20.5	26.0	31.5	37.0	42.5
Fishmeal	2.5	2.5	2.5	2.5	2.5	2.5
Steamed bonemeal	2.0	2.0	2.0	2.0	2.0	2.0
Ground oyster shells	.5	.5	.5	.5	.5	.5
Iodized salt	.5	.5	.5	.5	.5	.5
C-2054 (premix)	1.0	1.0	1.0	1.0	1.0	1.0
Soybean Oil	--	.2	.4	.6	.8	1.0

[1]The number of observations were not the same for each pen since observations at 600 and 1,300 gram weights also were obtained for all pens; for some pens these weights occurred at the same time as the regular weekly weighings, resulting in fewer observations in these particular pens.

(10.1) $G = .0331 + .4823C + .6415S - .0183C^2 - .0497S^2 - .0232CS$

(10.2) $G = 10.1730 - .2300C - .1775S + .3314\sqrt{C} + .5004\sqrt{S} + .0200\sqrt{C\ S}$

(10.3) $G = .9922C^{.5537}S^{.3371}$

(10.4) $G^{-1} = .1532 + .2700C^{-1} + .4512S^{-1}$

(10.5) $G = 17.908 - 11.442(.9238^{C}) - 14.343(.9257^{S}) + 8.323(.9238^{C})(.9257^{S})$

G refers to gain in pounds per broiler, C refers to pounds of corn per bird, and S refers to pounds of soybean oilmeal per bird. Statistics for these equations of the over-all production surfaces are presented in Table 10.3. Aside from the interaction term in the square root equation, the regression coefficients are all highly significant. A problem of autocorrelation arises in estimating the regression coefficients of the over-all production surface for this reason: The 12 and 13 observations for each pen are not independent. Hence, it can be claimed that the total degrees of freedom (df) is something less than the 140 remaining after estimating the regression coefficients. However, if the total number of degrees of freedom remaining after estimation of regression coefficients is considered to be only 6 (i.e., to correspond to 12 pens or independent observations), the regression coefficients for the quadratic, equation 10.1, power, equation 10.3, and resistance, equation 10.4, equations are still acceptable at the .01 and .05 probability levels. In all of the equations, the feed inputs account for over 99 per cent of the variance in gains. The over-all quadratic and power equations are used for estimates of this study in prescribing least-cost "average" rations and optimum marketing weights.

These particular functions were selected for estimating a broiler production surface because of their logical basis. Other studies for meat production indicate that output tends to increase at a decreasing rate (i.e., each additional input of feed usually results in less and less gain in weight), and feeds tend to substitute for each other at diminishing

Table 10.3. Values of R and t for Equations 10.1 to 10.5

Equation	R	Values of t for Regression Coefficients in Order Shown				
		b_1	b_2	b_3	b_4	b_5
10.1	.999*	39.82*	29.69*	7.42*	7.44*	3.28†
10.2	.999*	6.68*	3.01‡	6.10*	6.89*	.24§
10.3	.999*	43.72*	26.97*	--	--	--
10.4	.979*	23.48*	20.99*	--	--	--
10.5**	.910*	2.09‡	4.19†	3.52†	--	--

*$p < .01$ for both 140 and 6 df.
†$p < .01$ for 140 df and $p < .02$ for 6 df.
‡$p < .01$ for 140 df and $.01 < p < .05$ for 6 df.
§$p > .5$ for 140 df.
**Because of the nonlinear form of this equation, the listed statistics are somewhat inexact.

rates. These conditions are expected for broilers. As birds increase in size, more and more of each pound of feed is required for maintenance. Also, as broilers increase in size, the composition of the body changes. The changes in body composition are expected to cause changes in the rate of substitution between feeds. The structural portions of the body and the organs develop first, followed by muscle, tissue, and fat. Thus, rations of a higher protein content are required during the early periods of growth. As birds increase in size and body weight, less protein is required, and feeds containing more carbohydrates may be substituted for high protein feeds as the fattening stage is approached.

Production functions which permit estimation of the above relationships were desired. That is, they should permit (a) decreasing productivity per pound of a fixed ration (the two feeds increased in fixed proportion) as well as diminishing productivity of either feed alone, (b) diminishing rates of substitution along a particular gain isoquant, and (c) changing substitution rates along a ration line (i.e., changing substitution ratios for a particular ration) as the bird progresses in weight and higher gain isoquants are attained. Functions 10.1, 10.2, and 10.5 meet all of these qualifications. The Spillman-type function has marginal products for corn bearing a ratio of .9238 to each other; those for soybean oilmeal bear a ratio of .9257 to each other. Production functions 10.3 and 10.4 do not permit substitution rates along a ration line to change as the broilers increase in size. That is, they do not account for the fact that protein, in relation to carbohydrates, is of greater value to the young birds than to older birds. They force into the analysis the relationship that the rate of substitution must be constant along a ration line.

The difference between the functions such as the resistance, Spillman, Cobb-Douglas and quadratic equations with respect to conditions of substitution rates along a fixed ration line can be illustrated by Figures 10.1 to 10.3. (The relationships in these illustrations are assumed and do not represent actual predictions in this study.) In Figure 10.1, the negatively sloped curves are gain isoquants or contours, indicating all of the possible combinations of the two feeds which produce, respectively, a 1.0-, 2.0-, and 3.0-pound gain on broilers. The solid and positively sloped (straight) lines are isoclines.

The isoclines are expansion paths showing the path of rations which should be followed as a bird gains in weight if profits are to be maximized (i.e., if the least-cost ration is to be used for each particular level of gain). Thus, if the price of protein feed were twice the price of carbohydrate feed, the price ratio would be 2.0. Using equation 10.1 or 10.5 to express the necessary condition, the rations (proportions of the two feeds) along isocline 2.0 should be followed in Figure 10.1. Since the isoclines in Figure 10.1 do not intersect the origin, a different ration (i.e., a different proportion of the two feeds as read off the two axes) would be required for each fractional pound of change in gain. This path is biologically logical since the proportion of carbohydrates

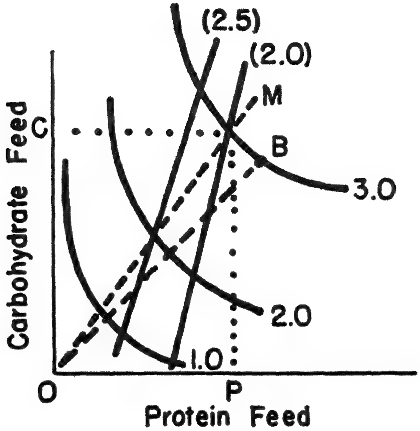


Figure 10.1. Isoclines for over-all quadratic and Spillman functions.(assumed).

to protein feed should increase with weight. (Rations higher in the plane along the 2.0 isocline include a greater proportion of carbohydrates.)

However, producers cannot practically change rations with each fractional pound of gain. Generally they feed the same ration, or change it only once, throughout the production period. If the optimum ration for a 3.0-pound gain were selected, through equating the substitution ratio with the price ratio as in equations 10.2 or 10.5, the optimum ration would include OC of the carbohydrate feed and OP of the protein feed. If the ration with the proportion of feeds at OC/OP, were fed throughout, the "feed path" would be OM.[2] This line does not indicate the least-cost ration. (Isocline 2.0 does.) Hence, another ration such as B, with feeds combined in the proportions read from the axes, could be less costly than the ration indicated by line OM, if fed over the entire production period. (Ration B would not equate substitution and price ratios for any particular levels of gain.) Thus, while the path traced by isocline 2.0 indicates the least-cost ration for all individual weights, the ration indicated by OB may be more practical and less costly than the ration indicated by OM.

The functions 10.3 and 10.4 provide isoclines of the nature of the positively sloped lines labeled 2.5 and 2.0 in Figure 10.2. Since they are linear and pass through the origin, these isoclines suggest that the rates of substitution of the two feeds do not change along a fixed ration line as a bird progresses to heavier weights. If they are used for decisions, they indicate that the same ration should be used from the

[2]A diagonal line could be drawn from each point when the 2.0 isocline intersects the 1.0-, 2.0-, and 3.0-pound isoquants to the origin. The slopes would indicate rations which could be fed in each of the three weight ranges: 0 to 1.0, 1.01 to 2.0, and 2.01 to 3.0 pounds, respectively. These rations would be "averages" for the intervals and would not equate substitution and price ratios for each gain isoquant within an interval.

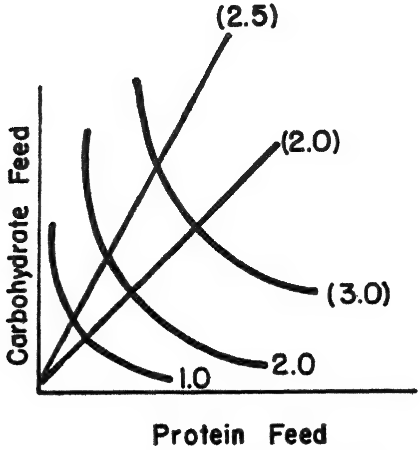


Figure 10.2. Isoclines for over-all resistance and Cobb-Douglas functions (assumed).

beginning to the end of the production period. Biologically, it is expected that rates of substitution do change along a ration line. Therefore, a linear isocline should not pass through the origin. However, while these equations may not provide the greatest degree of biological accuracy, they may provide a practical basis for selecting the one "average" ration to be fed over the entire production period for producers who use this method of feeding. Hence, with a price of protein feed twice as great as the price of carbohydrate feed, the single ration (proportion of the two feeds) indicated by isocline 2.0 in Figure 10.2 would be fed over the entire production period.

An alternative combining the advantages of the two types of functions has been used in this study. It includes the estimation of Cobb-Douglas functions for different weight intervals or segments of the production function. The effect on isoclines is that shown in Figure 10.3. The Cobb-Douglas functions of the various weight intervals provide isoclines for a particular segment of the production surface. They indicate the one best "average" ration to be fed over the particular weight interval; the "average" ration for the interval can be determined by equating the price and substitution ratios. The isoclines for each interval can be combined to give isoclines with linear segments as in Figure 10.3. In this case, with a price ratio of 2.0, the lower linear segment of isocline 2.0 indicates the "average," least-cost ration to be fed to 2 pounds of weight, and the middle segment indicates the "average" least-cost ration to be fed between 2 and 3 pounds of weight. Since the second segment, based on the second interval function, has a greater slope than the first segment, a ration containing a greater proportion of carbohydrate feed would be used for heavier weights.[3] Hence, on

[3] Actually all segments of the "combined isocline" in Figure 10.3 originate at zero for

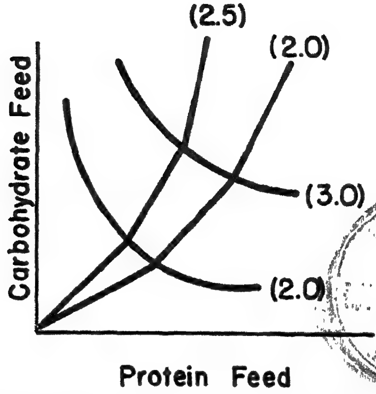


Figure 10.3. Isocline segments for interval functions (assumed).

practical grounds, the producer could use two or more rations over the production period (without changing feed proportions for each fraction of gain as indicated along the isoclines of Figure 10.1).

For the reasons outlined above, the over-all power and resistance functions are used to predict optimum feed quantities when a single ration is to be fed over the entire production period. Two interval power equations, presented later, have been used to provide data allowing one change in the ration where this practice is preferred. (Producers seldom use more than two rations.) However, since they assume constant elasticity over the entire production surface, the over-all power equations tend to overestimate the gains associated with total feed consumption during the latter part of the production period. Hence, they tend to overestimate the optimum marketing weight for a particular broiler price. For this reason the quadratic equation is used in predicting most profitable marketing weights. In effect, this procedure is one of using Cobb-Douglas functions to predict "average" rations to be used as practical alternatives up to weights of nearly 3 pounds. Beyond this weight, the quadratic function can be used to specify (a) "exact" feed combinations as marketing time approaches and (b) optimum marketing weights.

Production Surfaces From Over-all Functions

Production surfaces based on equations 10.1, 10.3, 10.4, and 10.5 are presented, respectively, in figures 10.4 to 10.7. Lines OD and OB in the feed plane are ration lines indicating 16 and 26 per cent protein

their particular weight interval. However, they can be spliced together as indicated to represent total gains and contours over the entire production surface, rather than gains within a single interval.

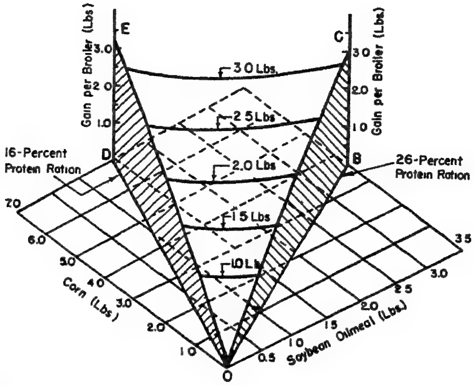


Figure 10.4. Production surface showing feed-gain relationships predicted from over-all quadratic function 10.1 for broilers fed 16- to 26-per cent protein rations.

rations, respectively. Curve OE above the 16 per cent ration line of OD indicates gain levels when various amounts of this particular ration are fed per bird. Curve OC above the 26 per cent ration line of OB indicates gain levels when various amounts of a 26 per cent ration are fed. Other ration lines (i.e., 18, 20, 22, etc.) such as OD and OB could be drawn in the feed plane, and each would have above it an input-output curve for the particular ration. These quantities, corresponding to equation 10.1 provide the later basis for determining the optimum marketing weight when a particular ration is fed; they are not used for predicting the optimum ration. Figures 10.6 and 10.7 are drawn to provide a somewhat different perspective of the surface than are figures 10.4 and 10.5.

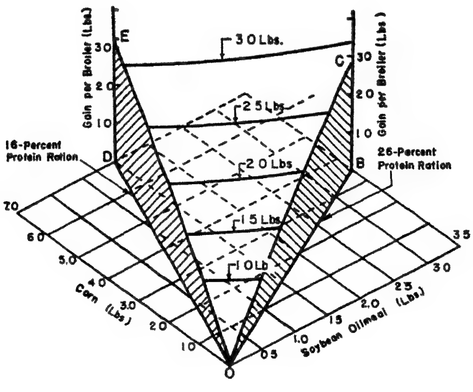


Figure 10.5. Production surface showing feed-gain relationships predicted from over-all Cobb-Douglas function 10.3 for broilers fed 16- to 26-per cent protein rations.

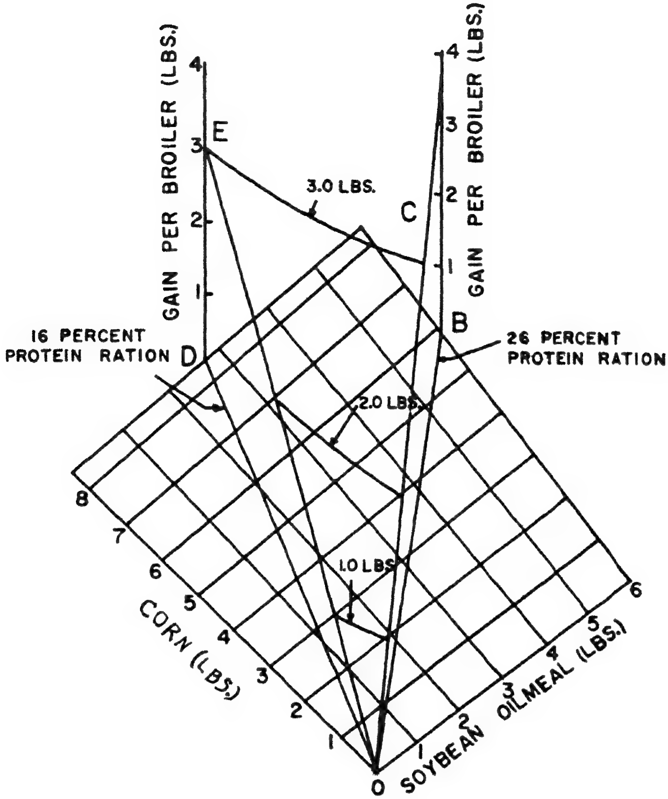


Figure 10.6. Production surface showing feed-gain relationships predicted from resistance formula 10.4 for broilers fed 16- to 26-per cent protein ration.

The contours on the surfaces are gain isoquants. For all functions, the gain contours are curved indicating that, as a greater proportion of one feed is used, the amount replaced of the other feed declines. The nature of the surfaces indicates diminishing marginal productivity (i.e., each pound of feed adds less to total weight than the previous pound) for particular rations.

Individual Growth Functions

It has been suggested that under certain conditions input-output curves and isoquants derived from a single equation estimate of the production surface might be spurious. Supposedly, this situation might occur where one portion of the surface drops discretely down to a ledge, or a "canyon" exists on one part of the function. The over-all equation

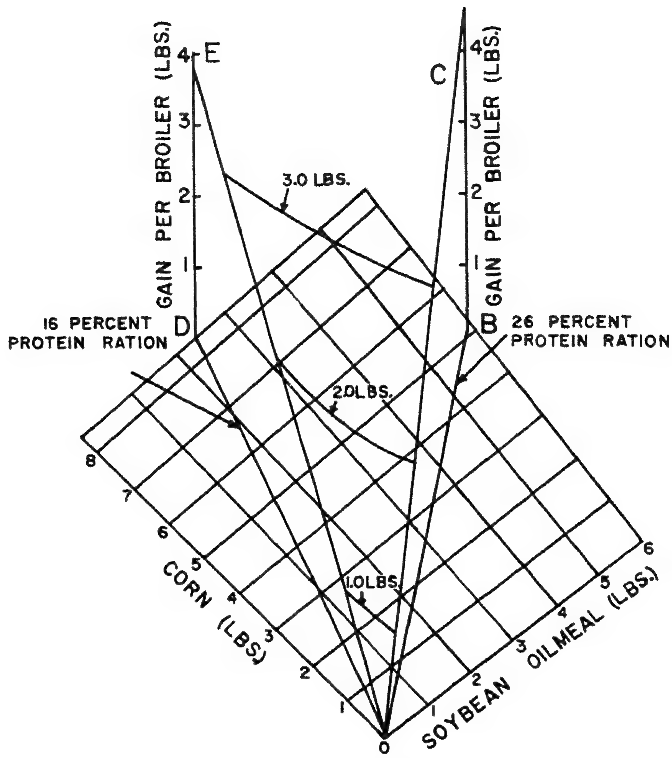


Figure 10.7. Production surface showing feed-gain relationships predicted from Spillman function 10.5 for broilers fed 16- to 26-per cent protein ration.

would be affected equally by all observations over the surface and would not allow prediction of this discrete depression in gains. Also, if the joint relationships involved were sufficient (gain depending on feed input and feed intake depending on weight), estimation by simultaneous equations might be appropriate. As a basis of comparison of the predictions made from the over-all functions, single-variable equations of the quadratic and Cobb-Douglas types were estimated for each ration included in the study. An input-output curve for each ration, independent of those for all other rations, was then predicted from the single-variable equations and compared with a similar estimate from the over-all function.

The single-variable functions have been derived with gain, G, as the dependent variable and corn, C, as the independent variable. This procedure can be used since the proportion of soybean oilmeal to corn is fixed for any one of the six different protein levels.

The derived polynomial functions, hereafter designated as quadratic single-variable functions for the indicated per cent protein levels, are:

$$G = -.0296 + .5984C - .0244C^2 \tag{10.6}$$ (16%)

$$G = .0370 + .6886C - .0323C^2 \tag{10.7}$$ (18%)

$$G = .0444 + .7183C - .0305C^2 \tag{10.8}$$ (20%)

$$G = .0256 + .8726 - .0520C^2 \tag{10.9}$$ (22%)

$$G = .0319 + 1.0030C - .0680C^2 \tag{10.10}$$ (24%)

$$G = .0377 + 1.0983C - .0868C^2 \tag{10.11}$$ (26%)

where G is pounds gain in weight per broiler, and C represents the pounds of corn fed in the various broiler rations. Implicit in each pound of C are other feed inputs as described earlier.

Single-variable ration functions of the Cobb-Douglas type for the various protein levels are:

$$G = .5878C^{.9029} \tag{10.12}$$ (16%)

$$G = .6669C^{.8905} \tag{10.13}$$ (18%)

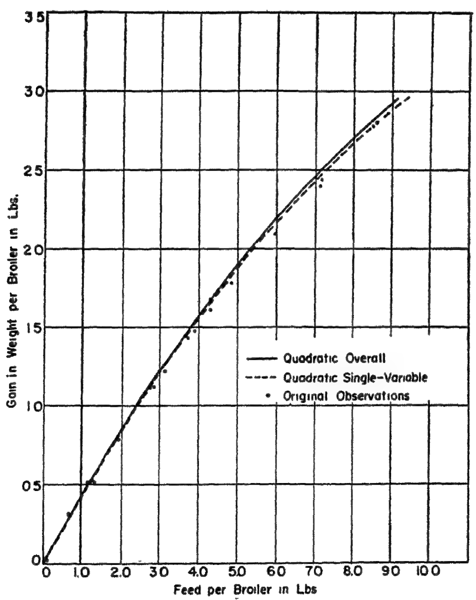


Figure 10.8. Comparison of input-output curves for broilers on a 16 per cent protein ration as predicted by quadratic over-all function 10.1 and quadratic single-variable function 10.6.

(10.14) $G = .7240C^{.8786}$ (20%)

(10.15) $G = 9.7997C^{.9172}$ (22%)

(10.16) $G = .9422C^{.8624}$ (24%)

(10.17) $G = 1.0048C^{.8723}$ (26%)

The input-output curves for particular rations derived from the single-variable ration functions were all plotted on scatter diagrams for comparison with their respective over-all functions. Comparisons of input-output curves derived from the single-variable and over-all quadratic functions are shown in figures 10.8 through 10.13 for the six rations. The similarity of these two sets of curves indicates that the over-all function does not give spurious results for any particular level of protein. Similar comparability existed for estimates from single-variable and over-all Cobb-Douglas equations.

Interval Functions

The interval functions of the Cobb-Douglas type used for predictions

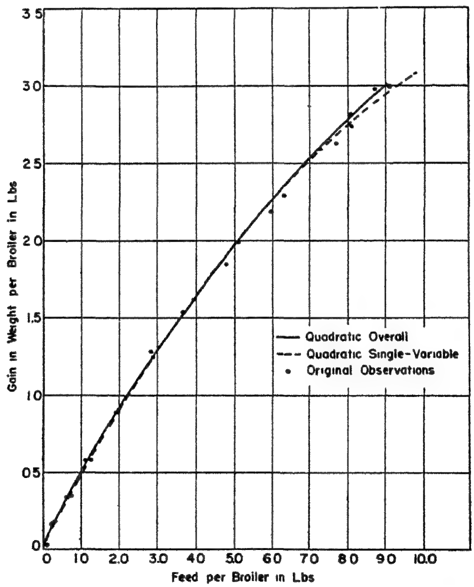


Figure 10.9. Comparison of input-output curves for broilers on an 18 per cent protein ration as predicted by quadratic over-all function 10.1 and quadratic single-variable function 10.7.

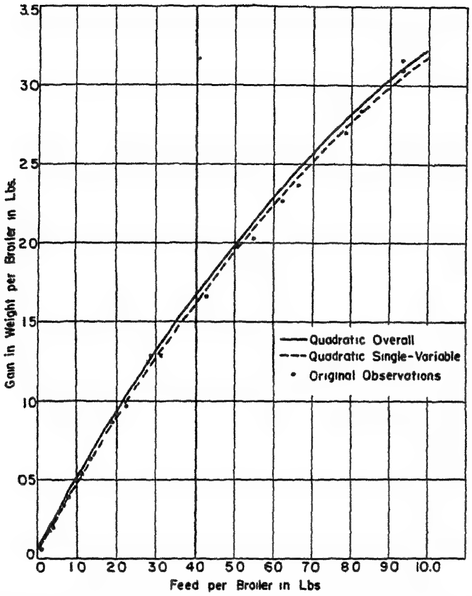


Figure 10.10. Comparison of input-output curves for broilers on a 20 per cent protein ration as predicted by quadratic over-all function 10.1 and quadratic single-variable function 10.8.

of "average" least-cost rations over two weight intervals are provided in equations 10.18 and 10.19. Equation 10.18 has been fitted to observations in the weight interval of 1.3 pounds (600 grams) or less while equation 10.19 has been fitted to observations in the interval of weights greater than 1.3 pounds.

(10.18) $G = 1.0754C^{.5425}S^{.3838}$ (up to 1.3 pounds liveweight)

(10.19) $G = .7021C^{.6163}S^{.2944}$ (over 1.3 pounds liveweight)

Statistics for the two interval functions are given in Table 10.4. Even for a number of individual pens (rather than of pens by number of weighings) the statistics are significant at the 1 per cent level of probability. As mentioned earlier, some pens were fed the same ration throughout the experiment. (These are the observations upon which the over-all functions and the single-variable functions are based.) Since some pens were switched to different rations at a liveweight level of 1.3 pounds, an analysis of variance was made for gains of birds in the second interval in relation to gains on rations fed in the first interval. It was found that gains in the second period did not differ significantly in terms of the ration fed in the first period. Gains in the second period did not appear to be associated with protein level in the first

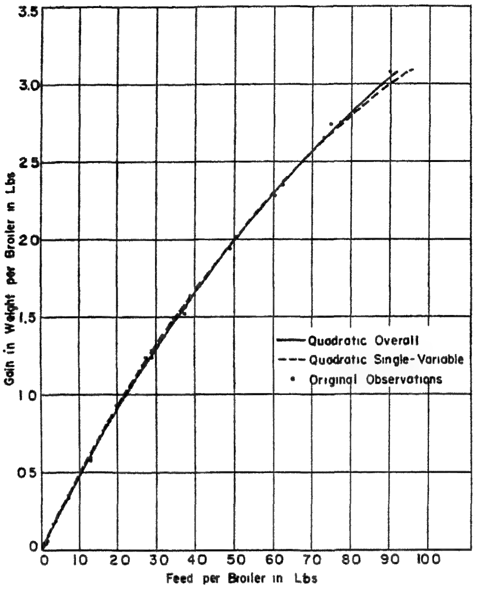


Figure 10.11. Comparison of input-output curves for broilers on a 22 per cent protein ration as predicted by quadratic over-all function 10.1 and quadratic single-variable function 10.9.

period. Hence, data for the "straight through" and "switched" pens were pooled, and each of the interval functions is based on observations for 30 pens averaging slightly over 6 weighings each (189 observations).

The constant elasticity of the over-all Cobb-Douglas function causes it to overestimate the gains associated with particular feed inputs as birds approach maturity as illustrated in Figure 10.14. The curves for the Cobb-Douglas over-all function fit the gain observations poorly at high feed inputs.

Table 10.4. Multiple Correlation Coefficients and Values of t for Interval Cobb-Douglas Functions 10.18 and 10.19

Interval	R Values	Value of t for Regression Coefficients in Order Given in Equation	
		b_1	b_2
.09-1.32 lbs. liveweight	.9956*	33.10*	46.49*
Above 1.32 lbs. liveweight	.9885*	38.09*	17.95*

*$p < .01$ for 186 or 27 degrees of freedom.

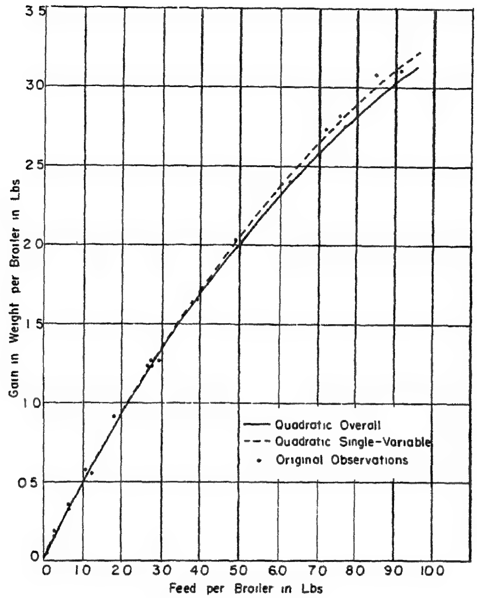


Figure 10.12. Comparison of input-output curves for broilers on a 24 per cent protein ration as predicted by quadratic over-all function 10.1 and quadratic single-variable function 10.10.

Input-output curves for the same three protein levels are provided in figures 10.15 and 10.16 when the estimates are made by "splicing" together the two interval Cobb-Douglas equations. The "spliced" input-output curves represent the portion for the second interval added (at the end of the first interval) to the portion for the first interval. Obviously, the problem of overestimation through use of Cobb-Douglas functions has been lessened by "splicing" together the two interval functions; "average" least-cost rations can be estimated as practical measures for the two intervals without a problem of overestimating gains.

ESTIMATION OF MARGINAL QUANTITIES AND OPTIMUM RATIONS

For the practical purposes of this study, for estimation of "average" least-cost rations over weight intervals and best marketing weights, we use estimates from the Cobb-Douglas and quadratic functions in this section. For the purpose of those who may prefer to use quantities from the Spillman-type and resistance formulas, we present added details on the latter two equations in a following section. Also, we later make some comparisons between these two functions and the quadratic.

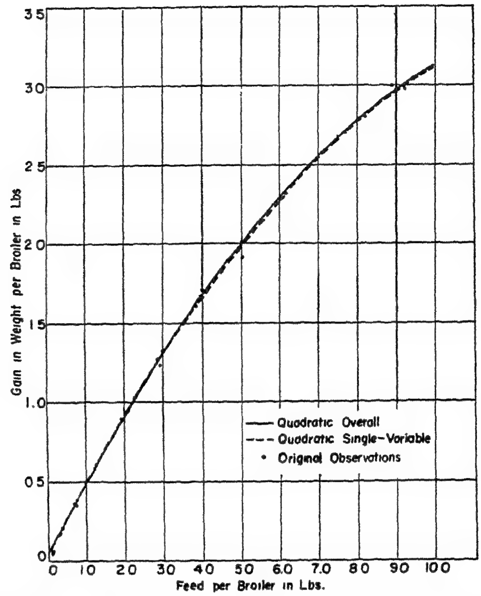


Figure 10.13. Comparison of input-output curves for broilers on a 26 per cent protein ration as predicted by quadratic over-all function 10.1 and quadratic single-variable function 10.11.

Total and Marginal Gains

Marginal product functions with corn and soybean oilmeal fixed are listed in equations 10.20 and 10.21, respectively, for the over-all quadratic production surface and in equations 10.22 and 10.23, respectively, for the over-all Cobb-Douglas function.

$$\frac{\delta G}{\delta C} = .4823 - .0366C - .0232S \tag{10.20}$$

$$\frac{\delta G}{\delta S} = .6415 - .0994S - .0232C \tag{10.21}$$

$$\frac{\delta G}{\delta C} = .5494C^{-.4463}S^{.3371} \tag{10.22}$$

$$\frac{\delta G}{\delta S} = .3345C^{.5537}S^{-.6629} \tag{10.23}$$

Table 10.5 includes total weights per bird and marginal gains per pound of feed when total feed input per bird is at specified levels for various rations. Diminishing productivity of feed is indicated in total

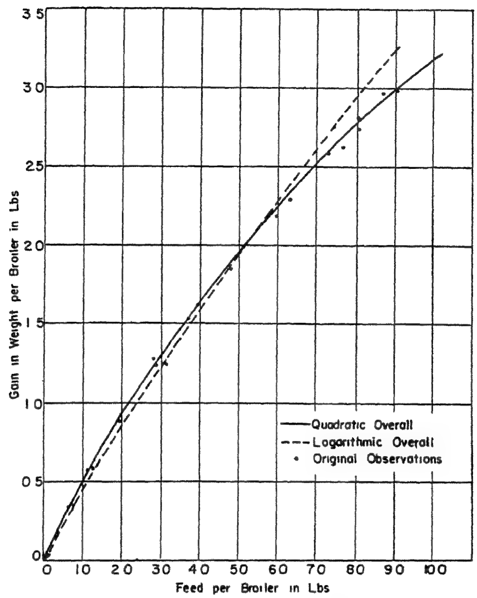


Figure 10.14. Comparison of various over-all functions for estimating input-output curves for an 18 per cent protein ration. Quadratic equation 10.1 and Cobb-Douglas equation 10.3.

Table 10.5. Total Weight per Bird and Marginal Gains for Various Levels of Feed Inputs per Bird, With Feed in Fixed Proportions for Specified Rations; Over-all Quadratic Function 10.1

Pounds of Feed	Total Weight in Pounds						Marginal Gain in Pounds*					
	Per cent protein levels						Per cent protein levels					
	16	18	20	22	24	26	16	18	20	22	24	26
1	.547	.556	.562	.569	.576	.582	.415	.421	.427	.434	.440	.445
2	.947	.964	.967	.989	.999	1.010	.392	.395	.400	.405	.409	.412
3	1.322	1.346	1.363	1.380	1.394	1.406	.368	.369	.374	.377	.379	.379
4	1.671	1.702	1.723	1.742	1.757	1.768	.344	.343	.347	.349	.348	.346
5	1.994	2.032	2.056	2.077	2.091	2.097	.321	.317	.320	.320	.318	.313
6	2.292	2.336	2.363	2.383	2.393	2.394	.297	.291	.293	.292	.288	.280
7	2.565	2.614	2.642	2.660	2.666	2.657	.273	.265	.266	.264	.257	.247
8	2.811	2.866	2.894	2.910	2.907	2.888	.250	.239	.239	.235	.227	.214
9	3.032	3.091	3.120	3.131	3.120	3.085	.226	.213	.212	.207	.196	.181
10	3.228	3.291	3.319	3.323	3.300	3.250	.203	.186	.185	.178	.166	.148
11	3.397	3.464	3.491	3.487	3.451	3.382	.179	.160	.158	.150	.136	.115

*Marginal gains are computed as a derivative of the over-all quadratic function and represent the marginal physical products at the feed quantities shown in the first column.

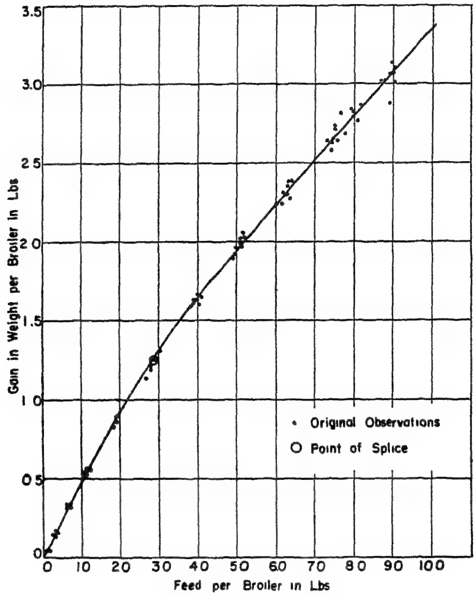


Figure 10.15. "Spliced" input-output curve for broilers fed an 18 per cent protein ration predicted from power interval functions 10.18 and 10.19.

weights; the amount added to total weight for each added pound of feed declines with total feed inputs. The maximum weight attained with 9 pounds of feed is with a 22 per cent protein ration. Rations with a greater percentage include relatively too much protein for greatest nutritional efficiency at heavier weights; rations with a smaller percentage include relatively too much carbohydrate for greatest efficiency at low weights. If extrapolations are used, the 20 per cent ration gives a maximum weight for 11 pounds of feed. Of course, the ration which gives maximum weight for a given total input of feed need not be the most profitable ration. The value of the greater gain from the particular ration must be compared with the prices of the two feeds and the quantity of each used in the ration.

The marginal gains per combined pound of feed for different rations again reflect the relative nutritive importance of the two feeds at different bird weights and total feed inputs. Up to a total feed input of 3 pounds, the marginal productivity of feed is greatest for a 26 per cent protein ration; between 3 and 5 pounds of total feed input, marginal products are greatest for a 22 per cent ration; between 6 and 8 total pounds of feed, a 20 per cent ration has the largest marginal products while for total feed inputs of 9 or more pounds, the 16 per cent ration has the greatest marginal productivity. These shifts in marginal productivity, as feed inputs become greater, parallel the total weights shown in the left-hand

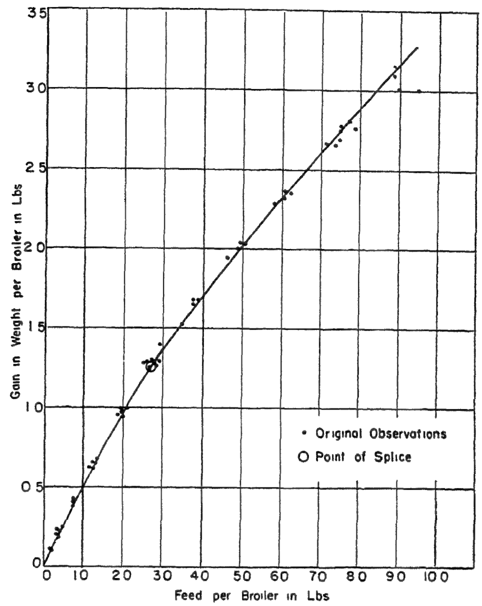


Figure 10.16. "Spliced" input-output curve for broilers fed a 22 per cent protein ration predicted from power interval functions 10.18 and 10.19.

portion of the table. The fact that marginal gains per pound of feed are greatest for (a) higher protein rations at light weights and (b) lower protein rations at high weights is illustrated graphically in Figure 10.17.

Gain Isoquants

The production functions of equations 10.1 and 10.3 are used to derive equations describing the various combinations of the two feeds which will produce a given level of gain:

$$(10.24) \qquad C = 13.1959 - .6350S \pm 27.3581\sqrt{.2351 + .0245S - .00310S^2 - .0731G}$$

$$(10.25) \qquad C = \left[\frac{G}{.9922S^{.3371}}\right]^{1.8032}$$

The gain isoquants derived from equations 10.24 and 10.25 are presented in figures 10.18 and 10.19. The contours from both equations for a given gain fall at about the same location in the feed plane for lower gains. However, for greater gains, the location of isoquants for the Cobb-Douglas function fall higher in the feed plane. (It was mentioned previously that the over-all power function tends to overestimate gains

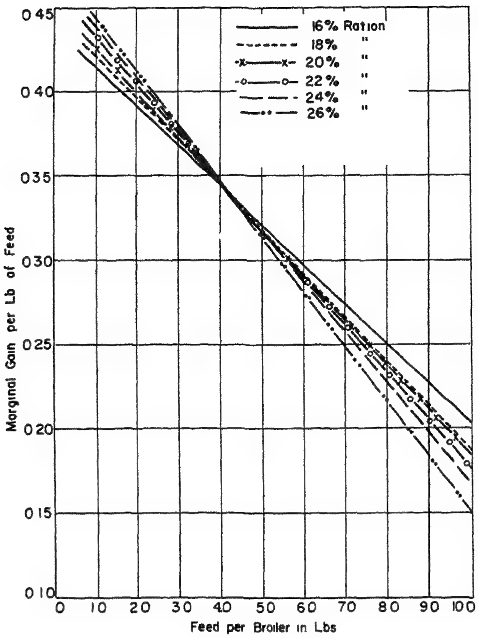


Figure 10.17. Marginal (additional) gains per pound of feed for broilers on various protein rations. Derived from over-all quadratic function 10.1.

for large feed inputs or weights per bird.) The figuration of the isoquants is most accurate for the quadratic function. However, since the slope of the isoquants along a line of given percentage protein is the same for the Cobb-Douglas function, it can serve in the practical manner mentioned earlier (i.e., it can be used to suggest the "average" least-cost over the entire growth period, although it is not best for indicating the least-cost ration for a particular increment of gain). The dots in figures 10.18 and 10.19 indicate the feed combinations and quantities necessary to give the specified gains when predictions are provided by the single-variable equations representing particular rations.

Isoquants for the lower and upper interval functions (Cobb-Douglas type) are given in equations 10.26 and 10.27, respectively. It should be remembered that within each gain interval each member of the family of gain isoquants will

(10.26) $$C = \left[\frac{G}{1.0754S^{.3838}}\right]^{1.8403}$$ (up to 1.3 pounds liveweight)

(10.27) $$C = \left[\frac{G}{.7021S^{.2944}}\right]^{1.5473}$$ (over 1.3 pounds liveweight)

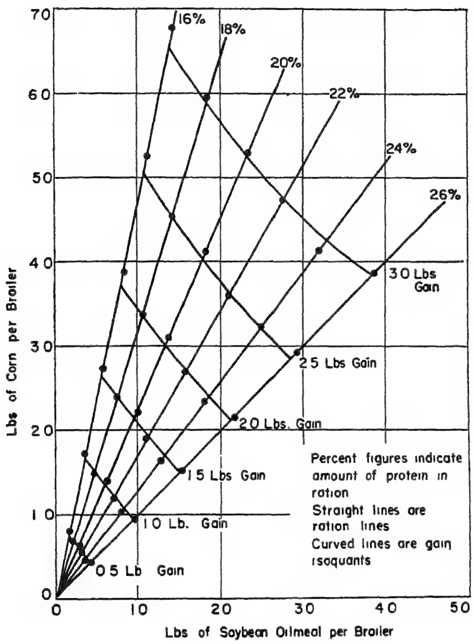


Figure 10.18. Gain isoquants predicted from over-all quadratic function 10.1. Dots show feed quantities required for same gains when predictions are based on quadratic single-variable equations for particular rations.

have the same slope along a fixed ration line for these equations. Hence, the predictions provide the basis for the practical recommendation of the "average" least-cost ration within the particular interval.

Substitution Rates for Corn and Soybean Oilmeal

Prediction of the substitution rates of soybean oilmeal for corn along the isoquants allow specifying least-cost feed combinations for particular gains. The marginal substitution rate, the slope of the iso-product curve at a particular point or for a particular feed combination, by the over-all power function is:

$$\frac{\delta C}{\delta S} = .6088\frac{C}{S}. \tag{10.28}$$

For producers who want to use only one ration during the production period, the substitution rates from the above equation would provide the "average" basis of ration selection. Where they desire to feed two rations during the production period, equations 10.29 and 10.30 can be used to express "average" substitution rates within the lower and upper

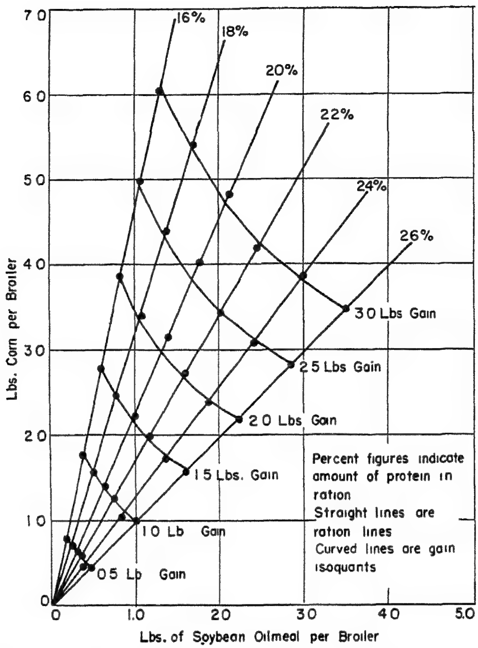


Figure 10.19. Gain isoquants predicted from power function 10.3. Dots show feed quantities required for same gains when predictions are based on power single-variable equations for particular rations.

interval, respectively. (They are based, respectively, on equations 10.26 and 10.27.)

$$\frac{\delta C}{\delta S} = .7075\frac{C}{S} \tag{10.29}$$

$$\frac{\delta C}{\delta S} = .4555\frac{C}{S} \tag{10.30}$$

Data in Table 10.6 derived from equations 10.18 and 10.19 show the various combinations of the two feeds which will produce 1 pound of gain at broiler weights of 1.32 and 3.09 pounds liveweight. Columns 4 and 7 provide the substitution quantities in tabular form and are derived from equations 10.29 and 10.30. Since the data in Table 10.6 are for power functions, the substitution ratio will not change between isoquants within a gain interval (i.e., for other broiler weights) when the feeds are combined in a fixed proportion to result in a given percentage of protein in the ration. In other words, 1 pound of soybean oilmeal substitutes for 1.62 pounds of corn when .58 pound of soybean oilmeal is combined with 1.33 pounds of corn, a total of 1.91 pounds, into a 20 per cent protein

Table 10.6. Combinations of Corn and Soybean Oilmeal for Producing a Pound of Gain and Marginal Substitution Rates for Broilers of 1.32 and 3.09 Pounds Liveweight. Estimates Based on Interval Cobb-Douglas Functions 10.18 and 10.19

Per Cent Protein In Ration	Lbs. Feed To Produce 1 lb. of Gain*		Marginal Rate of Substitution of Soybean Oilmeal for Corn‡	Lbs. Feed To Produce 1 lb. of Gain†		Marginal Rate of Substitution of Soybean Oilmeal for Corn‡
	Corn	Soybean oilmeal		Corn	Soybean oilmeal	
16	1.790	.378	3.349	2.456	.519	2.749
17	1.609	.418	2.720	2.301	.598	2.233
18	1.521	.476	2.260	2.171	.680	1.856
19	1.417	.527	1.903	2.087	.775	1.562
20	1.326	.578	1.622	1.957	.854	1.331
21	1.285	.631	1.396	1.867	.946	1.146
22	1.174	.686	1.211	1.786	1.044	.994
23	1.109	.744	1.054	1.710	1.148	.866
24	1.049	.805	.922	1.650	1.259	.756
25	.994	.871	.807	1.576	1.381	.662
26	.940	.940	.708	1.509	1.510	.581

*Derived from equation 10.26, lower weight interval.
†Derived from equation 10.27, upper weight interval.
‡Marginal rate of substitution or $\delta C/\delta P$ refers to the pounds of corn replaced by a pound of soybean oilmeal at the indicated weights. Rates for 1.32-pound weights are from equation 10.29 while those for 3.09-pound weights are from equation 10.30.

ration for a pound of gain on birds weighing 1.32 pounds; it will substitute at the same rate for corn when feeds are combined in the same proportions for other weights up to 1.32 pounds. However, it will require less of the feeds in this fixed proportion to produce a pound of gain when broilers are at weights lighter than 1.32 pounds; it will take more at heavier weights. This difference in feed requirements per pound of gain, while feeds are held in fixed proportions to give a constant substitution rate, comes about because of a decline in the rate at which feed is transformed into gain. Comparisons of feed quantities to produce a pound of gain at weights of 1.32 pounds and 3.09 pounds illustrate this fact. A pound of gain for birds at the latter weight, with a 20 per cent protein ration, requires 1.96 pounds of corn and .85 pound of soybean oilmeal, a total of 2.81 pounds.

Substitution rates for corresponding rations are lower for 3.09-pound than for 1.32-pound broilers; a pound of soybean oilmeal replaces less corn for heavier birds than for light birds when fed the same ration. For 1.32-pound broilers on a 20 per cent protein ration, a pound of soybean oilmeal replaces 1.62 pounds of corn, but it replaces only 1.33 pounds of corn for 3.09-pound broilers. This relationship conforms to the nutritional needs of broilers at different weights. At low weights, protein is relatively more important for growth and corn is a less efficient substitute for soybean oilmeal than at heavier weights where maturity is approached.

One-pound gain isoquants for broilers of 1.32- and 3.09-pound liveweights based on the data of Table 10.6 are shown in Figure 10.20. The isoquants in this figure are to be interpreted differently than the conventional isoquant maps such as shown in figures 10.18 and 10.19. The

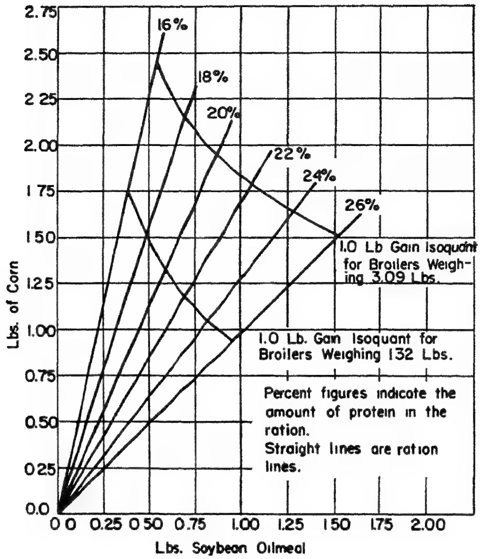


Figure 10.20. One-pound gain isoquants for broilers at 1.32 and 3.09 pounds liveweight as determined from power interval functions 10.26 and 10.27 respectively.

lower and upper curves in Figure 10.20 show the combinations of corn and soybean oilmeal required for 1 pound of gain when broilers have liveweights of 1.32 and 3.09 pounds, respectively. Conventional isoquant maps show accumulated gains (or weights) and feed inputs rather than feed inputs for a pound of gain at a specified weight. The gain isoquants shown in Figure 10.20 illustrate graphically the preceding discussion on diminishing substitution rates between rations and between weights. The fact that along a fixed ration line the 1-pound isoquant for a 3.09-pound weight has less slope than for a 1.32-pound weight indicates that soybean oilmeal substitutes at a lower rate at the heavier weight.

Least-Cost Interval Rations From Cobb-Douglas

Least-cost rations are determined by equating the marginal rate of substitution with the inverse price ratio. The isoclines, or points of equal substitution rates, lie on a straight line passing through the origin for a power function. These lines are also ration lines for the particular type of function. Thus, where the need is to predict one ration which "averages" least-cost over the entire feeding period, equating substitution rates from the over-all power function, equation 10.3, with the price ratio provides such a ration. Where the need is to change rations between two growth periods, equating the price ratio with substitution rates from interval equation 10.18 provides the average least-cost

ration for the first 6-7 weeks; equating the price ratio with equation 10.19 provides the least-cost average ration in the latter part of the feeding period. (These procedures are used as practical measures since most broiler producers change the ration not at all or only once during the production period. The quadratic function provides "biologically more accurate" isoclines, but is less practical. The isoclines do not indicate "average" rations to be fed over gain intervals.)

Data in Table 10.7 provide substitution rates to indicate least-cost rations as averages over two weight intervals or over the entire

Table 10.7. Marginal Rates of Substitution of Soybean Oilmeal for Corn for Specified Gains as Estimated by the Cobb-Douglas Over-All and Interval Functions

Per Cent Protein in Ration	1.23 Pounds Gain (1.32 Pounds Liveweight)		3.0 Pounds Gain (3.09 Pounds Liveweight)	
	Substitution rates for single ration over entire production period*	Substitution rates for first interval†	Substitution rates for single ration over entire production period*	Substitution rates for second interval‡
15.0	3.666	4.259	3.666	2.742
15.5	3.234	3.758	3.234	2.420
16.0	2.882	3.349	2.881	2.156
16.5	2.589	3.008	2.589	1.937
17.0	2.341	2.720	2.341	1.751
17.5	2.129	2.474	2.130	1.593
18.0	1.945	2.261	1.945	1.455
18.5	1.782	2.071	1.782	1.333
19.0	1.638	1.903	1.638	1.225
19.5	1.510	1.755	1.510	1.130
20.0	1.396	1.622	1.396	1.044
20.5	1.294	1.504	1.294	.968
21.0	1.202	1.397	1.202	.899
21.5	1.118	1.299	1.118	.837
22.0	1.042	1.210	1.042	.779
22.5	.972	1.129	.962	.727
23.0	.908	1.054	.908	.679
23.5	.848	.986	.848	.634
24.0	.793	.922	.793	.593
24.5	.742	.862	.742	.555
25.0	.695	.807	.695	.520
25.5	.650	.756	.650	.487
26.0	.609	.708	.609	.456

*Derivatives for over-all Cobb-Douglas function covering both weight intervals. Substitution rates do not change in the different weight intervals when the over-all function is used (see earlier discussion on logic of estimation).
†Derivatives for Cobb-Douglas function in first interval.
‡Derivatives for Cobb-Douglas function in second interval.

production period. If the price of oilmeal is 3 cents per pound and of corn is 2 cents per pound, the price ratio is 3/2 or 1.5; a 20.5 per cent protein ration gives the least-cost ration as an average over the first weight interval; for this price ratio, the least-cost ration falls between 17.5 and 18.0 per cent protein for the second weight interval. If the same ration were to be fed over the entire production period, the best "average" ration is 19.5 per cent protein. These are the rations where the marginal rates of substitution of soybean oilmeal for corn most nearly approximate the soybean oilmeal to corn price ratio of 1.5. If the price of soybean oilmeal increases to 4 cents, with corn remaining at 2 cents per pound, the price ratio becomes 2.0. An 18.5 per cent protein ration then averages least-cost for the first weight interval. This ration would be fed for a total gain of 1.23 pounds (1.32 pounds liveweight), and then a ration of 16.5 per cent protein would be fed through the second interval.

Substitution rates as averages for the over-all production period (based on the over-all power function) fall between those for the two

Table 10.8. Rations in Percent Protein Providing Least-Cost Combinations of Corn and Oilmeal With Different Feed Prices (for Broilers Fed a Fixed Percentage of Protein Throughout the Feeding Period; Cobb-Douglas Over-All Function 10.3 Used as a Basis of Feed Combinations)

Price of Corn in Cents per Pound*	Price of Soybean Oilmeal in Cents per Pound†							
	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5
1.6	18.0	17.5	16.5	16.0	15.5	15.5	--	--
1.8	19.0	18.0	17.5	16.5	16.0	15.5	15.5	15.0
2.0	19.5	18.5	18.0	17.0	16.5	16.0	16.0	15.5
2.2	20.0	19.0	18.5	17.5	17.0	16.5	16.5	16.0
2.4	20.5	19.5	19.0	18.0	17.5	17.0	16.5	16.5
2.6	21.0	20.0	19.5	18.5	18.0	17.5	17.0	16.5
2.8	22.0	20.5	20.0	19.0	18.5	18.0	17.5	17.0
3.0	22.5	21.0	20.5	19.5	19.0	18.5	18.0	17.5
3.2	22.5	21.5	20.5	20.0	19.5	18.5	18.0	18.0
3.4	23.0	22.0	21.5	20.5	19.5	19.0	18.5	18.0
3.6	23.5	22.5	21.5	20.5	20.0	19.5	19.0	18.5
3.8	24.0	23.0	22.0	21.0	20.5	20.0	19.0	18.5
4.0	24.5	23.5	22.5	21.5	20.5	20.0	19.5	19.0

*The price for corn includes the cost of grinding, mixing, and a proportionate share of the other feed ingredients included in the feed mixture other than soybean oilmeal.
†The price of soybean oilmeal includes a charge for mixing along with a proportionate share of the other feed ingredients included in the feed mixture other than corn.

intervals. With a 20 per cent protein ration, the rate of substitution of soybean oilmeal for corn is 1.622 for the first interval, 1.396 for the over-all period or function, and 1.044 for the second interval. If the ration which averages least-cost over the entire period is fed, it includes less protein for the first interval and more protein for the second interval than would be fed if separate rations averaging least-cost over the two weight ranges were used. Hence, the cost of gains to marketing would be greater for a single ration than for two different rations over the growth period. This difference must be compared to the equipment, labor and general practicality of feeding one ration throughout the period, or of shifting the ration to conform with changes in substitution rates with broiler growth.

Tables 10.8, 10.9, and 10.10 provide data showing the least-cost rations, respectively, (a) throughout the production period, (b) for the first interval of growth, and (c) for the second interval of growth when Cobb-Douglas functions are used as the basis for predicting "average" rations over the particular periods. Hence, with a "low" price for corn at 1.7

Table 10.9. Rations in Percent Protein Providing Least-Cost Combinations of Corn and Soybean Oilmeal With Different Feed Prices (for Broilers Fed a Fixed Percentage of Protein From .09 to 1.32 Pounds Liveweight; Cobb-Douglas Interval Function 10.18 Used as a Basis of Feed Combinations)

Price of Corn in Cents per Pound*	Price of Soybean Oilmeal in Cents per Pound†							
	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5
1.6	19.0	18.0	17.5	17.0	16.5	16.0	15.5	15.0
1.8	20.0	19.0	18.0	17.5	17.0	16.5	16.0	15.5
2.0	20.5	19.5	18.5	18.0	17.5	17.0	16.5	16.0
2.2	21.0	20.0	19.5	18.5	18.0	17.5	17.0	16.5
2.4	22.0	20.5	20.0	19.0	18.5	18.0	17.5	17.0
2.6	22.5	21.0	20.5	19.5	19.0	18.5	18.0	17.5
2.8	23.0	22.0	21.0	20.0	19.5	19.0	18.5	18.0
3.0	23.5	22.5	21.5	20.5	20.0	19.0	18.5	18.0
3.2	24.0	22.5	21.5	21.0	20.0	19.5	19.0	18.5
3.4	24.5	23.0	22.0	21.5	20.5	20.0	19.5	19.0
3.6	25.0	23.5	22.5	22.0	21.0	20.5	20.0	19.5
3.8	25.0	24.0	23.0	22.0	21.5	21.0	20.0	19.5
4.0	25.5	24.5	23.5	22.5	22.0	21.0	20.5	20.0

*The price for corn includes the cost of grinding, mixing, and a proportionate share of the other feed ingredients included in the feed mixture other than soybean oilmeal.
†The price of soybean oilmeal includes a charge for mixing along with a proportionate share of the other feed ingredients included in the feed mixture other than corn.

Table 10.10. Rations in Percent Protein Providing Least-Cost Combinations of Corn and Soybean Oilmeal With Different Feed Prices (for Broilers Fed a Fixed Percentage of Protein for All Weights Above 1.32 Pounds; Cobb-Douglas Interval Function 10.19 Used as a Basis of Feed Combinations)

Price of Corn in Cents per Pound*	Price of Soybean Oilmeal in Cents per Pound†							
	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5
1.6	16.5	16.0	15.5	--	--	--	--	--
1.8	17.0	16.5	16.0	15.5	--	--	--	--
2.0	18.0	17.0	16.5	16.0	15.5	--	--	--
2.2	18.5	17.5	17.0	16.0	16.0	15.5	15.0	--
2.4	19.0	18.0	17.5	16.5	16.0	15.5	15.5	15.0
2.6	19.5	18.5	17.5	17.0	16.5	16.0	15.5	15.5
2.8	20.0	19.0	18.0	17.5	17.0	16.5	16.0	15.5
3.0	20.5	19.5	18.5	18.0	17.5	17.0	16.5	16.0
3.2	20.5	19.5	19.0	18.0	17.5	17.0	16.5	16.5
3.4	21.0	20.0	19.5	18.5	18.0	17.5	17.0	16.5
3.6	21.5	20.5	19.5	19.0	18.5	17.5	17.5	17.0
3.8	22.0	21.0	20.0	19.0	18.5	18.0	17.5	17.0
4.0	22.5	21.0	20.5	19.5	19.0	18.5	18.0	17.5

*The price for corn includes the cost of grinding, mixing, and a proportionate share of the other feed ingredients included in the feed mixture other than soybean oilmeal.
†The price of soybean oilmeal includes a charge for mixing along with a proportionate share of the other feed ingredients included in the feed mixture other than corn.

cents and a "high" price for soybean oilmeal at 6 cents per pound, the least-cost ration to be fed over the entire period includes 15.0 per cent protein. With corn at 2 cents and soybean oilmeal at 4 cents, the least-cost ration in the first interval is 18.5 per cent protein; the least-cost ration for the second interval is 16.5 per cent protein. Hence, tables 10.8 to 10.10 can be used to determine the percentage of protein in the ration which gives lowest feed costs per pound of gain for any of the combinations of the prices shown. The rations indicated in the cells of the table are those where the marginal rate of substitution of soybean oilmeal for corn is equal to the price ratio obtained by dividing the soybean oilmeal price at the top by the corn price in the left-hand column of the table.

Graphic illustrations of the average least-cost rations for two different price ratios of soybean oilmeal and corn are given in figures 10.21 and 10.22 for the two weight intervals. Under a situation with a price ratio of soybean oilmeal to corn of 1.6, (e.g., $4.00 and $2.50 per hundred pounds, respectively, for the two feeds), the least-cost rations

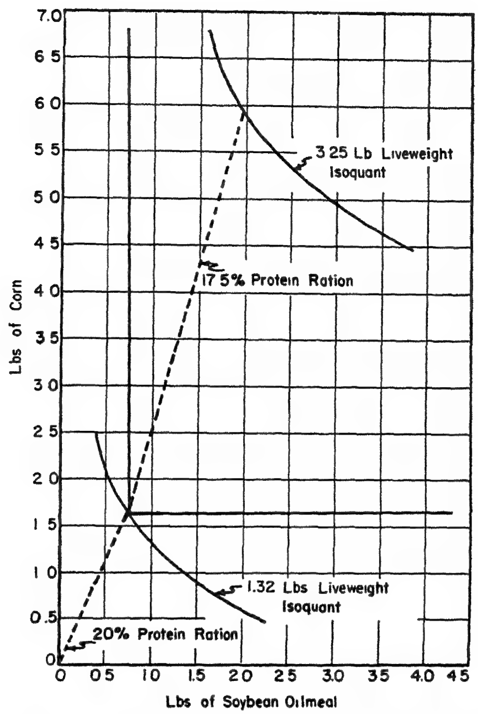


Figure 10.21. Least-cost rations for two weight intervals based on power functions 10.18 and 10.19 with a soybean oilmeal to corn price ratio of 1.6.

for the two periods are as shown in Figure 10.21. A 20 per cent protein ration provides the "average" least-cost ration until a weight of about 1.32 pounds is attained; then a 17.5 per cent ration provides the "average" least-cost ration for the remainder of the feeding period. An increase in the price ratio to 1.875, caused by an increase in soybean oilmeal prices, a decrease in corn price or a combination of these, would cause a new set of rations to become lowest in cost, as shown in Figure 10.22. The "average" least-cost rations now include 19.0 and 16.5 per cent protein levels for the first and second periods. Figure 10.23 shows the nature of *ration paths* over the two intervals when one change is made in feed combinations over the production period. The *break* in slopes of the isoclines comes at the end of the first interval. The corresponding isocline for the second interval is "spliced" on to indicate the "average" least-cost rations for the two weight ranges. Hence, the isocline labeled 2.0 would be followed for prices such as 4 cents for soybean oilmeal and 2 cents for corn, etc.

In figures 10.21 to 10.23, we should remember that the slope of the upper segment of the isocline starts from the origin of a new feed plane. In figures 10.21 and 10.22, for example, the boundaries of the new feed

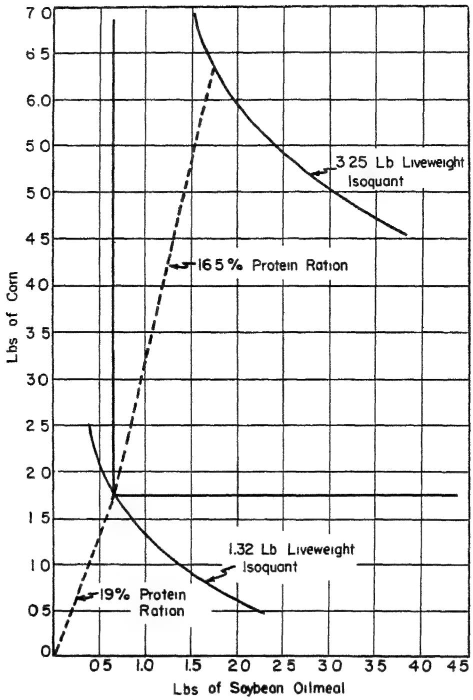


Figure 10.22. Least-cost rations for two weight intervals based on power functions 10.18 and 10.19 with a soybean oilmeal to corn price ratio of 1.875.

plane are formed by the two lines which intersect at the "splice" in the isocline. The scale for these new axes starts from zero and feeds are measured accordingly for the second interval. Although the "new axes" are not shown because of space limitations, a new origin actually occurs in Figure 10.23 at the point of "splice" for each pair of segments forming an isocline, and feeds must be measured accordingly.

Simplified Determination

Figure 10.24 provides a simplified basis for estimating the least-cost ration in either weight interval, or for the total production period. While it has been devised by relating substitution ratios to price ratios, it considers only discrete points on the production surface and specifies a single optimum ration for small ranges of price ratios. For example, it indicates a 24.5 per cent ration over the entire growth period for soybean oilmeal to corn price ratios between .7 and .8; for price ratios between 2.4 and 2.5, the optimum single ration over the entire production period is 16.5 per cent protein.

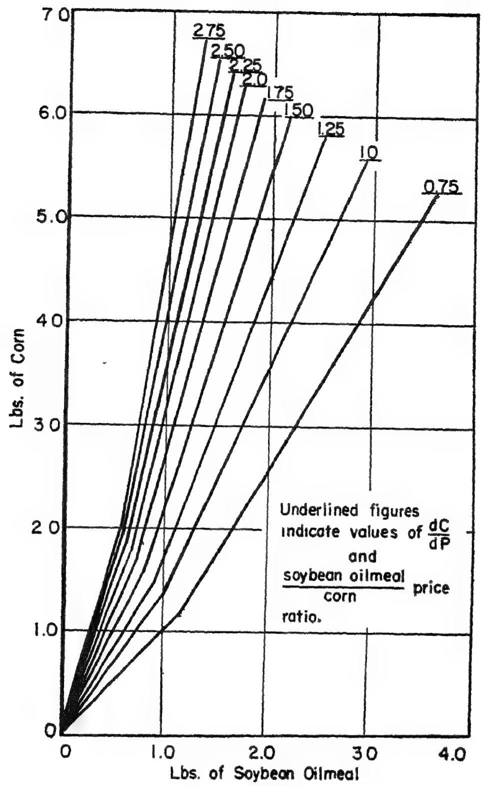


Figure 10.23. "Spliced" isoclines for two weight intervals showing path of least-cost rations when feed combinations are changed once during the production period. Based on power functions 10.18 and 10.19.

The graph can be used as follows: Suppose the price of soybean oilmeal is 6 cents per pound ($6 per hundredweight) while corn is 3 cents per pound ($3 per hundredweight). Follow across the horizontal "$6 line" for soybean oilmeal until it intersects the "$3 line" for corn. Then follow the diagonal line passing through this point of intersection to find the least-cost ration. It will include 18 per cent protein if a single ration is fed; it will include 19 per cent for the first interval and 16.5 per cent for the second interval if one change in rations is made during the growing period.

Most Profitable Weights for Broilers

While the procedure and tables outlined above allow specification of the optimum ration, they do not indicate the total amount of feed to be fed per broiler and, hence, the optimum marketing weight. However,

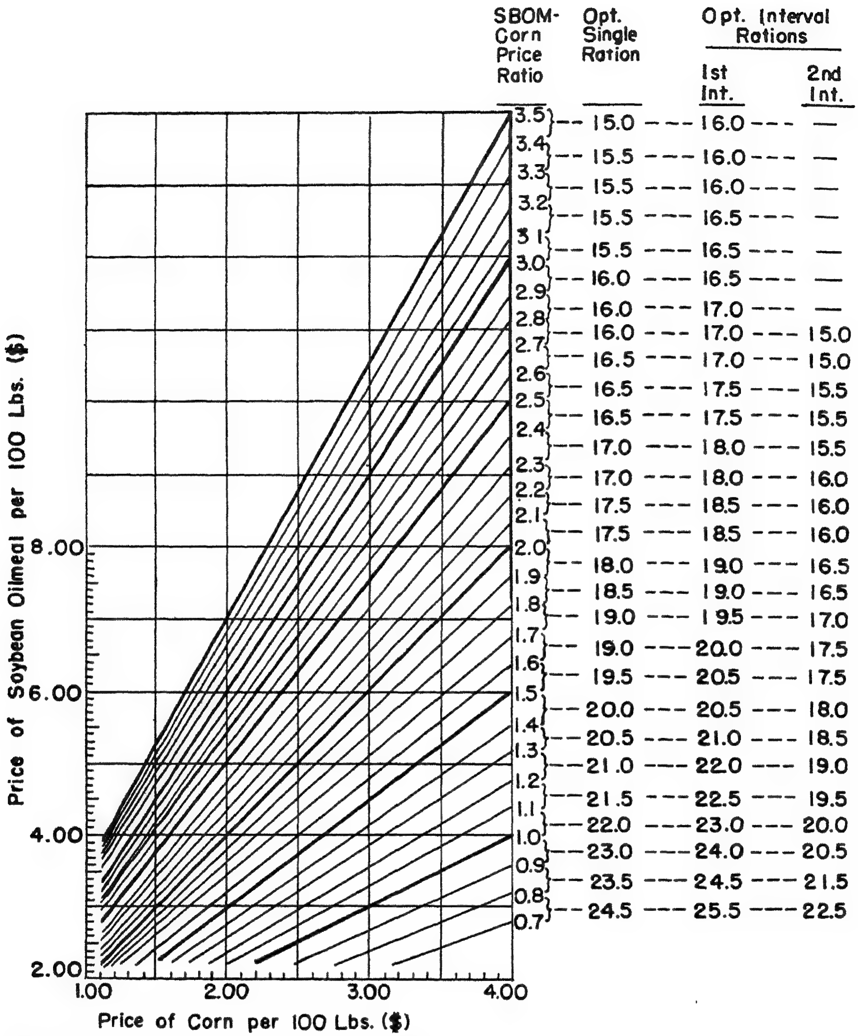


Figure 10.24. "Average" least-cost protein rations for broilers fed (a) a single ration and (b) two different rations during the production period for various prices of soybean oilmeal and corn. Equations 10.28, 10.29 and 10.30 are used as a basis for ration selections.

after the least-cost ration has been determined, it is possible to use the input-output equations to determine the optimum level of feeding and the most profitable marketing weight. The optimum market weight is determined by equating the derivative of the gain-feed function for a particular protein ration with the feed to broiler price ratio and solving for the relevant unknowns. The marginal physical products from feed are thus equated with the feed to broiler price ratio.

Quadratic function 10.1 has been used for obtaining the optimum weights for protein levels. Total weights for broilers estimated for various protein levels are shown in Table 10.11. Rations high in protein provide the greatest gains per unit of feed used for low weights; as feed intake increases, rations lower in protein content are more efficient. The marginal quantities in Table 10.12 illustrate this relationship more clearly. For the first few pounds of feed consumed, the marginal or additional gains per unit of feed input are highest at the 26 per cent protein level. As more feed is consumed, the rations giving the highest additional gains per unit of feed consumed are those with lower protein levels.

Table 10.13 indicates the optimum marketing weight for various ratios of feed and broiler prices when time and uncertainty, number of broods and capital limitations are not of concern. The least-cost ration might be determined first in tables 10.8, 10.9, and 10.10. Then Table 10.13 could be used to predict the total amount of the particular ration and the optimum marketing weight per broiler. By equating the derivative of each function with the feed-broiler ratio, (column 2 of Table 10.13) the optimum quantity of feed for a particular protein ration is obtained. (The feed to broiler ratio is the inverse of the broiler to feed ratio.) Broiler to feed price ratios (column 1 of Table 10.13) from 3.6 to 7.0 are used as a basis for determining optimum feed quantities for the various rations. Once the optimum quantity of feed is obtained, the

Table 10.11. Total Liveweight per Broiler for Indicated Pounds of Accumulated Feed Inputs When Fed Various Protein Rations — Predicted From Quadratic Equation 10.1*

Feed Inputs in Pounds	Per Cent Protein in Ration										
	16	17	18	19	20	21	22	23	24	25	26
.5	.337	.339	.342	.343	.345	.347	.349	.350	.352	.354	.356
1.0	.547	.551	.556	.559	.562	.566	.569	.572	.576	.579	.582
1.5	.750	.757	.763	.768	.772	.777	.782	.787	.792	.796	.800
2.0	.947	.956	.964	.970	.976	.982	.989	.994	.999	1.005	1.010
2.5	1.138	1.148	1.158	1.166	1.173	1.180	1.188	1.194	1.201	1.207	1.212
3.0	1.322	1.334	1.346	1.355	1.363	1.372	1.380	1.387	1.394	1.400	1.406
3.5	1.500	1.514	1.527	1.537	1.546	1.556	1.565	1.572	1.579	1.586	1.591
4.0	1.671	1.687	1.702	1.713	1.723	1.733	1.742	1.750	1.757	1.763	1.768
4.5	1.836	1.854	1.870	1.882	1.893	1.904	1.913	1.921	1.928	1.933	1.937
5.0	1.994	2.014	2.032	2.045	2.056	2.067	2.077	2.085	2.091	2.095	2.097
5.5	2.147	2.168	2.187	2.201	2.213	2.224	2.233	2.241	2.246	2.249	2.250
6.0	2.292	2.315	2.336	2.351	2.363	2.376	2.383	2.389	2.393	2.395	2.394
6.5	2.432	2.456	2.478	2.493	2.506	2.517	2.525	2.531	2.533	2.533	2.530
7.0	2.565	2.591	2.614	2.630	2.642	2.653	2.660	2.665	2.666	2.663	2.657
7.5	2.691	2.719	2.743	2.759	2.771	2.782	2.789	2.791	2.790	2.785	2.777
8.0	2.811	2.841	2.866	2.882	2.894	2.904	2.910	2.911	2.907	2.900	2.888
8.5	2.925	2.956	2.982	2.999	3.011	3.020	3.024	3.023	3.017	3.006	2.991
9.0	3.032	3.064	3.091	3.108	3.120	3.128	3.131	3.128	3.120	3.105	3.085
9.5	3.133	3.167	3.194	3.212	3.223	3.230	3.230	3.225	3.213	3.196	3.172
10.0	3.228	3.263	3.291	3.308	3.319	3.324	3.323	3.315	3.300	3.279	3.250
10.5	3.306	3.352	3.381	3.398	3.408	3.412	3.409	3.473	3.379	3.353	3.320
11.0	3.397	3.435	3.464	3.481	3.491	3.493	3.487	3.473	3.451	3.420	3.382

*Total liveweights obtained by adding initial weight of .09 pound for chicks, to gains estimated from quadratic over-all function 10.1.

Table 10.12. Marginal Gains (Pounds Gain per Added Pound of Feed) From Specified Feed Inputs per Broiler on Various Protein Rations Estimated From Quadratic Function 10.1*

Feed Inputs in Pounds	Per Cent Protein in Ration										
	16	17	18	19	20	21	22	23	24	25	26
.5	.426	.430	.434	.438	.441	.444	.448	.451	.455	.458	.461
1.0	.413	.417	.421	.424	.427	.431	.434	.437	.440	.442	.445
1.5	.400	.404	.408	.411	.414	.417	.420	.422	.424	.426	.428
2.0	.388	.392	.395	.398	.401	.403	.405	.407	.409	.411	.412
2.5	.375	.379	.382	.385	.387	.389	.391	.393	.394	.395	.395
3.0	.362	.366	.369	.372	.374	.376	.377	.379	.379	.379	.379
3.5	.349	.353	.356	.358	.360	.362	.363	.363	.364	.363	.362
4.0	.336	.340	.343	.345	.347	.348	.349	.349	.348	.347	.346
4.5	.324	.327	.330	.332	.333	.334	.334	.334	.333	.332	.329
5.0	.311	.314	.317	.319	.320	.320	.320	.319	.318	.316	.313
5.5	.298	.301	.304	.306	.306	.307	.306	.305	.303	.300	.296
6.0	.285	.288	.291	.292	.293	.293	.292	.290	.288	.284	.280
6.5	.272	.276	.278	.279	.279	.279	.278	.275	.272	.268	.264
7.0	.259	.263	.265	.266	.266	.265	.264	.261	.253	.253	.247
7.5	.247	.250	.252	.253	.253	.251	.249	.246	.242	.237	.231
8.0	.234	.237	.239	.239	.239	.238	.235	.232	.227	.221	.214
8.5	.221	.224	.226	.226	.226	.224	.221	.217	.212	.205	.198
9.0	.208	.211	.213	.213	.212	.210	.207	.202	.196	.189	.181
9.5	.195	.198	.200	.200	.199	.196	.193	.188	.181	.174	.165
10.0	.183	.185	.187	.187	.185	.183	.178	.173	.166	.158	.148
10.5	.170	.172	.173	.173	.172	.169	.167	.161	.154	.142	.132
11.0	.157	.159	.160	.160	.158	.155	.150	.144	.136	.126	.115

*Figures in body of table indicate added pounds of gain from each 1-pound added unit of feed, starting from the total feed inputs shown in the first column.

corresponding amount of gain is found by substituting the feed quantity into the appropriate ration function. Adding the initial weight of the chick, or about .09 pound, provides the optimum marketing weights for the various broiler and feed price combinations.

The predicted optimum marketing weights for broilers on rations of protein levels ranging from 16 to 26 per cent with various broiler and feed prices are shown in Table 10.13. These predicted weights are for situations where (a) capital is not limiting, (b) the weights provide maximum returns (or minimum losses) above feed costs, (c) risk and uncertainty are not considered, (d) time required for attaining optimum weights is not considered, and (e) the same ration is fed throughout the feeding period.

Using the broiler to feed price ratio of 4.8, the optimum marketing weights for broilers according to data in Table 10.13 would range from 2.92 to 3.15 pounds, depending on the ration fed. With a 20 per cent protein ration, the least-cost ration over this same period, a weight of 3.15 is optimum. Data from tables 10.12 and 10.13 have been used to develop graphical guides for determining optimum weights and corresponding feed inputs, but are not shown here.[4]

[4] See Heady, Earl O., Balloun, Stanley, and McAlexander, Robert. Least-cost rations and optimum marketing weights for broilers. Iowa Agr. Exp. Sta. Bul. 442. Ames. 1956.

Table 10.13. Weights (in Pounds) for Maximizing Returns Above Feed Costs for Broilers on Various Protein Rations With Specified Broiler to Feed (Feed to Broiler) Price Ratios Predicted From Quadratic Equation 10.1

Broiler to Feed Price Ratio	Feed to Broiler Price Ratio	Per Cent Protein in Ration										
		16	17	18	19	20	21	22	23	24	25	26
3.6	.278	2.38	2.43	2.48	2.51	2.52	2.53	2.52	2.51	2.48	2.45	2.39
3.8	.263	2.53	2.59	2.63	2.66	2.67	2.67	2.66	2.65	2.61	2.58	2.51
4.0	.250	2.66	2.72	2.76	2.78	2.80	2.80	2.78	2.76	2.72	2.68	2.61
4.2	.238	2.77	2.83	2.87	2.89	2.90	2.90	2.89	2.86	2.82	2.78	2.72
4.4	.227	2.87	2.93	2.97	2.99	3.00	2.99	2.97	2.95	2.90	2.86	2.80
4.6	.217	2.96	3.01	3.05	3.07	3.08	3.07	3.05	3.02	2.98	2.92	2.87
4.8	.208	3.03	3.08	3.13	3.15	3.15	3.14	3.12	3.08	3.04	2.98	2.92
5.0	.200	3.10	3.15	3.18	3.21	3.21	3.20	3.18	3.14	3.10	3.04	2.98
5.2	.192	3.15	3.20	3.25	3.27	3.27	3.26	3.23	3.19	3.15	3.08	3.02
5.4	.185	3.21	3.26	3.30	3.32	3.32	3.31	3.28	3.24	3.19	3.13	3.06
5.6	.179	3.26	3.31	3.35	3.37	3.36	3.35	3.32	3.28	3.23	3.17	3.10
5.8	.172	3.30	3.35	3.39	3.41	3.40	3.39	3.36	3.32	3.27	3.20	3.13
6.0	.167	3.34	3.39	3.42	3.44	3.44	3.42	3.40	3.35	3.29	3.23	3.16
6.2	.161	3.37	3.42	3.46	3.48	3.47	3.46	3.42	3.38	3.32	3.26	3.19
6.4	.156	3.41	3.45	3.49	3.51	3.50	3.48	3.45	3.41	3.35	3.29	3.21
6.6	.152	3.43	3.48	3.52	3.54	3.53	3.51	3.48	3.44	3.37	3.31	3.23
6.8	.147	3.46	3.51	3.54	3.56	3.56	3.54	3.50	3.46	3.40	3.33	3.25
7.0	.143	3.48	3.53	3.57	3.58	3.58	3.56	3.52	3.48	3.42	3.35	3.27

Time Requirements

Previous analysi dealt only with the cost of alternative rations. However, the broiler producer also is interested in the time required for gains. If he is faced with a seasonal or cyclical decline in broiler price, he may wish to use the least-time ration, rather than the least-price combination of feeds. If he is faced with the possibility of rising broiler prices, he will undoubtedly want to use the least-cost ration. The least-time ration need not be identical with the least-cost ration. The two will be identical under price situations where the cost of the feed ingredient providing the greatest timeliness is low, so that the two types of rations are identical. Under other price situations the costs of the "time-saving" feed may be relatively high. The least-time ration will then be more costly than the least-cost ration.

To provide information aiding these types of decisions an additional function showing time elapsed to consume various amounts of these rations has been computed. From the gain and time equations it is possible to compute time elapsed for a specific gain or weight level. Various algebraic functions were tried as expressions of the time relationship. The best function appeared to be equation 10.31 with square root terms where T is time in days and S and C

$$T = .6735 + 4.7974C + 9.4576S + 21.4617\sqrt{C} + 13.6188\sqrt{S} - 12.0287\sqrt{CS} \qquad (10.31)$$

are pounds of soybean oilmeal and corn per bird in pounds. The property which appears to qualify it over other functions tried was that the function allows a relatively sharp curvature for low feed inputs but tends to

more nearly approach linearity for high feed inputs. In other words, it is consistent with the growth of the bird's digestive capacity at the outset when proportionately less time is required to consume a given additional quantity of feed; it is consistent with the tendency for a limit in growth of the bird's digestive capacity as the bird approaches maturity. At heavy weights, digestive capacity is limited, and a bird consumes about a constant amount per day (i.e., each additional pound of feed is consumed in about the same period of time as the bird approaches maturity). All of the coefficients for this time function are acceptable at a 1 per cent level of probability, and 99 per cent of the variance in time required to consume various quantities of feed is explained by the variables in the equation.

Analysis of variance also was used to test the significance in differences in rate of gain up to 1.23 of total gain and up to a total of 3.0 pounds of gain for the six rations of the experiment. These tests showed the differences to be significant at the 5 per cent level of probability. However, there was no significant difference in rate of gain for the six rations between a total gain of 1.23 and 3.0 pounds. Evidently, the main effect of rations on rate of gain is in the earlier growing period.

Figures 10.25 and 10.26 show the predicted relationship between feed consumption and time for two rations of the study. Time curves for other rations are similar. Table 10.14 indicates the predicted amount of time required to attain weights of 1.32 pounds and 3.25 pounds for rations of different protein levels. For the lower weight range, a ration of slightly over 23 per cent protein is predicted to give a 1.32-pound liveweight in the minimum amount of time. For the entire weight range to 3.25 pounds liveweight, a protein percentage of slightly over 21.0 per cent is predicted to give most rapid gains. Rations containing

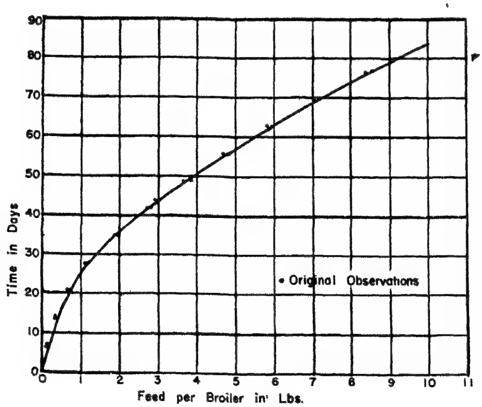


Figure 10.25. Estimated number of days required to consume various quantities of feed for broilers fed a 16 per cent protein ration. Time estimates are based on square root function 10.31.

a greater percentage of protein are predicted to give somewhat less rapid gains. From the data of substitution rates in earlier tables, it is obvious that these rations which give the most rapid gains do not also give the least-cost rations under normal price relationships. However, the cost of the least-time ration is only slightly above the least-cost ration when the price of soybean oilmeal is low relative to the prices of corn. In cases where the price of soybean oilmeal is relatively high, the broiler producer needs to compare the savings in feed from use of the least-cost ration with any possible gain in broiler price obtained from getting to market sooner under the least-time ration. The absolute differences in profits from least-cost and least-time rations will be very small for a few birds but can be quite large for a large operation when corn is low in price compared to soybean oilmeal.

Table 10.15 shows the predicted number of days for broilers to reach the optimum marketing weights shown in Table 10.14. The data of the two tables allow prediction of the weight which will allow maximum profit above feed costs. The sequence in considering ration costs, marketing weights and time to market might be this: (a) Select the least-cost ration in Table 10.9. (b) Select the most profitable marketing weight for the least-cost ration from Table 10.13. (c) Examine Table 10.15 for the time involved and, after considering the prospects for prices, determine whether or not feeding plans should be altered to fit prospects for price increases or decreases. Table 10.15 can be interpreted thus: With a broiler to feed price ratio of 4.4, and with an 18 per cent protein ration being the least-cost one shown in Table 10.9, the optimum marketing weight of 2.97 pounds shown in Table 10.13 can be obtained in 74.7 days. If broilers increase in price so that the broiler to feed price ratio becomes 5.0, with an 18 per cent

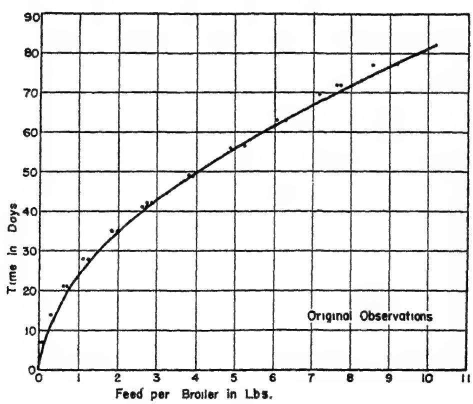


Figure 10.26. Estimated number of days required to consume various quantities of feed for broilers fed a 24 per cent protein ration. Time estimates are based on square root function 10.31.

ration giving lowest feed costs, the optimum marketing weight is 3.18 pounds per Table 10.13 and the time required is 79.5 days per Table 10.15.

Number of Flocks Per Year

A considerable variation exists in the number of flocks of birds raised by broiler producers each year. Some poultrymen raise only a flock or two each year. However, poultrymen who have broiler production as a major source of income usually raise at least three or more flocks each year. For producers who raise only three or less groups per year, it usually is possible to carry the broilers to their optimum weights without any time conflict. For broiler producers who desire to raise four groups per year, there would be no conflict on time for broilers fed on any of the rations when the broiler to feed price ratio is 5.5 or less. Raising four groups under the above price ratios would permit at least a week between each flock, depending on the broiler to feed price ratio and the protein ration being fed. When the broiler to feed price ratio is above 5.5, birds on the lower protein rations would require marketing at slightly less than optimum weights if a four flock schedule were rigidly followed. With a broiler to feed price ratio of 6.0, four flocks could be carried to optimum weights on rations containing a 19.5 or greater percentage of protein with a week between flocks; birds on lower protein rations could not be carried to optimum weights without a time conflict. Data in Table 10.13 can be

Table 10.14. Estimated Feed Requirements and Number of Days per Broiler for Specified Weights When Fed Various Protein Rations

Per Cent Protein in Ration	Starting to 1.32 Pounds Weight		Starting to 3.25 Pounds Weight	
	Pounds feed*	No. days†	Pounds feed‡	No. days†
15.0	3.31	46.3	10.27	86.1
16.0	3.11	44.5	9.82	82.6
17.0	2.96	43.4	9.41	80.1
18.0	2.86	42.5	9.29	78.7
19.0	2.79	41.6	9.17	77.6
20.0	2.74	41.1	9.11	77.0
21.0	2.71	40.7	9.09	76.7
22.0	2.69	40.5	9.11	76.8
23.0	2.68	40.4	9.18	77.3
24.0	2.68	40.5	9.28	78.1
25.0	2.70	40.8	9.44	79.3
26.0	2.74	41.0	9.62	80.7

*Predicted from equation 10.18.
†Predicted from equation 10.31.
‡Predicted from equations 10.18 and 10.19.

used to determine whether birds could be held until optimum weights are attained if four or more flocks are to be produced each year.

Risk and Uncertainty

Because of risk and uncertainty, broiler producers may not hold their birds until they attain the optimum market weights. The uncertainty of expected prices and death losses due to disease and other hazards may result in earlier marketing. However, modern techniques for prevention and treatment of diseases have done much to reduce this type of uncertainty. Also, insurance against hazards such as fire tends to reduce risk for the poultrymen. Prices usually provide the greatest source of uncertainty except where the producers have some type of forward pricing.

Obviously, the decision of the best marketing weight depends on many factors including (a) input-output relationships, (b) the previous commitments, i.e., contractual arrangements, (c) number of flocks per year, (d) price expectations, and (e) risk preference of the individual poultryman. However, the data provided in this study on input-output data, selection of rations, estimation of optimum market weights, and corresponding time required for attaining optimum weights provides information for reducing much of the uncertainty in broiler production.

Table 10.15. Predicted Time in Days Required for Broilers To Attain Optimum Marketing Weights Shown in Table 10.13; Estimated From Time Function 10.31

Broiler to Feed Price Ratio	Feed to Broiler Price Ratio	Per Cent Protein in Ration										
		16	17	18	19	20	21	22	23	24	25	26
		(days)	(days)	(days)	(days)	(days)	(days)	(days)	(days)	(days)	(days)	(days)
3.6	.278	64.9	65.0	65.1	64.9	64.7	64.5	64.3	64.0	63.6	63.2	62.7
3.8	.263	68.0	68.0	67.0	67.8	67.5	67.2	66.9	66.5	66.1	65.6	65.2
4.0	.250	70.7	70.6	70.5	70.2	69.9	69.6	69.2	68.8	68.3	67.8	67.3
4.2	.238	73.1	72.9	72.7	72.4	72.0	71.6	71.2	70.8	70.3	69.8	69.2
4.4	.227	75.2	75.0	74.7	74.3	73.9	73.5	73.0	72.5	72.0	71.5	70.9
4.6	.217	77.1	76.8	76.5	76.0	75.6	75.1	74.6	74.1	73.6	73.0	72.4
4.8	.208	78.8	78.5	78.1	77.6	77.1	76.6	76.1	75.6	75.0	74.4	73.8
5.0	.200	80.4	79.9	79.5	79.0	78.5	77.9	77.4	76.8	76.3	75.6	75.0
5.2	.192	81.8	81.3	80.9	80.3	79.7	79.2	78.6	78.0	77.4	76.8	76.2
5.4	.185	83.1	82.6	82.0	81.5	80.9	80.3	79.7	79.1	78.5	77.9	77.2
5.6	.179	84.3	83.7	83.2	82.6	81.9	81.3	80.7	80.1	79.5	78.8	78.2
5.8	.172	85.4	84.8	84.2	83.5	82.9	82.3	81.7	81.0	80.4	79.7	79.1
6.0	.167	86.4	85.7	85.1	84.4	83.8	82.2	82.5	81.9	81.2	80.5	79.9
6.2	.161	87.3	86.6	86.0	85.3	84.6	84.0	83.3	82.7	82.0	81.3	80.6
6.4	.156	88.2	87.5	86.8	86.1	85.4	84.7	84.0	83.4	82.7	82.0	81.3
6.6	.152	89.0	88.3	87.6	86.8	86.1	85.4	84.7	84.1	83.4	82.7	82.0
6.8	.147	89.7	89.0	88.3	87.5	86.8	86.1	85.4	84.7	84.0	83.3	82.6
7.0	.143	90.5	89.7	88.9	88.2	87.4	86.8	86.0	85.3	84.6	83.9	83.2

SOME FURTHER COMPARISONS

The quadratic and Cobb-Douglas equations were used in the previous section to provide certain estimates for practical use. Additional comparisons are made in this section of estimates provided by the resistance and Spillman-type functions. Some readers may be interested in the latter two functions, in substituting them for practical recommentations or in deriving added detail from them. Both differ from other equations examined since they provide gain asymptotes, a condition which appears realistic for meat production functions. However, since the broilers in the experiment reported were not carried to weights approaching maturity, use of an equation allowing this condition did not seem necessary for the foregoing analysis.

Total Gains

The total gains predicted from the resistance, equation 10.4, and Spillman, equation 10.5, equations are provided in Table 10.16. These can be compared with similar estimates from the quadratic equation 10.1 in Table 10.11.

The resistance formula, equation 10.4, provides total gain and marginal feed productivities more like those of the quadratic, equation 10.1, than does the particular form of the Spillman equation 10.5 used. However, equation 10.4 predicts smaller total weights at low protein levels and higher total weights at high protein levels than does equation 10.1. In general, equation 10.5 predicts larger total weights for a given feed input than do either of the other two equations.

Table 10.16. Total Liveweight per Bird Predicted From Equations 10.4 and 10.5 for Specified Feed Inputs and Rations

Feed Input in Pounds	Resistance (10.4)						Spillman (10.5)					
	Per cent protein						Per cent protein					
	16	18	20	22	24	26	16	18	20	22	24	26
1	.419	.479	.547	.584	.616	.633	.633	.750	.765	.838	.806	.774
2	.787	.892	1.009	1.073	1.126	1.155	1.056	1.066	1.123	.992	1.179	1.211
3	1.113	1.252	1.405	1.487	1.555	1.591	1.370	1.234	1.441	1.504	1.522	1.586
4	1.404	1.569	1.748	1.843	1.921	1.962	1.684	1.638	1.827	1.869	1.677	1.864
5	1.666	1.851	2.048	2.151	2.237	2.281	1.928	2.079	2.050	2.219	2.295	2.353
6	1.902	2.101	2.313	2.422	2.512	2.559	2.293	2.398	2.432	2.610	2.533	2.649
7	2.116	2.327	2.547	2.661	2.753	2.802	2.643	2.717	2.875	2.806	3.099	3.197
8	2.312	2.536	2.758	2.874	2.968	3.018	2.940	3.065	3.050	3.354	3.448	3.540
9	2.490	2.715	2.946	3.065	3.160	3.209	3.155	3.407	3.601	3.711	3.840	3.961
10	2.655	2.884	3.118	3.235	3.332	3.382	3.590	3.751	3.970	4.064	4.249	4.385
11	2.806	3.037	3.273	3.391	3.487	3.536	3.822	3.935	4.315	4.451	4.485	4.552

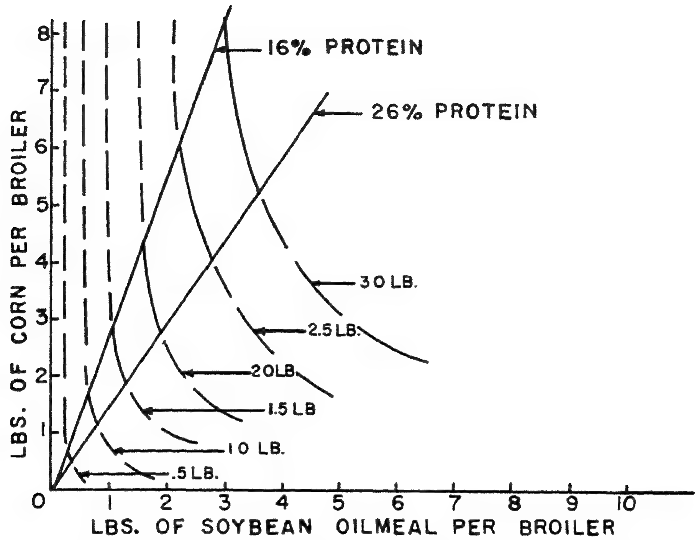


Figure 10.27. Gain isoquants predicted from resistance formula 10.4.

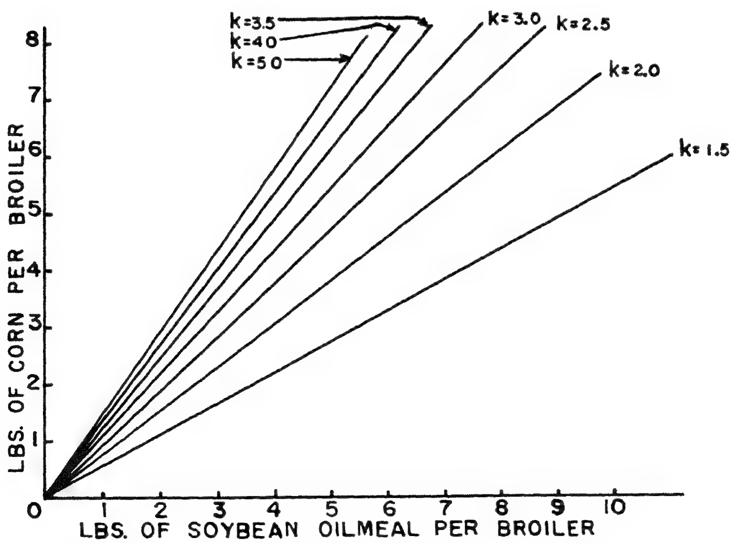


Figure 10.28. Isoclines for broilers from resistance formula 10.4.

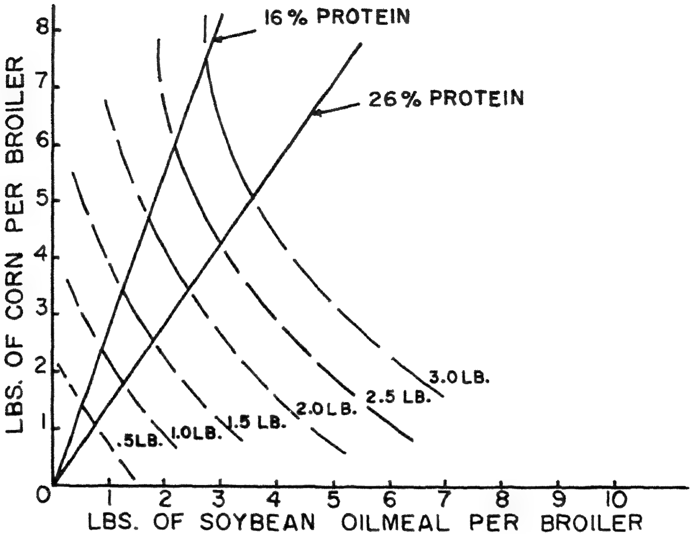


Figure 10.29. Gain isoquants predicted from Spillman function 10.5.

Isoquants and Isoclines

Isoquants and isoclines for the resistance, equation 10.4, and Spillman, equation 10.5, functions are illustrated in figures 10.27 to 10.30. They are derived from isoquant equations 10.32 and 10.33 below.

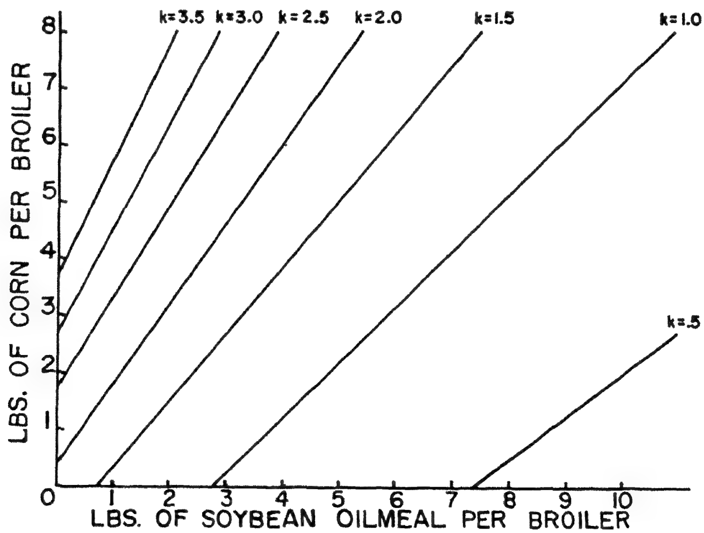


Figure 10.30. Isoclines for broilers from Spillman function 10.5.

$$(10.32) \qquad S = (.4512CG)(C - .1532CG - .2700G)^{-1}$$

$$(10.33) \qquad S = \log \left[\frac{G - 17.908 + 11.442(.9238^{C})}{8.323(.9238^{C}) - 14.343}\right] (\log .9257)^{-1}$$

The respective equations of marginal rates of substitution for these two equations are:

$$(10.34) \qquad \frac{\delta S}{\delta C} = .6112 \frac{S^2}{C^2}$$

$$(10.35) \qquad \frac{\delta C}{\delta S} = \frac{.926^{S} (.0772)[14.343 - 8.323(.9238^{C})]}{.9238^{C} (.0793)[11.442 - 8.323(.9257^{S})]}$$

Isoclines have been derived for the resistance, equation 10.4, and Spillman, equation 10.5, equations from equations 10.34 and 10.35, respectively, by setting the marginal rates of substitution equal to a constant, k, and solving for one feed variable in terms of the other. The isoclines for equations 10.4 and 10.5 are respectively:

$$(10.36) \qquad S = 1.2923Ck^{-.5}$$

$$(10.37) \qquad S = \log \left[\frac{(.9238^{C})(.9069)}{1.107k - .642k(.9238^{-C}) + .6597(.9238^{C})}\right] (\log .9257).^{-1}$$

The isoquants for the resistance and Spillman functions are relatively flat within the boundaries of the rations used in the study. The extrapolations outside of these boundaries in figures 10.27 and 10.29 are not used for prediction, but only to better suggest the general algebraic nature of these isoquants. Similarly, isoclines are included which are outside of the experimental observations.

Marketing Weights

The marketing weights predicted by the particular Spillman function were considerably above those for the quadratic equation. Predictions by the resistance formula were relatively similar to the quadratic equation, especially for broiler to feed price ratios which are reasonable. Hence, because of these comparisons and the greater ease in calculations, the function used in previous sections was retained as a basis for practical recommendations.

CHAPTER 11

Least-cost Rations and Optimum Marketing Weights for Turkeys

Stanley Balloun
Earl O. Heady
Gerald W. Dean

TURKEY PRODUCTION has become increasingly competitive and specialized in recent years. United States turkey production has nearly doubled since 1940, while the number of farms raising turkeys has steadily decreased. An increasing number of producers are looking toward savings in feed costs and better marketing methods as important aids in maintaining or increasing profits. The objective of this study was to provide turkey producers with useful predictions of least-cost rations and most profitable, or optimum, marketing weights for a wide range of price relationships.

As necessary information in attaining this final objective, the first step is one of predicting the feed-gain production function. From the production function, in turn, it is necessary to predict the gain isoquants showing the possible combinations of feed which will produce turkeys of a specified weight. From the gain isoquants can be predicted marginal substitution rates between the two major feed categories included in this study. Isoclines may also be computed, which, when related to feed price ratios, allow indication of rations which produce each pound of gain at lowest cost. Finally, marginal rates of transformation of feed into gain can be used to predict optimum marketing weights.

The emphasis of this study is on feed substitution and least-cost rations in the latter part of the production period for turkeys. Emphasis is placed on this portion of the production period because daily and total feed intake is greatest then; major savings in cost can be made during this period when feed consumption is largest. Also, it is during this period that the greatest opportunity exists for substitution between feed categories with least restriction in the growth of poults. Accordingly, the experiment was designed mainly to allow prediction of the production surface for the upper weight range (based on observations in the 12-24 week portion of the turkey production period). Though only a limited number of observations were obtained at lighter weights, it was also possible to predict production surfaces for weight intervals based on observations in the 0-6 week and 6-12 week periods of production. The authors look upon the estimates for lighter weights as having limitations which do not attach to the predictions for heavier weights. However, it is believed that these estimates are generally of more value for feeding recommendations than data which heretofore have been available.

Production functions and feed substitution possibilities involve corn and soybean oilmeal as the central resources of decision. However, opportunities for feed substitution exist primarily when the ration is properly fortified with the vitamins, minerals, and trace ingredients explained later.

EXPERIMENT AND BASIC DATA

This study is based on an experiment in which 600 bronze turkey poults were fed on alternative rations for a 24-week period. At the start of the experiment, the 600 turkeys were randomly allotted to 48 different pens of 12 or 13 birds each; individual pens contained approximately half males and half females. Eight pens (or replicates) of birds were fed on each of 6 protein rations (21, 23, 25, 27, 29, and 31 per cent) for the first 6 weeks of the experiment. At 6 weeks, the 600 birds were completely rerandomized into 24 pens of 24 to 25 birds each with 4 pens of birds fed on each of 6 protein rations (15, 17, 19, 21, 23, and 25 per cent) for the 6-12 week period. At the end of 12 weeks, the 600 birds were again rerandomized into 24 pens, with 4 pens fed on each of 6 protein rations (10, 12, 14, 16, 18, and 20 per cent) for the 12-24 week period of the experiment.

Previous experiments with broilers and hogs indicated that there were no important cumulative or "carry-over" effects of previous protein rations in meat production. That is, a bird or animal fed a 1 per cent protein diet in an early period and a different per cent protein diet in a later period tended, after a short adjustment period, to perform in the later period as if it had received the second protein level throughout the entire production period. For example, in the broiler study cited, statistical analysis indicated no significant difference occurred in gains for later periods between (a) birds carried through the entire production period on a single ration and (b) birds changed to this ration from one containing another percentage of protein. A comparable analysis for the present study was not possible, since no turkeys were fed on one protein ration for the entire 24-week period. However, it is not expected that the switch in rations causes outcomes to differ in later periods.

The average weight per poult at the start of the experiment was .11 pound, with each bird weighed thereafter at 3, 6, 12, 16, 20, and 24 weeks of age. The average gain per bird and the corresponding average feed inputs per bird were computed for each treatment and pen; these quantities provided the observations used in the regression analysis which follows.

Table 11.1 indicates the ingredients included in the various protein rations used for the 0-6 week period. Corn and soybean oilmeal were combined in various proportions with a fixed "basic" ration of other ingredients to provide protein levels ranging from 21 to 31 per cent. Table 11.2 shows the rations used for the 6-12 week period; the "basic" ration remained the same, while the quantities of corn and soybean

Table 11.1. Quantities of Various Ingredients Required for 100 Pounds of Different Protein Rations*

Ingredients	Per Cent Protein in Ration					
	21	23	25	27	29	31
Corn	45	40	35	30	25	20
Wheat middlings	10	10	10	10	10	10
Bran	5	5	5	5	5	5
Soybean oilmeal	15	20	25	30	35	40
Fish meal	5	5	5	5	5	5
Meat scraps	5	5	5	5	5	5
Alfalfa meal	5	5	5	5	5	5
Dried whey	5	5	5	5	5	5
Minerals	4	4	4	4	4	4
Vitamin mix	1	1	1	1	1	1

*These rations were fed to turkeys from 0 to 6 weeks of age.

Table 11.2. Quantities of Various Ingredients Required for 100 Pounds of Different Protein Rations†

Ingredients	Per Cent Protein in Ration					
	15	17	19	21	23	25
Corn	60	55	50	45	40	35
Wheat middlings	10	10	10	10	10	10
Bran	5	5	5	5	5	5
Soybean oilmeal	0	5	10	15	20	25
Fish meal	5	5	5	5	5	5
Meat scraps	5	5	5	5	5	5
Alfalfa meal	5	5	5	5	5	5
Dried whey	5	5	5	5	5	5
Minerals	4	4	4	4	4	4
Vitamin mix	1	1	1	1	1	1

†These rations were fed to turkeys from 6 to 12 weeks of age.

oilmeal were varied to provide rations ranging from 15 to 25 per cent in protein.

DERIVATION OF PRODUCTION FUNCTIONS

The production functions used for later predictions are regression equations for each interval over which specific rations were fed. Because of the way in which birds were reallocated to different rations, it was impossible to predict a single over-all production function. There was opportunity only for predicting either gain isoquants or production functions based on observations for each of the age intervals of 0-6, 6-12, and 12-24 weeks. Practical use of substitution data is consistent with estimation of production functions over particular intervals because (a) rations which average lowest in cost for the total gain in the weight interval (but which do not necessarily represent the lowest cost for

each ounce of gain) can be so predicted and (b) producers prefer to change the ration only a few times over the total production period (i.e., one ration is selected for an interval of time, then a shift is made to another ration to be used for some time, etc.). Because of the restricted number of weighings and the fact that there was little difference in gains among the rations over the lighter weight ranges, an alternative method was devised as a check on the accuracy of the production functions fitted to the 0-6 and 6-12 week observations. However, a greater number of weighings and considerable difference in rates of gain and total gain among rations caused this check to be unnecessary for functions fitted to the 12-24 week observations.

A problem of autocorrelation arises in estimating the production functions within each weight interval where several measurements were taken from each pen of birds. To have independent observations along a particular ration line, it would be necessary to feed different pens of birds on each ration, with each pen being weighed and used only once as an observation showing gains forthcoming from particular levels of feed input (in contrast to the method used whereby the same pen was employed in prediction of gains associated with several levels of feed input within an interval). The autocorrelation presents problems mostly for probability statements and fiducial limits, rather than in prediction of mean gains and substitution rates. That is, the presence of autocorrelation does not present problems of predicting the relationship between the dependent and independent variables, but does introduce problems in making tests of significance. The effect of autocorrelation is to reduce the effective number of independent observations; the number of degrees of freedom which can be used for tests of significance in uncorrelated series is greater than it is when autocorrelation is present. Hence, a problem exists in specifying the number of degrees of freedom upon which probability statements should be based. Calculation of the autocorrelation coefficient and approximation of the effective number of degrees of freedom can be avoided by basing significance tests on a minimum number of observations (to which the series might be reduced by calculating the autocorrelation coefficient). Since the observations on different pens are independent, the number of noncorrelated observations generally is equal to the number of pens. Where a null hypothesis is rejected using this minimum number of degrees of freedom, it would certainly be rejected for the greater number of degrees of freedom represented by all observations in the series.

Interval Functions

Because each pen of birds was fed on a constant protein ration from 12 to 24 weeks, it was possible to fit a production surface to this particular interval with greatest confidence. The 12-24 week interval is one in which feed consumption is great and is also the relevant period for marketing the birds. Thus, estimates were made of optimum

marketing weights, as well as of least-cost rations, using interval functions fitted to the observations in the 12-24 week period.

A limited amount of information was available for the period up to 12 weeks of age. Only two weighings per pen (at 3 and 6 weeks) were made before the birds were rerandomized and the protein levels changed at the end of 6 weeks; only the 12-week weighing was made in the 6-12 week interval before the birds were rerandomized and the protein levels changed again at 12 weeks. Hence, because of the limited number of observations available at lighter weights, two alternative methods were used in obtaining estimates for the 0-6 week and 6-12 week periods. The first method attempted to predict, in the usual manner, the entire production surface for each interval; from these surfaces isoquants and marginal rates of substitution between corn and soybean oilmeal were obtained. However, since the available observations tended to be "clustered," a second or alternative method was devised as a check on the production surfaces. With this alternative procedure, gain isoquants were computed for the average turkey weights at 3, 6, and 12 weeks, i.e., the isoquants were computed directly from the adjusted data, rather than being derived from a previously estimated production surface. Marginal rates of substitution between corn and soybean oilmeal were then obtained along the "directly computed" isoquants. The check procedure simply involved comparing particular gain isoquants derived by the two methods for (a) consistency of slopes or marginal rates of substitution and (b) consistency with respect to the various feed combinations required to produce the specified gains.

A limitation of the alternative procedure involving direct estimation of the gain isoquants is in deciding whether to minimize sums of squares relative to corn or protein. Generally, corn inputs have been derived as a function of protein inputs where direct estimation of gain isoquants is involved. Since the alternative procedure is not used for predictive purposes, but only as a check on the reasonableness of the production surface, this limitation is not particularly serious. The data for the 12-24 week period are adequate for obtaining a reliable estimate of the production surface. Hence, the alternative procedure described above is not used for this period.

Functions Fitted

Fewer regression models were applied in this study than in the broiler study reported in Chapter 10. The data, as explained earlier, were not so well adapted for these purposes. The various production functions and gain isoquants fitted as regression equations are explained below.

Regression equations for 0-3 week interval

Only the observations obtained from the 3-week weighing were

available for use in predictions over the 0-3 week interval. At this early stage in the development of the bird, very little difference (absolute or relative) occurred in the gains for birds fed on various protein rations. A regression equation which predicts gain as a function of the two categories of feed inputs gives a low coefficient of determination (R value).[1] Therefore, the alternative procedure of fitting gain isoquants directly to the data was used. With this procedure, all gains for the 0-3 week interval were adjusted to the average 3-week gain of .57 pound; total feed quantities associated with each gain were then adjusted in the same proportion and direction as the gain adjustment. Regression equations 11.1 and 11.2 were fitted to these adjusted data, where corn is expressed as a function of soybean oilmeal. These equations predict (for .57 pound of gain) the quantity of corn consumed as a function of the quantity of soybean oilmeal fed. In equations such as 11.1 and 11.2, where corn is predicted as a function of soybean oilmeal consumption, the symbols C and S refer to the pounds of corn and soybean oilmeal consumed relative to a given gain (.57 pound in this case). In later equations where gain is predicted as a function of the feed inputs, the symbols C and S refer to total quantities of corn and soybean oilmeal required to produce any specified gain.

$$C = .1671\, S^{-.7413} \tag{11.1}$$

$$C = .8141 - 1.5800\, S + .9757\, S^2 \tag{11.2}$$

Table 11.3 shows the coefficient of determination, R^2, and the t values for the regression coefficients in equations 11.1 and 11.2. The R^2 and t values for both equations are significant at the 1 per cent level, and little difference occurs in the R^2 values for the two equations. Given the limitations mentioned previously, either equation may be used, on a probability basis, for predicting substitution rates between corn and soybean oilmeal along the .57-pound gain isoquant. In addition to statistical "fit," the logic of nutritional requirements and practicality of feeding operations become the basis on which selection of a function is made.

Table 11.3. R^2 and t Values for Regression Equations 11.1 and 11.2 Using Adjusted 3-Week Gain and Feed Data

Equation	Value of R^2	t Values for Regression Coefficients in the Order Shown in Equations 11.1 and 11.2	
11.1	.9308	24.93*	
11.2	.9420	6.56†	2.85†

*Significant at the 1 per cent level with 46 degrees of freedom.
†Significant at the 1 per cent level with 45 degrees of freedom.

[1] The function $Y = .9162\, C^{.2005}\, S^{.2620}$, which was obtained for the 0-3 week interval, has a low R value of .7230.

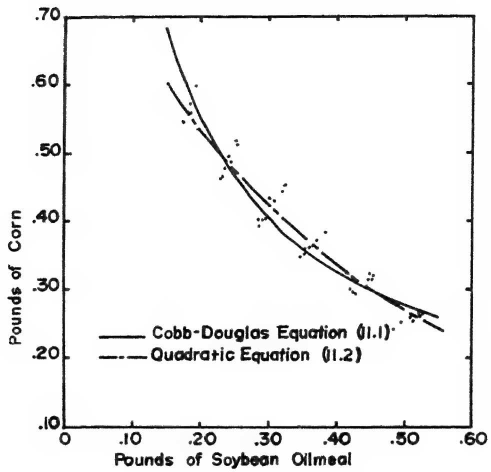


Figure 11.1. Comparison of 0.57-pound gain isoquants, predicted by regression equations 11.1 and 11.2.

Figure 11.1 shows the .57-pound gain isoquants computed from regression equations 11.1 and 11.2 plotted against the adjusted observations. It should be pointed out that the dots of the scatter diagram in Figure 11.1 represent the adjusted feed quantities required to produce .57 pound of gain for the various rations and pens. In this case both equations, which predict corn consumption as a function of soybean oilmeal consumption, have been fitted to these adjusted observations. Thus, the comparison of "closeness of fit" of the isoquants to the dots in Figure 11.1 is relevant. However, in later cases where a function is fitted as a production surface, the isoquants computed from this surface cannot be compared directly with the adjusted dots.

Regression equations for the 0-6 week interval

Production functions 11.3, 11.4, and 11.5 were fitted to the accumulated feed and gain quantities for the 0-6 week period. Since the average gain per poult at 6 weeks of age varied considerably with the protein ration fed, it was possible to fit a production surface indicating gain as a function of the corn and soybean oilmeal fed. Equation 11.3 is a Cobb-Douglas function of this type. Functions 11.4 and 11.5 were fitted by the same process explained for the .57-pound gain isoquants at 3 weeks of age, i.e., the gain and feed quantities used were adjusted to the mean gain of 2.33 pounds for the 6-week period.

(11.3) $$Y = 1.7167\ C^{.4422}\ S^{.3647}$$

(11.4) $$C = 1.9512\ S^{-.7506}$$

(11.5) $$C = 3.3915 - 1.6659\ S + .2668\ S^2$$

Table 11.4. R^2 and t Values for Regression Equations 11.3, 11.4, and 11.5 Using 6-Week Gain and Feed Data*

Equation	Value of R^2	t Values for Regression Coefficients in the Order Shown in Equations 11.3, 11.4, and 11.5	
11.3	.8640	12.44†	16.83†
11.4	.9324	24.95‡	
11.5	.9461	6.94†	3.21†

*Equation 11.3 is computed from the actual quantities of feed and gain for the 0-6 week period; equations 11.4 and 11.5 are computed from feed and gain quantities which have been adjusted to a gain of 2.33 pounds.
†Significant at the 1 per cent level with 45 degrees of freedom.
‡Significant at the 1 per cent level with 46 degrees of freedom.

Table 11.4 presents the R^2 and t values for regression equations 11.3, 11.4, and 11.5. While the R^2 values for equations 11.4 and 11.5 are larger than the value for equation 11.3, it should be noted that equation 11.3 is a production surface computed from unadjusted data while equations 11.4 and 11.5 are gain isoquants computed from adjusted data. For equation 11.4 the proper interpretation of R^2 is that 93.2 per cent of the sum of squares of the adjusted corn quantities is explained by the soybean oilmeal variable. For equation 11.3, however, R^2 should be interpreted as meaning that 86.4 per cent of the sum of squares of the true gains is explained by the corn and soybean oilmeal variables. Thus, the R^2 values of regression equations 11.4 and 11.5 should not be compared directly with the R^2 value of equation 11.3.

Regression equations 11.4 and 11.5 predict directly the 2.33-pound gain isoquants (i.e., isoquants representing a total turkey weight of 2.44 pounds, including the .11-pound initial weight). However, when using production function 11.3, which represents a surface or family of isoquants, an isoquant equation must be derived for predicting the 2.33-pound gain isoquant. Isoquant equation 11.6 is derived from production

$$C = \left[\frac{Y}{1.7167\ S^{.3647}}\right]^{2.2614} \tag{11.6}$$

function 11.3; the 2.33-pound gain isoquants resulting from equations 11.4, 11.5, and 11.6 are presented in Figure 11.2. While the isoquant computed from equation 11.6 does not appear to fit the observations as well as the isoquants of equations 11.4 and 11.5, the scatter dots shown in Figure 11.2 have been adjusted to a constant gain of 2.33 pounds and thus functions 11.4 and 11.5 are partially "forced into a better fit." Hence, the isoquant computed from equation 11.6 should not be compared, in "closeness of fit to the scatter dots," with the isoquants from equations 11.4 and 11.5. The relevant comparison for equation 11.6 would be that of a family of gain isoquants compared with a set of unadjusted observations. It is expected that the 2.33-pound gain isoquant

(i.e., an isoquant representing a total weight of 2.44 pounds, including the .11-pound initial weight) derived from equation 11.6 fits the unadjusted data better than either equation 11.4 or 11.5. Since equation 11.3 is a production surface based on unadjusted data, it will be used for predictive purposes; equations 11.4 and 11.5 provide some check on the reliability of this surface. The high degree of consistency between the 2.33-pound gain isoquants fitted by the three different equations (Figure 11.2) provides a basis for increased confidence in using equation 11.3 for predicting marginal rates of substitution and least-cost rations for the 0-6 week interval.

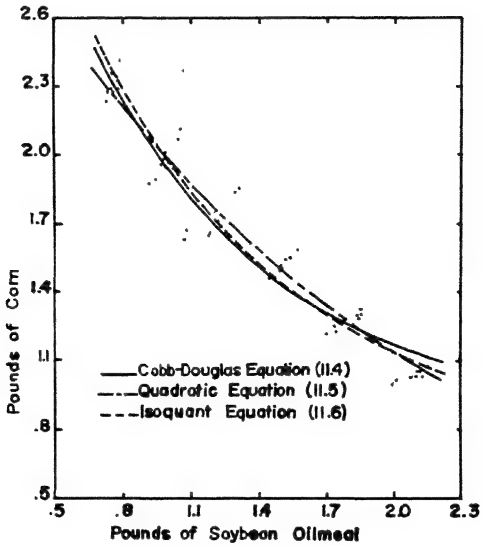


Figure 11.2. Comparison of 2.33-pound gain isoquants, predicted by regression equations 11.4 and 11.5 and by isoquant equation 11.6.

Regression equations for the 6-12 week interval

Regression equations 11.7 and 11.8 were fitted to the data obtained at the 12-week weighing, i.e., to the gains and feed quantities for the 6-12 week interval. Thus, predicted gains for the 6-12 week period are measured from a starting weight of 2.44 pounds, the average turkey

(11.7) $$Y = 1.7291\ C^{.4998}\ S^{.2531}$$

(11.8) $$C = 9.6199 - 3.3851\ S + .5219\ S^2$$

weight at 6 weeks. Equation 11.7 predicts a production surface for the 6-12 week period computed from unadjusted data, with gain the dependent

variable and corn and soybean oilmeal the independent variables. Equation 11.8, however, was computed by the alternative check procedure of deriving a gain isoquant directly from adjusted data, i.e., the equation uses observations adjusted to an average gain of 4.45 pounds over the 6-12 week interval. (The 4.45-pound gain isoquant for the 6-12 week interval represents a total turkey weight of approximately 6.89 pounds; 2.44 pounds weight at 6 weeks plus 4.45 pounds gain.) Again, equation 11.8 does not predict a production surface, but only a 4.45-pound gain isoquant in the 6-12 week period. Comparison of this isoquant with the 4.45 pound gain isoquant derived from equation 11.7 provides a check on the production surface predicted from equation 11.7.

Observations for four pens of birds fed on a 15 per cent protein ration were omitted in fitting the Cobb-Douglas function, equation 11.7. The four observations were not used because, at the 15 per cent protein level, the quantity of soybean oilmeal in the ration is zero, i.e., the observation points fall directly on the corn axis. One of the mathematical restrictions of the Cobb-Douglas function is that the gain isoquants cannot intersect either the corn or the soybean oilmeal axis, i.e., the isoquants must be asymptotic to both axes.[2] The quadratic function, equation 11.8, allows the 4.45-pound gain isoquant (4.45 pounds gain in the 6-12 week period) to intersect the corn and soybean oilmeal axes. This equation, then, using all of the observations for the 6-12 week period (including those for the 15 per cent ration) adjusted to a common gain of 4.45 pounds was computed as a check on function 11.7. The R^2 and t values for the regression equations 11.7 and 11.8 are given in Table 11.5.

Table 11.5. R^2 and t Values for Regression Equations 11.7 and 11.8 Using Feed and Gain Data for the 6-12 Week Period*

Equation	Value of R^2	t Values for Regression Coefficients in the Order Shown in Equations 11.7 and 11.8	
11.7	.9340	5.64†	14.50†
11.8	.9628	11.59‡	5.44‡

*Equation 11.7 was fitted to actual data for the 6-12 week period; equation 11.8 was fitted to feed and gain quantities adjusted to a constant gain of 4.45 pounds for the 6-12 week period.
†Significant at the 1 per cent level with 17 degrees of freedom.
‡Significant at the 1 per cent level with 21 degrees of freedom.

Equation 11.9 is the isoquant equation derived from production function 11.7. The 4.45-pound gain isoquants (average gain from 6 to 12 weeks) derived from isoquant equation 11.9 and directly from equation

[2] An alternative method was devised in an attempt to use the 15-per cent protein ration observations. A very small quantity of soybean oilmeal (1 per cent of the ration) was assumed for the 15-per cent ration in order that no observation points would fall directly on the corn axis. Because the observation points were extremely close to the corn axis, however, the shape of the gain isoquants was distorted when these observations were used.

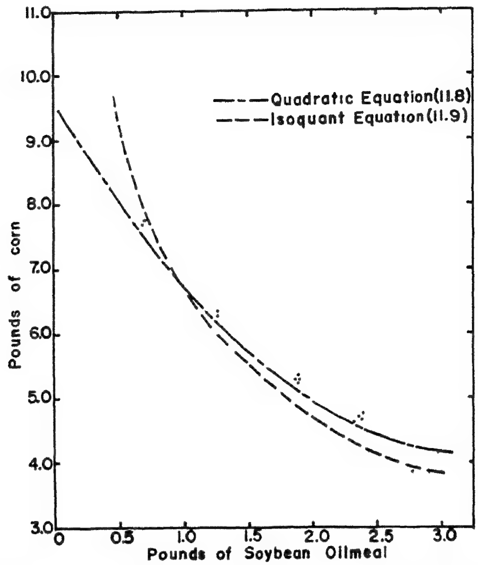


Figure 11.3. Comparison of 4.45 pound gain isoquants (gains on birds weighing 2.44 pounds), predicted by quadratic equation 11.8 and by isoquant equation 11.9.

11.8 are plotted in Figure 11.3. These two contours have quite consistent slopes except at the extreme upper ends (for protein rations of 15 to 17

$$(11.9) \qquad C = \left[\frac{Y}{1.7291\ S^{.2531}}\right]^{2.0008}$$

per cent). The influence of the adjusted observations for the 15 per cent protein ration (falling on the corn axis) forces the upper portion of the isoquant from function 11.8 down, relative to the isoquant from equation 11.9. However, the divergence is probably exaggerated because gains are extremely low on the 15 per cent protein ration. Hence, considerable inaccuracy may arise in the method of adjusting the gain and feed data for this ration to a common gain of 4.45 pounds for the 6-12 week period. For example, assume that 5.0 pounds of corn (with no soybean oilmeal) is required to produce 2.225 pounds of gain on the 15 per cent ration. Using the adjustment procedure, 10.0 pounds of corn are then assumed to produce 4.45 pounds of gain, a doubtful conclusion. Diminishing returns to corn are more consistent with nutritional theory than the constant returns used in the above adjustment. Thus, the observations for the 15 per cent ration should probably fall at greater quantities on the corn axis, forcing the upper portion of the isoquant from equation 11.8 to become more consistent with the isoquant from equation 11.9. Remember that the dots of the scatter diagram in Figure 11.3 are not

the observations to which equation 11.7 is fitted; these dots represent only the adjusted means to which equation 11.8 was fitted. Observations for equation 11.7 would have, if they could be presented simply, a scatter more consistent with the 4.45-pound gain contour derived from isoquant equation 11.9. Or, again, the relevant comparison for equation 11.9 would be a family of contours related to the set of unadjusted observations. Because an entire production surface for the 6-12 week period is given by equation 11.7, this function will serve as a basis for prediction in the 6-12 week interval. It should be remembered, however, that this function has, starting at the origin of feed inputs for the period, linear isoclines. Hence, the marginal substitution rates and least-cost rations so specified are those which are "average" for the interval. However, specification of optimum "average" rations over weight intervals is one objective of this study.

Regression equations for the 12-24 week interval

The data for the 12-24 week period is adequate for estimation of an interval production surface, with gain as a function of the two feed categories. Hence, the simple contour equations estimated as a check procedure for the 0-6 week and 6-12 week periods have not been computed for the 12-24 week interval. Three different types of functions were fitted to the gain and feed data for the 12-24 week interval: a Cobb-Douglas function, equation 11.10, a square root function, equation 11.11, and a quadratic function, equation 11.12. The gains predicted are those beyond the average 12-week weight of 6.93 pounds.

$$Y = 1.0764\, C^{.5108}\, S^{.2517} \tag{11.10}$$

$$Y = -2.8884 + .0450\, C - .2966\, S + .9894\sqrt{C} + 2.4592\sqrt{S} + .1284\sqrt{CS} \tag{11.11}$$

$$Y = .0148 + .1838\, C + .8837\, S + .0001C^2 - .0214\, S^2 - .0040\, CS \tag{11.12}$$

The 12-24 week observations for the 10 per cent protein ration (with no soybean oilmeal included) are not used in computing the Cobb-Douglas function, equation 11.10 for the reason given previously; use of observations falling on the corn axis distorts the gain isoquants. However, all the data are used in computing functions 11.11 and 11.12.

Table 11.6 shows the t and R^2 values for regression equations 11.10, 11.11, and 11.12. Equations 11.10 and 11.11 will be used in predicting economic quantities at later points in the study. Quadratic crossproduct function 11.12 is not used for later predictions because it contains one term which is statistically nonsignificant even at the 50 per cent level and it explains less of the deviations of the dependent variable, Y, than equations 11.10 and 11.11, as shown by the R^2 values of Table 11.6. While one of the regression coefficients in equation 11.11 is significant at the 20 per cent level of probability, all five terms are used, on grounds of nutrition logic, for estimating the production surface and making predictions. Dropping the nonsignificant term and recomputing the equation gives a slightly lower R^2 value, with all terms highly significant.

However, predictions differ only by minute quantities when the term is or is not used.

Table 11.6. R^2 and t Values for Regression Equations 11.10, 11.11, and 11.12 Using Feed and Gain Data for the 12-24 Week Period

Equation	Value of R^2	t Values for Regression Coefficients in the Order Shown in Equations 11.10, 11.11, and 11.12				
11.10	.9838	29.70*	21.10*			
11.11	.9936	1.40**	9.12†	3.80†	15.23†	2.87†
11.12	.9730	4.96†	14.71†	0.16‡	7.36†	1.67**

*Significant at the 1 per cent level with 17 degrees of freedom.
†Significant at the 1 per cent level with 18 degrees of freedom.
**Significant at the 20 per cent level with 18 degrees of freedom.
‡Nonsignificant at the 50 per cent level with 18 degrees of freedom.

Gain isoquant equations for the three production functions 11.10, 11.11, and 11.12 are shown, respectively, in equations 11.13, 11.14, and 11.15. Equations 11.13 and 11.14 were used in predicting the gain isoquants of Figure 11.4, which shows three pairs of isoquants for the average turkey gain (in the 12-24 week interval) at 16, 20, and 24 weeks of age. The contours shown in Figure 11.4 are for gains starting from an average weight of 6.93 pounds at 12 weeks of age.

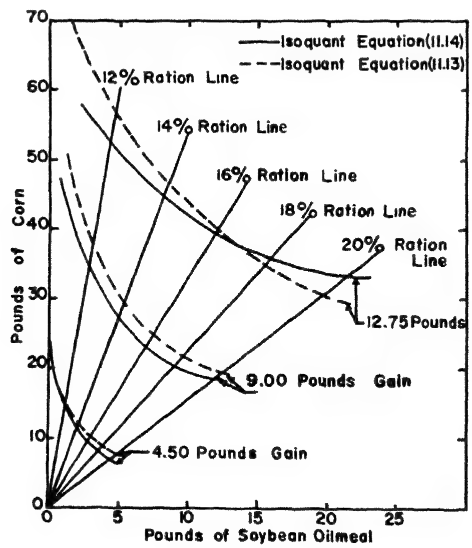


Figure 11.4. Comparison of gain isoquants predicted by isoquant equations 11.13 and 11.14.

$$(11.13)\quad C = \left[\frac{Y}{1.0764\ S^{.2517}}\right]^{1.9577}$$

$$(11.14)\quad C = (-10.9933 - 1.4267\sqrt{S} \pm 11.1111\sqrt{.0699\ S - .1886\sqrt{S} + .1800\ Y + 1.4988})^2$$

$$(11.15)\quad C = -712.5504 + 15.4961\ S \pm 3875.9690\sqrt{.000027\ S^2 - .0019\ S + .0005\ Y + .0338}$$

In connection with the isoquants of Figure 11.4, it should be remembered that the Cobb-Douglas function requires equal slopes for all isoquants along any straight line through the origin (ration line). Hence, this function tends to "average out" fluctuations over the input-output surface. The square root and quadratic functions are not subject to the restriction of constant slope along ration lines. Consequently, these types of functions provide a closer "fit" to data which are not consistent with the assumptions of constant slopes of isoquants at the points where they are intersected by any one ration line. The above restriction on the Cobb-Douglas function helps explain the difference in slopes along the two 12.75-pound gain isoquants shown in Figure 11.4. At high protein levels, to conform to the above-mentioned restriction, the isoquant computed from the Cobb-Douglas function is "pulled down" relative to the isoquant computed from the square root function. The slopes of contours from the two functions are quite similar at the lower protein levels; least-cost rations predicted from them would also be similar. If interest is in predicting a least-cost ration which changes with increasing weight within the 12-24 week interval, the square root function should be used since it allows the slope of the isoquants to change along a ration line. Too, it expresses, as is generally believed to be the case, lower rates of substitution of soybean oilmeal for corn as the bird approaches maturity.

Comprehensive tables of least-cost rations computed from Cobb-Douglas function, equation 11.10, are given in the following section since it is believed that the majority of turkey producers are interested in a single "average" least-cost ration to be fed for the entire 12-24 week interval. However, because turkey production is becoming more and more a specialized enterprise, an increasing number of producers are interested in changing rations more frequently to obtain small savings in feed costs per bird.

Table 11.7 summarizes the marginal rates of substitution and combinations of corn and soybean oilmeal required for various gains in the third weight interval, as predicted from square root function, equation 11.11. Columns (7), (8), and (9) in Table 11.7 show marginal rates of substitution of soybean oilmeal for corn along gain isoquants of 4.50, 9.00, and 12.75 pounds in the third weight interval. Since curvature is allowed in the isoclines of the square root function, the substitution

Table 11.7. Marginal Rates of Substitution and Combinations of Corn and Soybean Oilmeal for Various Gains in the Third Weight Interval — Computed from Equation 11.11

Per cent Protein in the Ration	Pounds of Corn and Soybean Oilmeal for 4.50 Pounds Gain*		Pounds of Corn and Soybean Oilmeal for 9.00 Pounds Gain*		Pounds of Corn and Soybean Oilmeal for 12.75 Pounds Gain*		Marginal Rates of Substitution Along Gain Isoquants of		
	Corn	Soybean Oilmeal	Corn	Soybean Oilmeal	Corn	Soybean Oilmeal	4.50 lbs.*	9.00 lbs.*	12.75 lbs.*
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
10	34.7	0	74.5	0	113.4	0	--	--	--
11	18.2	0.7	41.0	1.6	64.2	2.6	8.41	7.29	6.62
12	14.8	1.2	34.1	2.8	54.3	4.5	5.39	4.42	3.86
13	12.8	1.7	30.1	3.9	48.5	6.3	4.03	3.16	2.66
14	11.4	2.1	27.2	5.0	44.5	8.1	3.23	2.43	1.95
15	10.2	2.4	25.0	6.0	41.7	9.9	2.70	1.93	1.47
16	9.4	2.8	23.3	7.0	39.1	11.7	2.28	1.56	1.13
17	8.6	3.2	21.8	8.0	37.2	13.7	1.98	1.28	0.86
18	8.0	3.6	20.6	9.2	35.5	15.8	1.72	1.05	0.64
19	7.3	3.9	19.4	10.3	34.3	18.2	1.52	0.86	0.45
20	6.8	4.3	18.5	11.5	33.3	20.8	1.33	0.70	0.30

*Gains measured from an average weight of 6.93 pounds at 12 weeks.

rates along particular ration lines change as weight increases. The least-cost ration for attaining each of the three levels of gain in Table 11.7 is determined by locating the marginal rates of substitution in columns (7), (8), and (9) which most nearly equal the soybean oilmeal to corn price ratio. Thus, the least-cost ration for a particular price ratio may change three times as weight increases over the upper weight range.

Input-output curves and choice of function

The degree of conformity of the several functions in predicting single ration input-output curves is suggested in figures 11.5 and 11.6 for 14 and 20 per cent protein levels. The slopes of the curves are most nearly comparable for low protein levels. At higher protein levels, the quadratic function "rounds off" too rapidly, while the Cobb-Douglas may "overshoot."

The choice of a particular function to be used in making various estimates is based on several considerations. Foremost among these considerations are (a) the statistical and plotted "fit" of the functions to the data and (b) the practical aspects of applying the results to farm conditions. No single function for the 12-24 week interval appeared to best meet these considerations for all types of predictions. Consequently, in following sections the Cobb-Douglas function is used to predict the best "average" least-cost rations over the 12-24 week interval. Since this function has a constant slope along a particular ration line, it gives a single least-cost ration, as an "average" over the feeding period, for any given price ratio between corn and soybean oilmeal. Functions

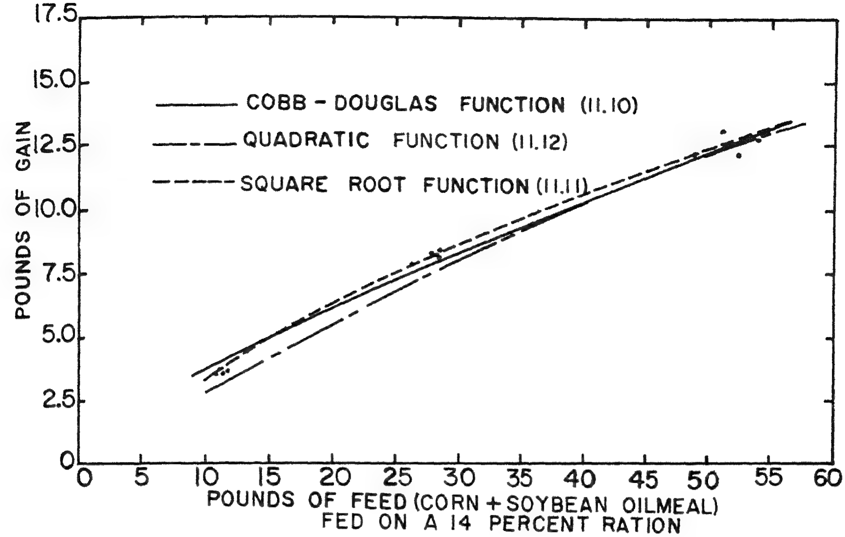


Figure 11.5. Comparison of input-output curves for a 14 per cent protein ration by three equations.

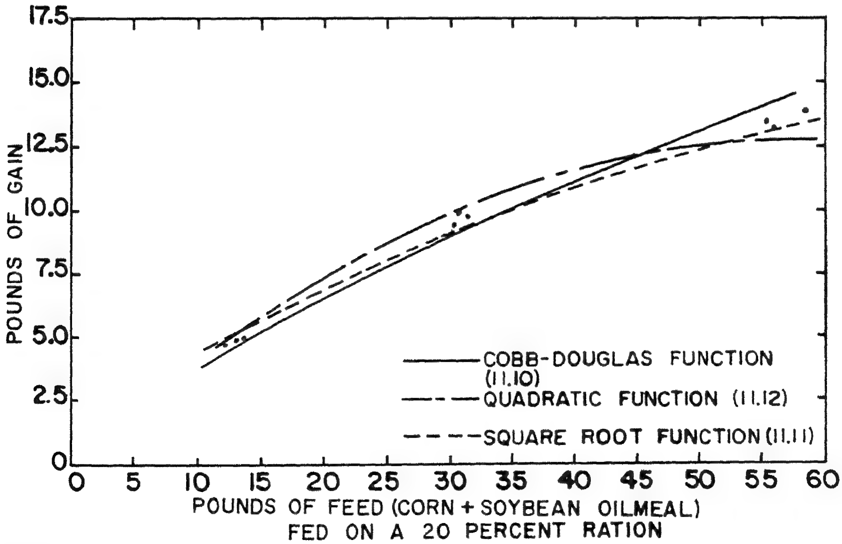


Figure 11.6. Comparison of input-output curves for 20 per cent protein ration by three equations.

11.11 and 11.12 provide more accurate least-cost rations than Cobb-Douglas function 11.10 if the per cent of protein is changed several times within the 12-24 week interval. However, because of the cost and inconvenience of frequently adjusting the protein level within a relatively short time period, many producers probably prefer to feed only one ration in the upper weight range.

While the Cobb-Douglas function is used to predict average least-cost rations in the 12-24 week interval, the square root function is used in predicting optimum marketing weights: it fits the plotted input-output data more closely than either quadratic or Cobb-Douglas functions. While the Cobb-Douglas function appears to be satisfactory in predicting the average slope or curvature of the gain isoquants, it tends to overestimate the slope for large feed inputs.

AVERAGE MARGINAL RATES OF SUBSTITUTION FOR COBB-DOUGLAS

Returning to one objective of this study, predicting rations which average least-cost over a feeding interval, we use quantities from the Cobb-Douglas equations in this section. Marginal rates of substitution and total feed quantities to produce total gains over each of the weight intervals are shown in tables 11.8, 11.9, and 11.10. The substitution and isoquant data are derived from the Cobb-Douglas function for each feeding and weight interval indicated below the tables.

Table 11.8. Combinations of Corn and Soybean Oilmeal Required to Produce Turkeys Weighing 2.44 Pounds, and Marginal Rates of Substitution Between Feeds in This Weight Range*

Per Cent Protein in the Ration	Pounds of Feed to Produce Turkeys Weighing 2.44 Pounds		Marginal Rates of Substitution of Soybean Oilmeal for Corn
	Corn	Soybean oilmeal	
21	2.40	0.80	2.47
22	2.18	0.90	2.00
23	2.00	1.01	1.65
24	1.84	1.10	1.37
25	1.70	1.21	1.15
26	1.57	1.33	0.97
27	1.46	1.46	0.82
28	1.35	1.60	0.70
29	1.25	1.75	0.59
30	1.16	1.93	0.49
31	1.07	2.14	0.41

*The figures in this table are derived from Cobb-Douglas function 11.3, computed from observations for the 0-6 week period.

Table 11.9. Combinations of Corn and Soybean Oilmeal Required to Increase Turkeys From 2.44 Pounds to 6.93 Pounds Liveweight and Marginal Rates of Substitution Between Feeds in This Weight Range*

Per Cent Protein in the Ration	Pounds of Feed to Increase Turkey Weight From 2.44 Pounds to 6.93 Pounds		Marginal Rates of Substitution of Soybean Oilmeal for Corn
	Corn	Soybean oilmeal	
16	10.11	0.43	11.65
17	7.86	0.72	5.57
18	6.75	0.97	3.55
19	6.03	1.21	2.53
20	5.50	1.45	1.92
21	5.08	1.69	1.52
22	4.73	1.95	1.23
23	4.43	2.23	1.01
24	4.17	2.50	0.84
25	3.93	2.81	0.71

*The figures in this table are derived from Cobb-Douglas function 11.7, computed from observations for the 6-12 week period.

Table 11.10. Combinations of Corn and Soybean Oilmeal Required to Increase Turkeys From 6.93 Pounds to 19.84 Pounds Liveweight and Marginal Rates of Substitution Between Feeds in This Weight Range*

Per Cent Protein in the Ration	Pounds of Feed to Increase Turkey Weight From 6.9 Pounds to 19.84 Pounds		Marginal Rates of Substitution of Soybean Oilmeal for Corn
	Corn	Soybean oilmeal	
11	74.03	2.96	12.32
12	58.18	4.85	5.91
13	50.17	6.54	3.78
14	44.89	8.16	2.71
15	41.09	9.79	2.07
16	38.06	11.42	1.64
17	35.58	13.11	1.34
18	33.44	14.86	1.11
19	31.57	16.71	0.93
20	29.88	18.68	0.79

*The figures in this table are derived from Cobb-Douglas function 11.10, computed from observations for the 12-24 week period.

Table 11.11. Combinations of Corn and Soybean Oilmeal Required To Produce 1 Pound of Gain on Turkeys Weighing 2.44, 6.93, and 19.84 Pounds

Per Cent Protein in the Ration	Pounds Feed for 1 Pound Gain on 2.44 Pound Turkeys*		Pounds Feed for 1 Pound Gain on 6.93 Pound Turkeys†		Pounds Feed for 1 Pound Gain on 19.84 Pound Turkeys‡	
	Corn	Soybean oilmeal	Corn	Soybean oilmeal	Corn	Soybean oilmeal
11	--	--	--	--	7.71	0.31
12	--	--	--	--	6.06	0.50
13	--	--	--	--	5.22	0.68
14	--	--	--	--	4.68	0.85
15	--	--	--	--	4.28	1.02
16	--	--	3.12	0.13	3.96	1.19
17	--	--	2.43	0.22	3.71	1.37
18	--	--	2.09	0.30	3.48	1.55
19	--	--	1.86	0.37	3.29	1.74
20	--	--	1.70	0.45	3.11	1.94
21	1.34	0.45	1.60	0.52	--	--
22	1.21	0.50	1.46	0.60	--	--
23	1.11	0.56	1.37	0.68	--	--
24	1.02	0.61	1.29	0.77	--	--
25	0.95	0.68	1.21	0.87	--	--
26	0.88	0.74	--	--	--	--
27	0.81	0.81	--	--	--	--
28	0.75	0.89	--	--	--	--
29	0.70	0.98	--	--	--	--
30	0.65	1.08	--	--	--	--
31	0.59	1.19	--	--	--	--

*Feed quantities predicted from Cobb-Douglas function 11.3.
†Feed quantities predicted from Cobb-Douglas function 11.7.
‡Feed quantities predicted from Cobb-Douglas function 11.10.

As is expected from nutritional logic and previous knowledge, predictions from the interval functions show that, for a given ration, the marginal rates of substitution of soybean oilmeal for corn decline as the bird increases in weight. This point is shown in substitution equations 11.16, 11.17, and 11.18 where the constants are -.8247, -.5065, and -.4928 for the three successive weight intervals. A pound of soybean oilmeal replaces 2.47 pounds of corn for turkeys fed on a 21 per cent protein ration in the first weight interval (Table 11.8); on this same ration a pound of soybean oilmeal replaces only 1.52 pounds of corn for turkeys in the second weight interval (Table 11.9). Tables 11.9 and 11.10 indicate that with 18 per cent of protein in the ration, one pound of soybean oilmeal replaces 3.55 pounds of corn for birds in the second weight interval, but replaces only 1.11 pounds of corn for birds in the third weight interval. These results occur because the bird requires more protein relative to carbohydrates in the early growing

stages and more carbohydrates relative to protein as maturity and the finishing period approach.

$$\frac{\delta C}{\delta S} = -.825\ CS^{-1} \tag{11.16}$$

$$\frac{\delta C}{\delta S} = -.507\ CS^{-1} \tag{11.17}$$

$$\frac{\delta C}{\delta S} = -.493\ CS^{-1} \tag{11.18}$$

Too, as each of the tables above show for the Cobb-Douglas functions, and also as is shown in Table 11.7 for the square root function, the marginal rate of substitution of soybean oilmeal for corn declines as relatively more protein is included in the ration for a particular level of gain. (Or, conversely, the marginal rate of substitution of corn for soybean oilmeal declines as the ration contains relatively less protein and relatively more carbohydrates.) Since the marginal rates of substitution between the two feed inputs are diminishing, unique rations can be found which minimize the cost of feed for a particular level of gain.

Table 11.11, based on Cobb-Douglas functions 11.3, 11.7, and 11.10, shows the amount of feed to produce one pound of gain at three weight levels. With a 22 per cent protein ration, only 1.71 pounds of feed (1.21 pounds of corn and .50 pound of soybean oilmeal) are required to produce an additional pound of gain on birds weighing 2.44 pounds. However, to produce an additional pound of gain on birds weighing 6.93 pounds using a 22 per cent protein ration, 2.06 pounds of feed (1.46 pounds of corn and .60 pound of soybean oilmeal) are required. Greater quantities of feed per pound of additional gain are required as birds reach 19.84 pounds liveweight.

Table 11.12. Per Cent Protein in Rations Which Are Least-Cost for Turkeys From .11 Pound to 2.44 Pounds, With Various Corn and Soybean Oilmeal Prices*

Price of Corn in Cents per Pound	Price of Soybean Oilmeal in Cents per Pound														
	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00	6.25	6.50
1.4	22.0	21.5	21.5	21.0	--	--	--	--	--	--	--	--	--	--	--
1.6	22.5	22.0	21.5	21.5	21.0	--	--	--	--	--	--	--	--	--	--
1.8	23.0	22.5	22.0	22.0	21.5	21.5	21.0	--	--	--	--	--	--	--	--
2.0	23.5	23.0	22.5	22.5	22.0	21.5	21.5	21.0	21.0	--	--	--	--	--	--
2.2	24.0	23.5	23.0	23.0	22.5	22.0	22.0	21.5	21.5	21.0	21.0	--	--	--	--
2.4	24.5	24.0	23.5	23.5	23.0	22.5	22.5	22.0	22.0	21.5	21.5	21.0	21.0	--	--
2.6	25.0	24.5	24.0	24.0	23.5	23.0	23.0	22.5	22.0	22.0	21.5	21.5	21.5	21.0	21.0
2.8	25.5	25.0	24.5	24.5	24.0	23.5	23.0	23.0	22.5	22.5	22.0	22.0	21.5	21.5	21.5
3.0	26.0	25.5	25.0	24.5	24.0	24.0	23.5	23.5	23.0	23.0	22.5	22.5	22.0	22.0	21.5
3.2	26.0	26.0	25.5	25.0	24.5	24.5	24.0	23.5	23.5	23.0	23.0	22.5	22.5	22.0	22.0
3.4	26.5	26.0	25.5	25.5	25.0	24.5	24.5	24.0	23.5	23.5	23.0	23.0	22.5	22.5	22.0
3.6	27.0	26.5	26.0	25.5	25.0	25.0	24.5	24.5	24.0	23.5	23.5	23.0	23.0	23.0	22.5
3.8	27.5	27.0	26.5	26.0	25.5	25.0	25.0	24.5	24.5	24.0	23.5	23.5	23.0	23.0	23.0
4.0	27.5	27.0	26.5	26.5	26.0	25.5	25.0	25.0	24.5	24.5	24.0	24.0	23.5	23.5	23.0

*Computed from substitution equation $\delta C/\delta S = -0.825\ CS^{-1}$.

Table 11.13. Per Cent Protein in Rations Which Are Least-Cost for Turkeys From 2.44 Pounds to 6.93 Pounds, With Various Corn and Soybean Oilmeal Prices*

Price of Corn in Cents per Pound	Price of Soybean Oilmeal in Cents per Pound														
	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00	6.25	6.50
1.4	20.0	19.5	19.5	19.0	19.0	18.5	18.5	18.5	18.0	18.0	18.0	18.0	17.5	17.5	17.5
1.6	20.0	20.0	19.5	19.5	19.0	19.0	18.5	18.5	18.5	18.5	18.0	18.0	18.0	18.0	17.5
1.8	20.5	20.5	20.0	19.5	19.5	19.5	19.0	19.0	18.5	18.5	18.5	18.5	18.0	18.0	18.0
2.0	21.0	21.0	20.5	20.0	20.0	19.5	19.5	19.5	19.0	19.0	18.5	18.5	18.5	18.5	18.5
2.2	21.5	21.0	21.0	20.5	20.0	20.0	19.5	19.5	19.5	19.0	19.0	19.0	19.0	18.5	18.5
2.4	22.0	21.5	21.0	21.0	20.5	20.5	20.0	20.0	19.5	19.5	19.5	19.0	19.0	19.0	19.0
2.6	22.5	22.0	21.5	21.0	21.0	20.5	20.5	20.5	20.0	20.0	19.5	19.5	19.5	19.0	19.0
2.8	22.5	22.5	22.0	21.5	21.5	21.0	20.5	20.5	20.5	20.0	20.0	20.0	19.5	19.5	19.5
3.0	23.0	22.5	22.5	22.0	21.5	21.5	21.0	21.0	20.5	20.5	20.0	20.0	20.0	19.5	19.5
3.2	23.5	23.0	22.5	22.5	22.0	21.5	21.5	21.0	21.0	20.5	20.5	20.5	20.0	20.0	20.0
3.4	23.5	23.5	23.0	22.5	22.0	22.0	21.5	21.5	21.0	21.0	20.5	20.5	20.5	20.0	20.0
3.6	24.0	23.5	23.0	23.0	22.5	22.5	22.0	21.5	21.5	21.5	21.0	21.0	20.5	20.5	20.5
3.8	24.5	24.0	23.5	23.0	23.0	22.5	22.0	22.0	21.5	21.5	21.0	21.0	21.0	20.5	20.5
4.0	24.5	24.0	24.0	23.5	23.0	23.0	22.5	22.0	22.0	21.5	21.5	21.5	21.0	21.0	20.5

*Computed from substitution equation $\delta C/\delta S = -0.507\ CS^{-1}$.

Table 11.14. Per Cent Protein in Rations Which Are Least-Cost for Turkeys From 6.93 Pounds to Finished Weight, With Various Corn and Soybean Oilmeal Prices*

Price of Corn in Cents per Pound	Price of Soybean Oilmeal in Cents per Pound														
	3.00	3.25	3.50	3.75	4.00	4.25	4.50	4.75	5.00	5.25	5.50	5.75	6.00	6.25	6.50
1.4	15.0	15.0	14.5	14.5	14.0	14.0	13.5	13.5	13.5	13.5	13.0	13.0	13.0	13.0	12.5
1.6	15.5	15.0	15.0	14.5	14.5	14.0	14.0	13.5	13.5	13.5	13.5	13.0	13.0	13.0	13.0
1.8	16.0	15.5	15.5	15.0	14.5	14.5	14.5	14.0	14.0	14.0	13.5	13.5	13.5	13.5	13.0
2.0	16.5	16.0	15.5	15.5	15.0	15.0	14 5	14.5	14.5	14.0	14.0	14.0	13.5	13.5	13.5
2.2	17.0	16.5	16.0	16.0	15.5	15.5	15.0	15.0	14.5	14.5	14.5	14.0	14.0	14.0	13.5
2.4	17.5	17.0	16.5	16.5	16.0	15.5	15.5	15.0	15.0	15.0	14.5	14.5	14.5	14.0	14.0
2.6	18.0	17.5	17.0	16.5	16.5	16.0	16.0	15.5	15.5	15.0	15.0	15.0	14.5	14.5	14.5
2.8	18.0	18.0	17.5	17.0	16.5	16.5	16.0	16.0	15.5	15.5	15.0	15.0	15.0	15.0	14.5
3.0	18.5	18.0	17.5	17.5	17.0	17.0	16.5	16.0	16.0	16.0	15.5	15.5	15.0	15.0	15.0
3.2	19.0	18.5	18.0	18.0	17.5	17.0	17.0	16.5	16.0	16.0	16.0	15.5	15.5	15.5	15.0
3.4	19.5	19.0	18.5	18.0	17.5	17.5	17.0	17.0	16.5	16.5	16.0	16.0	15.5	15.5	15.5
3.6	19.5	19.5	19.0	18.5	18.0	17.5	17.5	17.0	17.0	16.5	16.5	16.0	16.0	16.0	15.5
3.8	20.0	19.5	19.0	18.5	18.0	18.0	17.5	17.5	17.0	17.0	16.5	16.5	16.0	16.0	16.0
4.0	20.0	20.0	19.5	19.0	18.5	18.5	18.0	17.5	17.5	17.0	17.0	16.5	16.5	16.5	16.0

*Computed from substitution equation $\delta C/\delta S = -0.493\ CS^{-1}$.

LEAST-COST RATIONS

The data of tables 11.12, 11.13, and 11.14, predicted from the substitution equations derived from the Cobb-Douglas functions of the preceding section, provide estimates of rations which "average" least in cost (for various corn and soybean oilmeal prices) over each of the three specified weight intervals. The figures of Table 11.12, predicted from substitution equation 11.16, are estimates of the least-cost rations in the .11-pound to 2.44-pound weight interval. Substitution equations 11.17 and 11.18 are used, respectively, to provide the least-cost estimates given in tables 11.13 and 11.14 for the 2.44-pound to 6.93-pound

interval and the 6.93-pound to finished weight interval. The least-cost ration in each of these weight intervals is found by equating the marginal rate of substitution over the appropriate weight range with the inverse price ratio of the feeds, then solving for the ratio of corn to soybean oilmeal in the ration.

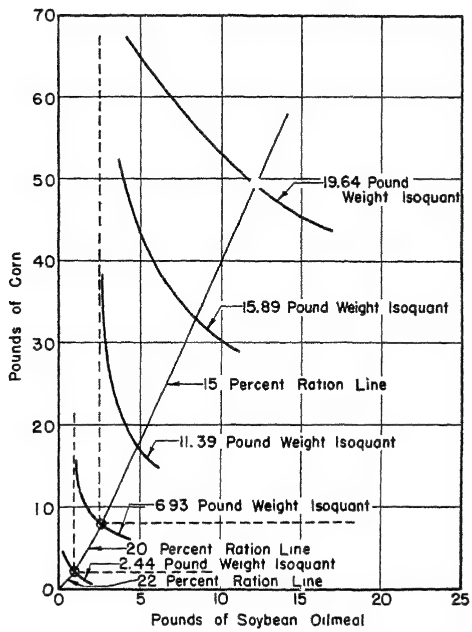


Figure 11.7. Least-cost rations for three weight intervals, predicted by Cobb-Douglas functions with a soybean oilmeal to corn price ratio of 2.0.

The estimate of tables 11.12, 11.13, and 11.14 may be used by turkey producers as follows: with corn at \$1.23 per bushel (2.2 cents per pound) and soybean oilmeal at \$4.50 per hundred pounds (4.5 cents per pound), the inverse price ratio is -4.5/2.2 or -2.05.[3] With this price ratio, the predicted least-cost ration contains 22.0 per cent protein for the first weight interval (Table 11.12), 20.0 per cent protein for the second weight interval (Table 11.13), and 15.0 per cent protein for the third weight interval (Table 11.14). In the first weight interval, the producer might choose to feed a slightly higher level of protein than given by the least-cost ration. From a practical standpoint, savings by a least-cost ration in the first weight interval are small, and the producer might not want to risk slower gains from a low protein ration. However, in the second

[3]As pointed out previously, a negative sign is attached to the price ratio because the price line (on a typical two-dimensional diagram) is negatively sloping.

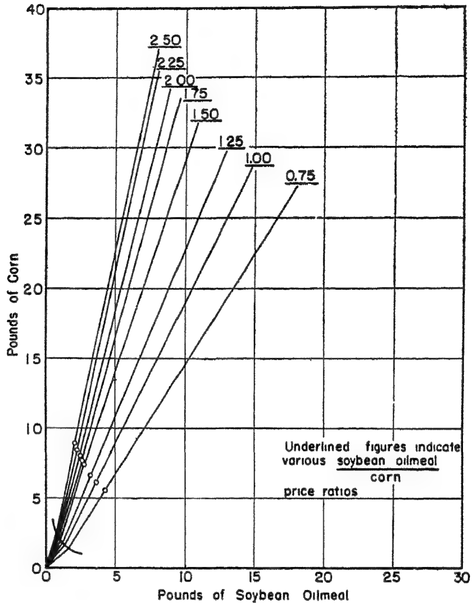
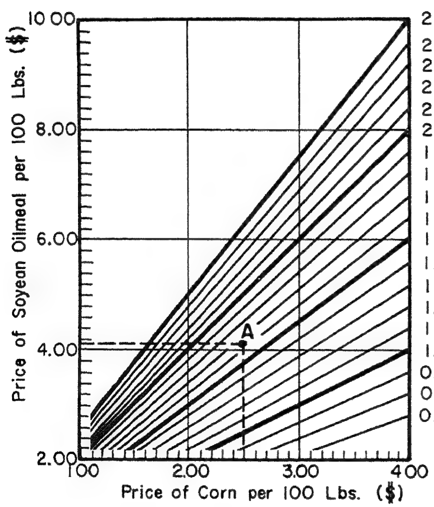


Figure 11.8. Least-cost rations for three weight intervals, predicted by Cobb-Douglas functions with various soybean oilmeal to corn price ratios.



Soybean Oilmeal-Corn Price Ratio	Optimum Percent Protein in the Ration	Estimated Quantities of Corn and Soybean Oilmeal per 100 Lbs. of Feed for Various Protein Rations: Corn (pounds)	Soybean Oilmeal (pounds)
2.5	21.0	45.00	15.00
2.4	21.0		
2.3	21.5	43.75	16.25
2.2	21.5		
2.1	22.0	42.50	17.50
2.0	22.0		
1.9	22.5	41.25	18.75
1.8	22.5		
1.7	23.0	40.00	20.00
1.6	23.5	38.75	21.25
1.5	23.5		
1.4	24.0	37.50	22.50
1.3	24.5	36.25	23.75
1.2	25.0	35.00	25.00
1.1	25.5	33.75	26.25
1.0	26.0	32.50	27.50
0.9	27.0	30.00	30.00
0.8			
0.7	27.5	28.75	31.25

Figure 11.9. Least-cost rations for the 0.11-pound to 2.44-pound weight interval, predicted by Cobb-Douglas function 11.3 with various corn and soybean oilmeal prices.

weight interval, and particularly in the third weight interval, substantial savings in feed costs may be realized by using a least-cost ration rather than one which produces faster gains. Under certain price relationships, of course, a least-cost ration may also produce the most rapid gains.

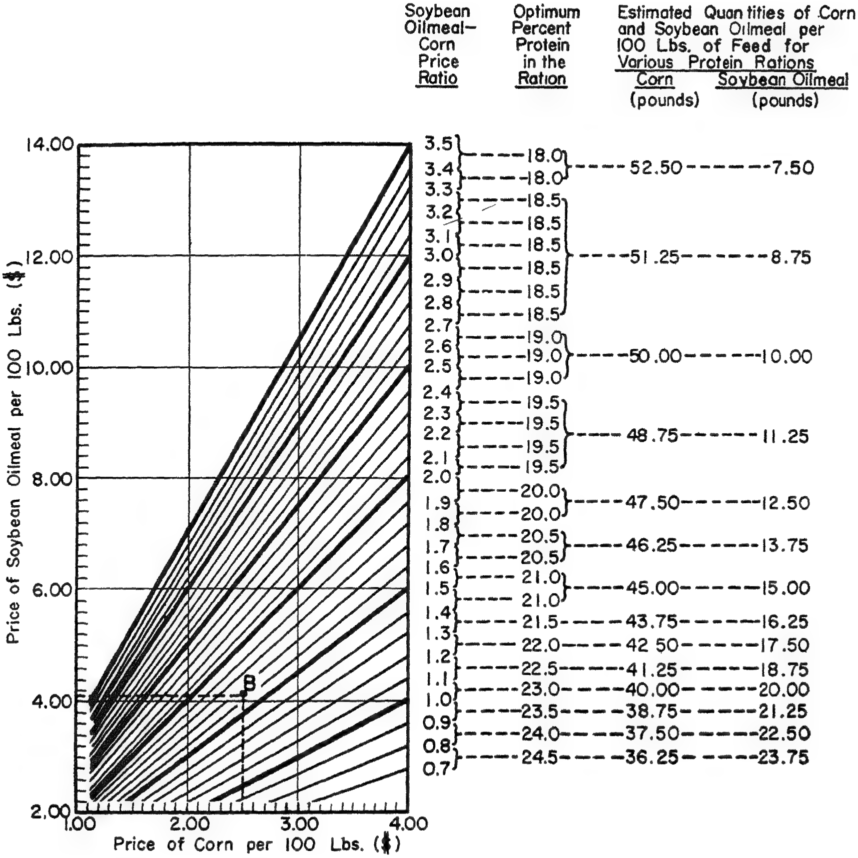


Figure 11.10. Least-cost rations for the 2.44-pound to 6.93-pound weight interval, predicted by Cobb-Douglas function 11.7 with various corn and soybean oilmeal prices.

The producer may wish to make further adjustments within the third weight interval to reduce feed costs. For example, during the first few weeks of the third weight interval, the producer may wish to feed a slightly higher protein level than prescribed by the least-cost ration; he may wish to decrease this protein level as the birds increase towards marketing weight. Table 11.7, and equation 11.11 upon which it is based, can be used for this purpose.

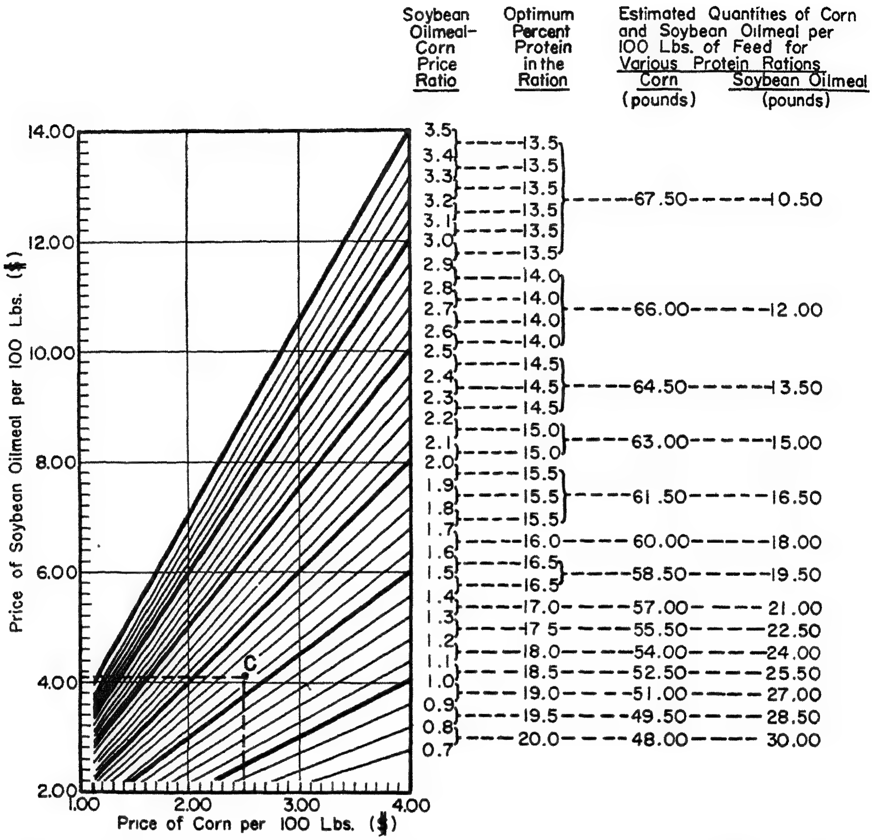


Figure 11.11. Least-cost rations for the 6.93-pound to finished weight interval, predicted by Cobb-Douglas function 11.10 with various corn and soybean oilmeal prices.

Referring again to tables 11.12, 11.13, and 11.14, if the price of soybean oilmeal should rise to 5.5 cents per pound, with corn remaining at 2.2 cents per pound, the price ratio becomes -2.50. Least-cost rations then contain 21.0 per cent protein for the first interval, 19.0 per cent for the second interval and 14.5 per cent protein for the third interval. With a price of 2.0 cents per pound for corn and 5.0 cents for soybean oilmeal, the least-cost rations also would contain 21.0, 19.0, and 14.5 per cent protein since the price ratio is still -2.50.

Graphic illustration of changes in average least-cost rations between weight intervals for a price ratio of -2.0 is produced in Figure 11.7. The line passing through the origin extending to and intersecting the 2.44-pound weight isoquant represents a 22 per cent protein ration line. Using the intersection point on the 2.44-pound isoquant as a new origin (circled), the least-cost ration for the second weight interval contains

20 per cent protein. Again using the intersection point of the 20 per cent ration line with the 6.93-pound isoquant as a new origin (circled), the least-cost ration for the third weight interval contains 15 per cent protein.[4] Figure 11.8 illustrates the least-cost ration path (expansion path) for the entire feeding period for various price ratios. As expected from nutritional logic, the percentage of protein in the ration consistently decreases with each higher weight interval, regardless of the existing price ratio.

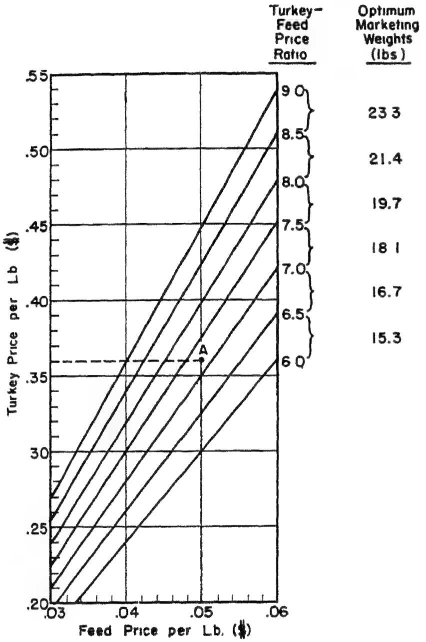


Figure 11.12. Optimum marketing weights for turkeys fed on a 13 per cent protein ration, predicted by square root function.

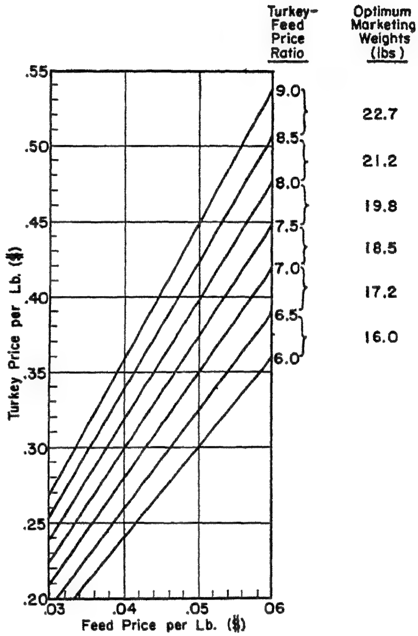


Figure 11.13. Optimum marketing weights for turkeys fed on a 15 per cent protein ration, predicted by square root function.

Simple Graphical Indication of Least-Cost Rations

Figures 11.9, 11.10, and 11.11 have been included to allow simple graphical selection of least-cost rations over the three weight intervals. These graphs assume linear segments along gain isoquants and indicate

[4]The reason the slope of the 15 per cent ration line in Figure 11.7 is not steeper relative to the 20 and 22 per cent ration lines is as follows: some of the high protein ingredients of the "basic" ration are reduced in quantity or removed entirely at the start of the third weight interval, thus requiring more soybean oilmeal relative to corn for a given percentage of protein in the ration.

least-cost rations for price ratios falling within the diagonal price rays shown. Figures 11.9, 11.10, and 11.11 may be used as follows: suppose the price of corn is 2.5 cents per pound and the price of soybean oilmeal is 4.1 cents per pound. These prices are located at point A in Figure 11.9. Following to the right of the diagram between the two diagonal lines, it is found that the least-cost ration over the first weight interval contains 23.0 per cent protein. One hundred pounds of the 23.0 per cent ration may be formulated by mixing 40.0 pounds of corn, 20.0 pounds of soybean oilmeal and 40.0 pounds of the basic ingredients shown in Table 11.1. The above feed prices are also found at point B, Figure 11.10, and specify a least-cost ration containing 20.5 per cent protein for the second weight interval. One hundred pounds of a 20.5 per cent ration contains 46.25 pounds of corn, 13.75 pounds of soybean oilmeal, and 40.00 pounds of other basic ingredients shown in Table 11.2. Point C, Figure 11.11, indicates a least-cost ration of only 16.0 per cent protein for the third weight interval, with the corn and soybean oilmeal prices assumed. One hundred pounds of the 16.0 per cent protein ration is composed of 60.0 pounds corn, 18.0 pounds of soybean oilmeal, and 22.0 pounds of other basic ingredients. The recommended rations resulting from use of figures 11.9, 11.10, and 11.11 are identical with those of tables 11.12, 11.13, and 11.14 and are included only as a simple alternative method of presenting the same results.

Throughout the analysis, the criterion for selecting rations has been one of minimum cost. However, a ration other than the least-cost ration for prevailing prices may be better suited for producing the most rapid gains over a given weight range. For example, if the producer anticipates a fall in turkey prices, he may be interested in getting the poults to market weight as rapidly as possible, rather than in minimizing feed cost for a given gain.

OPTIMUM MARKETING WEIGHTS

The preceding sections provided estimates of the least-cost combinations of corn and soybean oilmeal over the three weight intervals. Once the least-cost ration has been determined, the next question is one of finding the most profitable, or optimum marketing weight for the turkeys. The marketing weight which maximizes returns above feed costs is determined by equating the marginal product of feed for the least-cost ration with the feed to turkey price ratio. (Also, it can be determined by equating the partial derivatives of gain with respect to corn and soybean oilmeal with the proper price ratios. The latter is the more refined method. However, the other method is used here as a practical measure.) In other words, the most profitable marketing weight above feed costs is attained under the condition of equation 11.19, where dY/dR is the marginal product of the particular ration, showing the amount added to gain by each small added quantity of the ration, i.e., dY/dR is the derivative of gain with respect to feed inputs predicted

Table 11.15. Turkey Marketing Weights for Maximum Profits With Various Protein Rations and Turkey to Feed Price Ratios*

Turkey to Feed Price Ratio	Feed to Turkey Price Ratio	Per Cent Protein in the Ration															
		12.5	13.0	13.5	14.0	14.5	15.0	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5	20.0
6.0	.167	14.2	14.6	15.0	15.2	15.3	15.4	15.4	15.4	15.4	15.3	15.2	15.1	15.0	14.9	14.7	14.6
6.2	.161	14.7	15.2	15.5	15.7	15.8	15.8	15.8	15.9	15.8	15.7	15.6	15.5	15.4	15.2	15.0	14.9
6.4	.156	15.3	15.7	16.0	16.3	16.3	16.3	16.3	16.3	16.3	16.2	16.0	15.9	15.7	15.6	15.4	15.2
6.6	.152	15.8	16.2	16.5	16.8	16.8	16.8	16.8	16.8	16.7	16.6	16.4	16.3	16.1	15.9	15.7	15.5
6.8	.147	16.4	16.8	17.1	17.4	17.3	17.3	17.3	17.2	17.1	17.0	16.8	16.6	16.4	16.2	16.0	15.8
7.0	.143	17.0	17.4	17.7	17.9	17.9	17.8	17.8	17.7	17.6	17.4	17.2	17.0	16.8	16.5	16.3	16.0
7.2	.140	17.5	18.0	18.2	18.5	18.4	18.3	18.2	18.2	18.0	17.8	17.6	17.4	17.1	16.8	16.6	16.3
7.4	.135	18.2	18.6	18.8	19.1	19.0	18.9	18.7	18.6	18.4	18.2	18.0	17.7	17.4	17.2	16.9	16.6
7.6	.132	18.8	19.2	19.4	19.7	19.5	19.4	19.2	19.1	18.9	18.6	18.4	18.1	17.8	17.5	17.2	16.9
7.8	.128	19.4	19.9	20.1	20.3	20.1	19.9	19.8	19.6	19.3	19.0	18.8	18.5	18.1	17.8	17.5	17.2
8.0	.125	20.1	20.6	20.8	20.9	20.7	20.5	20.3	20.1	19.8	19.5	19.2	18.8	18.5	18.1	17.8	17.4
8.2	.122	20.8	21.2	21.4	21.6	21.3	21.0	20.8	20.6	20.3	19.9	19.6	19.2	18.8	18.4	18.1	17.7
8.4	.119	21.6	22.0	22.1	22.2	21.9	21.6	21.3	21.1	20.7	20.3	20.0	19.6	19.2	18.8	18.4	18.0
8.6	.116	22.3	22.7	22.8	22.9	22.6	22.2	21.9	21.6	21.2	20.8	20.3	20.0	19.5	19.1	18.7	18.2
8.8	.114	23.1	23.5	23.6	23.6	23.2	22.8	22.4	22.1	21.7	21.2	20.8	20.3	19.8	19.4	18.9	18.5
9.0	.111	23.9	24.2	24.3	24.3	23.8	23.4	23.0	22.6	22.2	21.6	21.2	20.7	20.2	19.7	19.2	18.8
9.2	.109	--	--	--	--	--	--	23.5	23.1	22.6	22.1	21.5	21.0	20.5	20.0	19.5	19.0
9.4	.106	--	--	--	--	--	--	--	23.7	23.1	22.5	22.0	21.4	20.8	20.3	19.8	19.3
9.6	.104	--	--	--	--	--	--	--	--	23.6	22.9	22.4	21.8	21.2	20.6	20.1	19.5
9.8	.102	--	--	--	--	--	--	--	--	--	23.4	22.8	22.2	21.5	20.9	20.4	19.8
10.0	.100	--	--	--	--	--	--	--	--	--	23.8	23.2	22.5	21.8	21.2	20.6	20.0

*Computed from square root function 11.11.

from the production function. In equation 11.19, P_r is price per pound of ration and P_y is price per pound of turkey.

$$(11.19) \qquad \frac{dY}{dR} = \frac{P_r}{P_y}$$

For practical purposes, it is supposed that the least-cost ration will be determined for each of the three weight intervals by the methods of the previous sections. In the third weight interval, the marginal products for the least-cost ration will be used in determining the optimum or most profitable marketing weight.

Square root function equation 11.11 is used in predicting optimum marketing weights for turkeys fed on different rations, under a wide range of feed to turkey price ratios (the feed to turkey price ratio is the reciprocal of the turkey to feed price ratio, Table 11.15). The square root function is used for these predictions because, as was mentioned previously, it fits the input-output observations for the various rations more closely than the other functions for the third weight interval. Table 11.15 indicates, for each ration and price ratio, the marketing weight which maximizes returns above feed costs. The practical marketing weight range for female birds is 12 to 18 pounds; for males the range is about 18 to 30 pounds. Thus, in a mixed or "straight run" flock for which the predictions of this study apply, the practical marketing weight range is from approximately 15 to 24 pounds. Separate production functions (computed from observations on all males or all females) would be required to provide a guide to optimum marketing weights for the producer feeding a flock of predominately one sex. However, the figures in Table 11.15 should be relevant for that majority of producers who feed "straight run" flocks.

Before using Table 11.15, the average least-cost ration for the third weight interval is determined from Table 11.14 or Figure 11.11 in the previous sections. Table 11.15 can then be used to predict, for any particular least-cost ration, the marketing weight which is optimum for a given price per pound of the ration and of the finished turkeys. Suppose that the least-cost ration for the third weight interval (predicted from Table 11.14) contains 15.0 per cent protein. If the price of turkeys is 32 cents per pound, and the price of the 15.0 per cent protein ration is 4 cents per pound (a turkey to feed price ratio of 8.0), the predicted optimum marketing weight for birds on this ration is 20.5 pounds (Table 11.15). If the turkey to feed price ratio is only 7.0, the optimum marketing weight is reduced to 17.8 pounds (Table 11.15).

Turkey to feed price ratios outside of the 6.0 to 10.0 range shown in Table 11.15 predict optimum marketing weights which do not fall within the practical marketing weight range for mixed flocks. However, the turkey to feed price ratios of recent years have been characterized by wide fluctuations, resulting in ratios which have frequently been higher than the upper range of 10.0 shown in Table 11.15. The average United States ratio was 11.0 in 1949, with a January high of 16.9 in Iowa. The average turkey to feed price ratio on the Pacific Coast was 6.4 in 1954.

Where risk and uncertainty are important, turkey producers may wish to market their birds previous to the weight at which marginal cost equals marginal return; use of these limited resources in the latter phases of turkey production may provide lower returns than their use in some other alternative.

The data from Table 11.15 have been used in deriving figures 11.12 through 11.15, which provide convenient graphical approximations of the marketing weights which maximize returns above feed costs for various rations and turkey to feed price ratios. Figures are used as follows: First, find the figure for which the protein level corresponds most closely with that for the least-cost ration predicted for the third weight interval. Second, locate the intersection of the turkey and feed prices on the graph. Third, follow along between the diagonals to the right side of the graph and read off the approximate optimum marketing weight. For example, assume a least-cost ration of 13 per cent protein, an expected turkey price of 36 cents per pound, and a feed price of 5 cents per pound. The intersection of the two feed prices is found at point A, Figure 11.12. Moving to the right side of the diagram between the two diagonals which bracket point A, the optimum marketing weight is found to be 18.1 pounds.

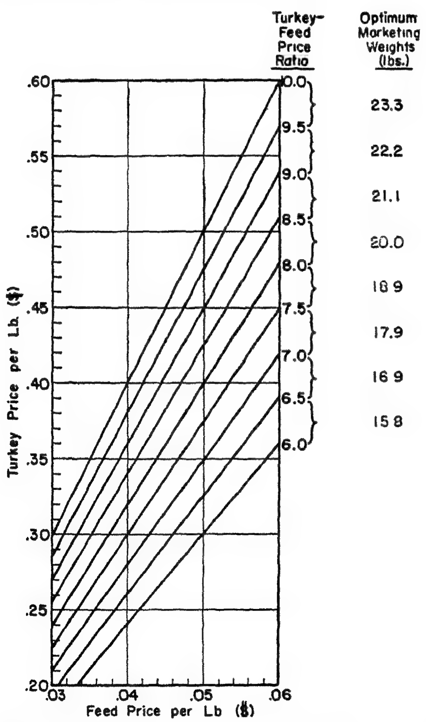


Figure 11.14. Optimum marketing weights for turkeys fed on a 17 per cent protein ration, predicted by square root function.

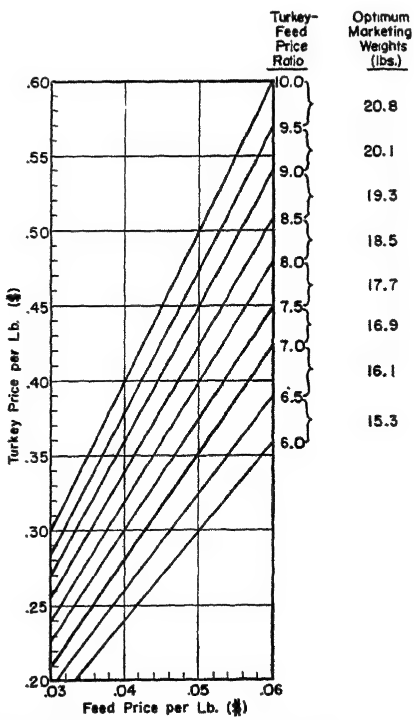


Figure 11.15. Optimum marketing weights for turkeys fed on a 19 per cent protein ration, predicted by square root function.

Earl O. Heady
Norman L. Jacobson
John A. Schnittker
Solomon Bloom

CHAPTER 12

Milk Production Functions and Marginal Rates of Substitution Between Forage and Grain

Interest in possibilities of forage-grain substitution in the dairy cow ration has been increased by recent agricultural developments. One development is acreage control which allows farmers to grow forage as a replacement crop for grain. Another is the continuing interest in conservation; erosion control plans ordinarily require an increased acreage of grasses and legumes and fewer acres of grains and row crops. Both of these developments increase the supply of forages relative to grains and give rise to questions of using forage profitably. One possibility is the substitution of forage for grain in terms of ruminants. The feasibility of this adjustment depends, however, on the rate at which the various classes of feeds substitute for each other.

Changes in price structures, with dairy product prices depressed relative to feed and labor costs, also have caused farmers to examine substitution possibilities as a means of lowering costs and increasing profits. Then, too, yearly and geographic differentials in the costs of concentrates relative to forages and to the prices of milk give rise to questions of the most profitable ration under particular economic circumstances. New technologies in dairy farm management and the growing concentration of production on specialized farms also add interest in this direction. To what extent should the grain-forage ration be varied as the price ratio of grain to forage changes? To what extent should the most profitable ration differ between grain surplus and grain deficit areas or other areas where concentrates are priced at different levels? These questions can be answered only if information is available on substitution ratios. The optimum ration, in terms of profit maximization, can be determined only by relating substitution ratios to price ratios. Finally, determining the nature of the milk production surface with its expression of feed to milk transformation ratios and feed substitution coefficients is a central problem in dairy cow nutrition.

OBJECTIVES

The experiment on which this report is based was designed to provide estimates of the milk production function and feed substitution ratios in alternative dairy cow rations. The experiment provides

predictions of the milk production surface and milk isoquants indicating marginal rates of substitution between the two classes of feeds — concentrates and hay. The experiment reported was relatively small in sample size, but large in terms of facilities. It was methodological in nature but is being continued on a somewhat larger scale.

The primary objectives are to establish (1) the rates at which grains and forages substitute under specific technical conditions and (2) the rate at which feeds are transformed into milk for various production levels and rations. An auxiliary objective is to investigate the economic potential of substituting forage for grain. Hence, details are provided for (a) explaining the models which serve as a basis for the experimental design, (b) illustrating the procedure used in predicting feed substitution and transformation rates, and (c) determining the particular ration and level of grain feeding which results in the least-cost ration and the most profitable level of production per cow.

The study reported was restricted in magnitude because of limitations in funds, cows, barn space, and other facilities. Because of its limited magnitude, it should be looked upon as an exploratory study, to be supplemented by later investigations now underway. The over-all objectives of this study are of a methodological nature. The central predictions revolve more nearly around estimation of the milk production function and feed transformation and substitution coefficients than around use of the particular principles in determining economic optima in dairy rations. It is hoped that this fundamental study will provide the basis for further work and encourage other studies which allow more refined predictions of the milk production function and of the economic gains in using the profit-maximizing principles outlined.

BASIC NATURE OF FUNCTIONS

Milk production is a complex process involving many resources, of which feeds represent but one class. The milk production function is of the general form:

$$M = f\,(C, F, X_1, X_2, X_3, X_4 \,.\,.\,.\,.\,.\,.\,.\, X_n). \tag{12.1}$$

Here, M refers to milk production per cow in a specified time period, C refers to concentrate intake, F refers to forage intake, X_1 refers to body size, X_2 refers to inherent breed qualities of the cow, X_3 refers to labor used, and X_4 through X_n refer to unspecified resources or inputs. While all of these resource or input categories are variable, most nutrition studies (and this investigation specifically) are carried on in the framework of a production relationship such as that represented by equation 12.2.

$$M = f\,(C, F \mid X_1, X_2, X_3, X_4 \,.\,.\,.\,.\,.\,.\,.\, X_n) \tag{12.2}$$

Here, only the resources or inputs to the left of the vertical bar are considered variable. Labor required to handle cows under different rations is necessarily varied in an experiment. For experimental purposes, however, labor is assumed to be available in unlimited quantities at no cost. While labor as a variable must be considered in terms of its cost or price in profit decisions on the farm (e.g., more labor may be required to feed a specific grain to hay ratio than to hand-feed grain and self-feed hay), this step is unnecessary in technical experiments involving only estimation of feed substitution rates or feed-milk transformation ratios.

POSSIBLE GEOMETRIC FORM

A milk production function involving two variable categories of feed can be represented as a three-dimensional diagram or surface. Milk output per cow in the relevant production period is measured on the vertical axis while each category of feed is measured on the respective horizontal axis. Each point in the feed plane represents a different ration and level of feeding and will correspond to a particular level of milk represented on the milk surface. The particular nature of this surface, including the slope of the isoclines over it and the slope of the contours around it, will determine (1) the forage to grain ration which gives the lowest cost for any stated level of milk output, (2) the level of feeding which will result in maximum profit per cow over feed costs, and (3) the extent to which conventional ENE or TDN evaluations of the energy or heat transformation of feed are appropriate for evaluations concerned with milk transformation of these same feeds.

Unfortunately, little is known about the nature of the milk production surface. Numerous feeding standards suppose that the milk production function is homogeneous of degree 1.0; the surface is implicitly assumed to be linear up to the limits of the cow's milk producing capacity. Some standards, such as the *total digestible nutrient* (TDN) basis for rations consider the milk isoquants and input-output curves to be straight lines since they are not varied to consider the proportion and level of feeding or milk production. Other nutrition recommendations suppose the milk surface to have nonlinear isoquants and isoclines. This is because ration recommendations seldom include only one class of feed (i.e., grain or forage), a recommendation which would be the most profitable one if milk isoquants were linear. Some possible hypotheses about the nature of the milk production surface were outlined in Chapter 3.

PREVIOUS RESEARCH RELATING TO NATURE OF ISOQUANTS AND MILK SURFACE

While no previous experiment has been designed to predict the

nature of the milk production surface, the results of other studies do lead to hypotheses about substitution and transformation coefficients. Huffman and Duncan discussed the possible stimulating effect of a small amount of grain when a cow has been fed forage alone.[1] They stated that a cow receiving forage alone will not produce as much milk (FCM) as when a small amount of grain is substituted for an equal amount of TDN from hay. These results lead to the supposition that the milk isoquant may curve rather sharply at the forage end and is, therefore, nonlinear.

Jensen *et al.* predicted grain input-output curves which are nonlinear.[2] Although the report provides no postulates about milk isoquants, it appears that curved isoquants should be associated with curved input-output curves for any mathematical function used to define the optimum or maximum milk production per cow. Beach indicated that as more grain is fed to a cow, the maintenance requirements (i.e., at the zero milk isoquant) in terms of TDN become less and less.[3] As the ration approaches an all-grain ration, TDN from grain eventually become less efficient in maintaining health and activities of the cow. While the Huffman-Duncan experiment suggested that input-output curves may be nonlinear and that the milk isoquants are curved on the hay extremity, the work of Beach suggested curved isoquants towards the grain extremity. Of course, the isoquants might have greater curvature at the ends but be nearly linear in the middle.

In a study based on a sample of dairy farms, Ashe concluded that the input-output curves follow a near linear relationship up to about 4,000 pounds of grain per cow; between 4,000 and 6,000 pounds of grain, milk increases only slightly; and over 6,000 pounds of grain, milk does not increase at all.[4] Yates and others summarized several research reports implying a diminishing rate of transformation of feed into milk. These studies included both European and American data and an "economic curve" derived from Danish experiments indicating considerable decline in marginal milk yields at high feeding levels.[5]

Recommended daily energy allowances for milk production were summarized by the Committee on Animal Nutrition of the National Research Council.[6] Because of lack of appropriate input-output data, its recommendations did not consider the concept of diminishing returns to dairy cow rations. The committee's approximate guide gave .32 pound TDN as the requirement for each additional pound of 4 per cent

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[1] Huffman, C. F. and Duncan, C. W. The nutritive value of alfalfa hay. Jour. Dairy Sci., 32: 465-71. 1949.

[2] Jensen, E. *et al.* Input-output relationships in milk production. Tech. Bul. 815. USDA, Washington, D.C. 1942.

[3] Beach, C. L. The facility of digestion of foods as a factor in feeding. Conn. Agr. Exp. Sta. Bul. 43. New Haven. 1906.

[4] Ashe, A. J. Response of milk production to increased grain feeding. Farm Econ., No. 174. Cornell Univ. 1950. Pp. 4474-76.

[5] Yates, F., Boyd, D., and Pettit, G. Influence of changes in level of feeding on milk production. Jour. Agr. Sci., 32: 428-56. 1942.

[6] National Research Council. Recommended nutrient allowances for dairy cattle. Washington, D.C. 1950.

fat-corrected milk (FCM) above maintenance, but it did not indicate any changes in the TDN transformation due to the combination of feeds and/or to the level of feeding. The Morrison recommendations, used extensively in this country, have the same over-all recommendations for TDN, whereas recommendations based upon the net-energy system (NE) estimate .30 therms for each additional pound of 4 per cent FCM above maintenance requirements. Blaxter, in a review of the energy standards for dairy cattle, stated that energy standards should express the productivities of feeds as they are.[7] However, he did not indicate that feeding values should vary with the proportions of feed fed.

The dependence of the ruminant upon the catabolic and anabolic activities of the rumen microflora implies that certain combinations of hay and grain may stimulate or depress microbiological activity and, hence, exert an effect on the nature or curvature of the isoquants. This problem was pointed out in studies by Hamilton and Swift, *et al.*, who reported a depression of ration digestibility at a hay-to-grain ratio of 1:1.[8] The concepts of the "stomach capacity" and "physiological limit" lines mentioned in Chapter 3 have further ramifications in dictating the size and feed capacity of the cow which can be used most economically under varying economic conditions. The problem of genetic ability or "dairy merit," which is defined by the percentage of consumed energy that is converted into milk energy, and its basic interrelationships with the plane of nutrition involved in the lactational level desired, must also be considered in determining the milk production surface.

DESIGN OF EXPERIMENT

The basic experiment for the predictions of this study included 36 cows and is explained in detail below. This experiment, conducted without the use of pasture, extended over a 17-month period in 1953 and 1954. For certain predictions and estimates, data from an earlier experiment including 15 cows were also used. Both experiments are explained below. The experimental design and over-all study represent an interdisciplinary approach by dairy nutritionists and production economists, with the aid of specialists in statistics. Individuals involved held exploratory seminars to discuss the logical models, based on previous knowledge in nutrition and economics. The design of the experiment and the prediction methods were then selected to conform to these models. It should be pointed out, however, that the researchers involved do not hold that the design used is optimum. The experiment is a relatively small one, but one feasible with the limited resources and facilities available.

[7] Blaxter, K. L. Energy feeding standards for dairy cattle. Nutr. Abst. and Rev., 20: 1-18. 1950.

[8] Hamilton, T. S. The effect of added glucose upon the digestibility of protein and of fiber in rations for sheep. Jour. Nutr., 23: 101-10. 1942; and Swift, R. W., Thacker, E. J., Black, A., Bratzler, J. W., and Jones, W. H. Digestibility of rations for ruminants as affected by proportion of nutrients. Jour. Anim. Sci., 1: 447-61. 1918.

Basic 1953-54 Experiment With 36 Cows

The Holstein herd at the Iowa State University Dairy Farm was the source of the 36 cows used from March, 1953, to September, 1954. From the date of calving, each cow remained under experimental conditions for an initial 14-day adjustment period. A fixed ratio of 7 pounds of hay to 4 pounds of concentrates, initiated during the adjustment period, was maintained throughout the preliminary period.[9] The preliminary period provided the basis for dividing the animals into high-, medium-, and low-producing ability groups for their subsequent random allotment to the experimental period. In general, the production ability ranges for the animals in terms of pounds of 4 per cent FCM were as follows: (a) "high" — 10,500 pounds and over, (b) "medium" — 9,000 to 10,500 pounds, and (c) "low" — 9,000 pounds or less.

The design for the experimental period of 182 days is illustrated in Table 12.1. Four of the hay-to-concentrate ratios were chosen, ranging from a ration in which 75 per cent of the energy (ENE) was derived from hay and 25 per cent from concentrates, to one in which 15 per cent of the energy was derived from hay and 85 per cent from concentrates. The four hay-to-concentrate ratios were 75:25, 55:45, 35:65, and 15:85. Each of these hay-to-concentrate ratios was fed at high, medium, and low levels. For each of the 12 "hay-to-concentrate ratio-feeding level" treatments, there were three cows — one of high-, one of medium-, and one of low-producing ability. At the start of the experimental period, each cow, after its assignment to an ability group, was randomly allotted to one of the 12 "hay-to-concentrate ratio-feeding level" positions.

Auxiliary Experiment

While the main analysis of this study rests on the basic experiment

Table 12.1. Nature of Allocation of Animals by Level of Feeding and Ability (Milk Production in Pounds)

Level of Feeding	Ability	Hay-to-Concentrate Ratio			
		H-75:C-25	H-55:C-45	H-35:C-65	H-15:C-85
High	High	2,553	3,157	2,710	2,982
	Medium	2,392	2,649	3,529	2,976
	Low	3,632	3,263	3,160	3,538
Medium	High	3,266	2,600	2,378	3,142
	Medium	3,469	3,444	2,643	3,493
	Low	3,440	3,597	3,432	3,516
Low	High	3,272	3,291	2,963	3,174
	Medium	3,450	3,483	3,128	2,606
	Low	3,302	3,294	3,439	2,159

[9] For a detailed explanation of the experiment, see: Bloom, Solomon. Effects of various dietary hay-concentrate ratios on nutrient utilization and production responses of dairy cows. Unpublished Ph.D. thesis. Iowa State University Library. Ames. 1955.

explained above, cows from the following experiment were used for predictions explained later. The auxiliary experiment is that reported by Martin *et al.*[10] Fifteen cows were used from this experiment to evaluate four levels of alfalfa hay feeding. Rates of hay feeding during the experimental period were at .5, 1.17, 1.83, and 2.5 pounds per day per 100 pounds of body weight. A 2-week preliminary period was used in which the cows were fed hay at a daily rate of 2 pounds per 100 pounds of body weight and grain to supply 100 per cent of Morrison's recommended levels. Cows were on the auxiliary experiment for 16 weeks.

BODY WEIGHT CHANGES

Table 12.2 summarizes the body weight changes from two aspects: the total changes over the entire 182-day experimental period and the changes from the end of the initial 4 weeks of the experimental period to its conclusion. The results of the weight changes over the two periods are presented in the analyses of variance in tables 12.3 and 12.4. When the entire experimental period was considered (Table 12.3), the effects of ration, feeding level, and ability upon body weight changes either approached significance or were significant at the 5 per cent level. However, when the last 22 weeks of the experimental period were considered these effects disappeared (Table 12.4). From the first to the second case, the components of variance for ration, feeding level, and ability decreased markedly. These results indicate that the body weight changes, after the first 4 to 5 weeks of the experimental period, were largely independent of the ration fed, the level of feeding, and the ability of the animals.

Table 12.2. Body Weight Changes During Two Time Intervals of the Experimental Period

Level of Feeding	Ability	H-75:C-25*			H-55:C-45*			H-35:C-65*			H-15:C-85*		
		Cow	Change (lbs.) 0-26 wks.	Change (lbs.) 4-26 wks.	Cow	Change (lbs.) 0-26 wks.	Change (lbs.) 4-26 wks.	Cow	Change (lbs.) 0-26 wks.	Change (lbs.) 4-26 wks.	Cow	Change (lbs.) 0-26 wks.	Change (lbs.) 4-26 wks.
High	High	2,553	99	200	3,157	-69	-48	2,710	42	79	2,982	-80	-21
	Med.	2,392	59	125	2,649	-33	-46	3,529	92	179	2,976	-82	-38
	Low	3,632	-45	-47	3,263	-24	-7	3,160	110	69	3,538	51	20
Med.	High	3,266	-22	7	2,600	-78	-31	2,378	14	64	3,142	-102	3
	Med.	3,469	40	57	3,444	36	26	2,643	46	22	3,493	-56	10
	Low	3,440	26	39	3,597	-25	-32	3,432	55	29	3,516	58	120
Low	High	3,272	-116	53	3,291	2	40	2,963	-153	-21	3,174	-178	-31
	Med.	3,450	-67	-24	3,483	-61	-5	3,128	23	89	2,606	-164	-91
	Low	3,302	92	55	3,294	-2	19	3,439	-70	-4	2,159	21	36

*Hay-to-concentrate ratio

[10] Martin, T. G., Stoddard, C. E., and Allen, R. S. Effects of varied rates of hay feeding on body weight and production of lactating dairy cows. Jour. Dairy Sci., 37: 1233-40. 1954.

Table 12.3. Analysis of Variance: Body Weight Changes During the Entire Experimental Period

Source of Variation	D.F.	Sum of Squares	Mean Squares	F	Component of Variance (Per Cent)
Ration	3	33,167	11,056	3.30†	9
Level	2	30,207	15,104	4.51*	15
Ability	2	32,906	16,453	4.91*	15
R x L	6	22,949	3,825	1.14	2
R x A	6	30,826	5,138	1.53	9
A x L	4	11,792	2,948	.88	0
R x L x A	12	40,200	3,350		51
Total	35	202,047			

*P < .05
†Approaches P < .05

MILK PRODUCTION FUNCTIONS FOR 36 COWS IN 26-WEEK BASIC EXPERIMENT IN THE OVER-ALL PERIOD

Two sets of functions have been derived from the basic 36-cow experimental data: (1) those where a single time period is included and (2) those where time is considered as a variable. While predictions for a single and constant time period are included as methodological materials, it is believed that the production functions which include time

Table 12.4. Analysis of Variance: Body Weight Changes From the End of the Initial Four Weeks of the Experimental Period to Its Conclusion

Source of Variation	D.F.	Sum of Squares	Mean Squares	F	Component of Variance (Per Cent)
Ration	3	26,268	8,756	2.54	6
Level	2	8,951	4,476	1.30	0
Ability	2	705	353	.10	0
R x L	6	32,062	5,344	1.55	14
R x A	6	25,333	4,222	1.23	5
A x L	4	10,441	2,610	.76	0
R x L x A	12	41,301	3,442		75
Total	35	145,061			

as a variable provide the most efficient estimates. The time variable allowed for some changes in body weight, as well as the normal trend in milk output over the lactation period. As the data in Table 12.2 indicate, cows attained a near-equilibrium in weight after a month of the experiment.

Initial Equations for 36-Cow Basic Experiment Over 26 Weeks (Time Not a Variable)

Since little is known about the milk production function, three initial types of algebraic equations which do not include time as a variable were fitted to the data. A power equation was selected as one of a general type (including those such as exponential, Mitcherlich, etc.) which does not require specification of a single maxima in milk production. It is known that such a function may not conform adequately to a milk production surface since it assumes constant elasticity of production, linear isoclines, and increasingly wide ranges of rations for higher levels of milk production. However, it was thought that the function might allow reasonable "average" estimates of substitution ratios and transformation coefficients in the midsection of the milk surface. The other two equations allow specification of one ration consistent with maximum milk production per cow and of isoclines which converge to this point. These two functions were a quadratic and a square root quadratic.

Variables and Regression Equations

Three variables were used in estimating the initial milk production functions. Two of these are the concentrate and hay feeds discussed earlier. The third is cow ability since this variable was found to be highly associated with milk production in the experimental period. Hence, the variables for the functions which follow are those explained below where the milk output and feed input measurements are aggregates extending over the 26-week experimental period of the basic experiment:

M is production in pounds per cow of 4 per cent fat-corrected milk in the 26-week experimental period.

H is pounds of alfalfa hay measured in pounds per cow over the 26-week experimental period.

G is concentrate mix, called grain hereafter, measured in pounds per cow over the 26-week experimental period.

A is cow ability measured in pounds of 4 per cent fat-corrected milk produced in the 50-day preliminary period when all cows were fed the same ration as detailed earlier.[11]

[11] The simple correlation coefficient (r) between the ability term used and milk produc-

The 26-week initial functions for the 36-cow basic experiment are shown in equations 12.3, 12.4, and 12.5 for the logarithmic, quadratic, and square root functions, respectively.

$$(12.3) \quad M = 15.749\ H^{.1213}\ G^{.2758}\ A^{.3659}$$

$$(12.4) \quad M = 3{,}787.56 - .1288\ H + .9842\ G - 1.0991\ A + .000042 H^2 - .000064\ G^2 + .000353\ A^2 + .000000032\ HGA$$

$$(12.5) \quad M = 19{,}356.40 + 1.7855\ H + 1.1258\ G + 3.2040\ A - 300.0230\ \sqrt{H} - 183.0605\ \sqrt{G} - 226.5986\ \sqrt{A} + 2.6626\ \sqrt{HG}$$

The relevant statistics for these three functions are given in Table 12.5. The t values are for the regression coefficients in their respective order within the equations above. From 73 to 78.6 per cent of the total variance in milk production is accounted for in the three variables, depending on the function. All of the regression coefficients in the power function are acceptable at the probability level of 1 per cent. However, none of the individual coefficients for the quadratic or square root functions are significant at this probability level, even though a larger portion of the total variance in milk production is accounted for by the variables of the latter two functions. In a pure probability sense, equation 12.3 might be accepted for prediction purposes while equations 12.4 and 12.5 might be dropped. However, even though their regression coefficients have relatively larger standard errors, the general forms of equations 12.4 and 12.5 might provide predictions which conform better to the logic of the milk production surface than does equation 12.3, particularly since the latter function (a) does not cause milk level to approach a maximum, (b) causes the milk surface to pass up a "ridge" of wide ration latitudes, rather than to narrow to a peak of a single ration for the maximum milk production per cow, (c) does not allow low-level isoquants to be attained with hay or grain alone, and (d) causes linear isoclines which "fan out," rather than converge at the maximum milk

Table 12.5. Correlation Coefficients and t Values for Equations 12.3, 12.4, and 12.5

Equation	R	Values of t in Order of b's in Equation						
		b_1	b_2	b_3	b_4	b_5	b_6	b_7
12.3	.8545*	2.84*	5.65*	5.13*	--	--	--	--
12.4	.8832*	.18	.94	.60	.74	.54	.99	.64
12.5	.8864*	1.21	.45	.91	.84	.39	.66	.73

*$P < .01$.

tion is .64. The comparable statistic using age-corrected fat in previous lactations as the ability term yielded a correlation coefficient of .33.

production level. As is indicated in Table 12.6, substitution ratios become extremely large or small at extreme rations for the logarithmic function. However, equation 12.3 may provide useful estimates of average marginal substitution rates over a narrow range of the surface surrounding the mean level of feeding and the mean ratio of feeds in the experiment.

Marginal Equations for Logarithmic Function

Equations 12.6 and 12.7, which are derived from equation 12.3, define the marginal or incremental productivity of hay and grain, respectively, when one is variable and one is fixed in quantity. Both indicate diminishing productivity since the exponent on both the H and G variables is less than 1.0 in equation 12.3. Also, since the sum of the exponents is less than 1.0, the function indicates a diminishing feed to milk transformation as any fixed ratio of hay and grain is increased in quantity. Equation 12.8, also derived from 12.3, provides the isoquants paralleling those outlined in the earlier section on logic. Equation 12.9 defines the slopes of the isoquants and, therefore, indicates the marginal rate of substitution of grain for hay if equation 12.3 is used for predictions. Obviously, the substitution ratio will be predicted to

Table 12.6. Feed Combinations and Marginal Rates of Substitution ($\delta H/\delta G$) Predicted From Logarithmic Equation 12.3 for Milk Isoquants of 5,300, 6,300, and 7,300 Pounds per Cow Over the 26-Week Period (Ability Fixed at Mean for the 36 Cows)

5,300 Pounds Milk*			6,300 Pounds Milk†			7,300 Pounds Milk‡		
Lbs. grain§	Lbs. hay§	$\frac{\delta H}{\delta G}$‖	Lbs. grain§	Lbs. hay§	$\frac{\delta H}{\delta G}$‖	Lbs. grain§	Lbs. hay§	$\frac{\delta H}{\delta G}$‖
1,000	5,815	13.22	--	--	--	--	--	--
1,500	2,313	3.51	1,500	9,308#	14.11	--	--	--
2,000	1,204	1.37	2,000	5,001	5.68	2,000	16,843#	19.15
2,500	724#	.66	2,500	3,008	2.74	2,500	10,152#	9.23
3,000	478#	.36	3,000	1,991	1.51	3,000	6,709#	5.08
--	--	--	3,500	1,401	.91	3,500	4,714	3.06
--	--	--	4,000	1,034#	.59	4,000	3,489	1.98
--	--	--	4,500	792#	.40	4,500	2,661	1.34
--	--	--	--	--	--	5,000	2,098	.95

*Columns 1 and 2 show feed combinations which will produce 5,300 pounds of milk; column 3 shows marginal rates of substitution of grain for hay for these combinations.
†Columns 4 and 5 show feed combinations which will produce 6,300 pounds of milk; column 6 shows marginal rates of substitution for these combinations.
‡Columns 7 and 8 show feed combinations which will produce 7,300 pounds of milk; column 9 shows marginal rates of substitution for these combinations.
§Derived from equation 12.8.
‖Derived from equation 12.9.
#Outside of range of observations in experiment.

change as hay to grain ratios change, if function 12.3 is used for estimation.

$$\frac{\delta M}{\delta H} = 1.906\ H^{-.8787}\ G^{.2758}\ A^{.3659} \tag{12.6}$$

$$\frac{\delta M}{\delta G} = 4.347\ H^{.1213}\ G^{-.7242}\ A^{.3659} \tag{12.7}$$

$$H = \left(\frac{M}{15.75\ G^{.2758}\ A^{.3659}}\right)^{8.2440} \tag{12.8}$$

$$\frac{\delta H}{\delta G} = (-2.278)\frac{H}{G} \tag{12.9}$$

Isoquants for three levels of milk production predicted from the logarithmic function (i.e., isoquant equation 12.8) are shown in Figure 12.1. The straight lines denoted as a, b, and c are isoclines indicating all feed combinations which give a specified rate of substitution between hay and grain, if predictions are based on equation 12.3. Line a shows, for example, all quantities of grain and hay where 1 pound of grain substitutes for 3 pounds of hay. The slope of the isoquants does not change greatly above isocline a with a substitution ratio of 3. The isoquants

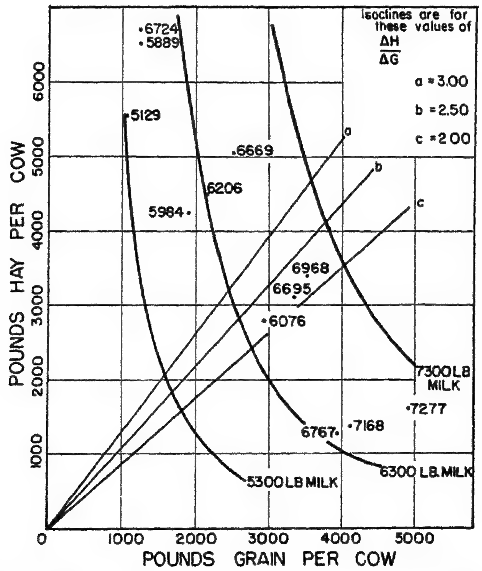

Figure 12.1. Milk isoquants for a 26-week period predicted from equation 12.3 with ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

bend rather sharply below line c. These same phenomena are illustrated in Table 12.6, the tabular counterpart of Figure 12.1, since the substitution ratios become very large for rations with a small proportion of grain and very small with a large proportion of grain. Isoquants with sharply changing slopes (rapid changes in substitution rates) at the extremes, such as found in the logarithmic equation, may actually be consistent with physiological processes of milk production. However, isoclines which fan out such as those in Figure 12.1 are inconsistent with a maximum milk output per cow, a condition possible only with converging isoclines.

If the power functions were accepted as the best predicting equations, the isoclines shown would indicate the least-cost ration for particular price ratios. For example, isocline b shows the points on the milk isoquants where 1 pound of grain substitutes for 2.5 pounds of hay. Hence, if the price of grain divided by the price of milk is 2.5, the points of intersection of the isoclines and isoquants show the least-cost rations for milk production levels of 5,300, 6,300, and 7,300 pounds per cow in the 26-week period. The dots in Figure 12.1 represent the 12 cows at the medium level of ability for 4 ration combinations and 3 feed levels. These dots provide a visual indication of whether the predicted isoquants seem consistent with the data. The average milk production for these 12 cows was 6,463 pounds. (All 36 cows served as the basis for predicting

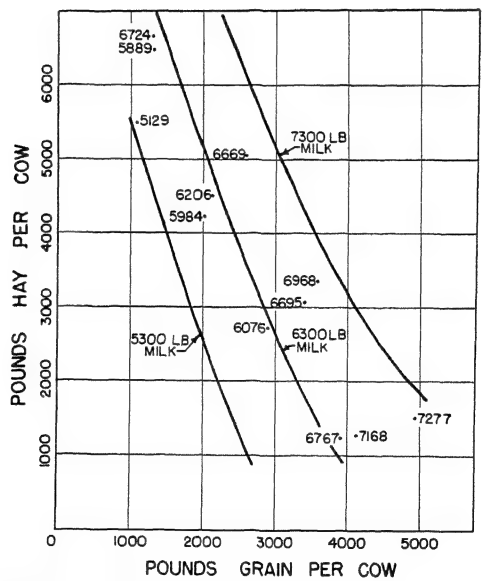

Figure 12.2. Milk isoquants for a 26-week period predicted from equation 12.18 with ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

the isoquants in Figure 12.1.) The marginal rate of substitution of grain for hay at approximately the midpoint (2,800 pounds of hay and 2,600 pounds of grain) of the 6,300-pound milk isoquant is 2.45. This figure indicates that, at the particular feed combination, one more pound of grain would replace 2.45 pounds of hay, with milk output held constant at 6,300 pounds. Conversely, 1 pound of hay would substitute for .40 pound of grain if the particular equation were used for the predictions.

Marginal Equations for Quadratic and Square Root Estimates

The marginal milk product functions for grain and hay, as single variables, are indicated as equations 12.10 and 12.11, respectively, for quadratic equation 12.4; they are indicated as equations 12.14 and 12.15 respectively, for the square root function of equation 12.5.

$$(12.10)\quad \frac{\delta M}{\delta H} = -.1288 + .000085\,H + .000000032\,GA$$

$$(12.11)\quad \frac{\delta M}{\delta H} = .9842 - .000129\,G + .000000032\,HA$$

$$(12.12)\quad H = 1{,}524.0 - .9556\,G \pm 11{,}838.0\sqrt{-.5309 - .000187\,G + .000000017\,G^2 + .000169\,M}$$

$$(12.13)\quad \frac{\delta H}{\delta G} = \frac{.9842 - .000129\,G + .000000032\,HA}{-.1288 + .000085\,H + .000000032\,GA}$$

$$(12.14)\quad \frac{\delta M}{\delta H} = 1.7855 - 150.0115\,H^{-.5} + 1.3313\,G^{5}\quad H^{-.5}$$

$$(12.15)\quad \frac{\delta M}{\delta G} = 1.1258 - 91.5302\,G^{-.5} + 1.3313\,H^{.5}\,G^{-.5}$$

$$(12.16)\quad H = \left[84.01 - .7456\sqrt{G} \pm .2800\sqrt{7.1421\,M - 290.2395\sqrt{G} - .9513\,G - 24{,}469.10}\right]^2$$

$$(12.17)\quad \frac{\delta H}{\delta G} = \frac{1.1258 - 91.5302\,G^{-.5} + 1.3313\,H^{.5}\,G^{-.5}}{1.7855 - 150.0115\,H^{-.5} + 1.3313\,G^{.5}\,H^{-.5}}$$

The milk isoquant functions for the quadratic and square root functions are indicated, respectively, as equations 12.12 and 12.16; the marginal rate of substitution equations are 12.13 and 12.17.

Equations 12.4 and 12.5 appear unsatisfactory for prediction purposes since the signs of the H terms in equations 12.10 and 12.14 are such that the marginal product of hay increases with higher levels of

feeding. In other words, each pound of hay would add more to milk production than the previous pound over all possible feeding levels. If equation 12.4 is used as the basis for predicting a milk isoquant of 6,300 pounds for the 26-week experimental period, the figures in Table 12.7 result. Similar results are forthcoming for predictions based on equation 12.5. In Table 12.7, the marginal rates of substitution of grain for hay change slowly between the extremes of the ration and do not differ significantly from the substitution rate of 2.3 from the linear equations presented later.

Table 12.7. Milk Isoquant of 6,300 Pounds and Marginal Rates of Substitution From Quadratic Equation 12.4: Ability Fixed at Mean for 36 Cows

Feed to Produce 6,300 Pounds Milk		Marginal Rate of
Lbs. grain	Lbs. hay	Substitution: $\frac{\delta H}{\delta G}$
1,000	7,711	2.45
1,500	6,490	2.43
2,000	5,279	2.41
2,500	4,083	2.37
3,000	2,908	2.32
3,500	1,769	2.23
4,000	688	2.09

Other Functions for 36 Cows

Other quadratic or square root equations similar to equations 12.4 and 12.5 appear logical in estimating a milk production surface, since (a) the t values are low for equations 12.4 and 12.5 and (b) the marginal product equation for hay is increasing. The t values are low in equations 12.4 and 12.5 partly because the size of the sample is small relative to the variance of milk production and because a relatively large proportion of the degrees of freedom are exhausted in the many coefficients estimated for the equations. A larger experiment might qualify a quadratic or square root function in estimating the surface. In a simple attempt to increase the number of degrees of freedom for the small sample, equation 12.18 was estimated with the variables outlined earlier. In this case, A^2 was dropped so that ability enters into the function in linear form only. Only three regression coefficients (in contrast to the seven coefficients of equation 12.4) were estimated, including one for the term $H - .00001\ H^2$ and one for the term $G - .00007\ G^2$. The coefficients before the H^2 and G^2 were simply estimated from previous nutrition studies and the data of this study.

$$(12.18) \quad M = .4304(H - .00001\ H^2) + 1.5008(G - .00007\ G^2) + .9265\ A - 665.49$$

The regression coefficients for this "adjusted quadratic" equation can be accepted in a probability sense at the 1 per cent level even if two added degrees of freedom are dropped to compensate for direct estimation of the constants for H^2 and G^2 (see Table 12.8).

Equations 12.19 and 12.20 have been derived from equation 12.18 and are, respectively, the milk isoquant and substitution rate equations. The milk isoquants in Figure 12.2 are based on equation 12.19; equation 12.20 indicates the slopes of the isoquants at particular points in the feed plane.

$$H = 50{,}000.0 \pm (-116{,}165.61)\sqrt{.2136 + .0000258\,G - .0000000018\,G^2 - .0000172\,M} \tag{12.19}$$

$$\frac{\delta H}{\delta G} = \frac{1.5008 - .000210\,G}{.4304 - .0000086\,H} \tag{12.20}$$

If used to estimate milk isoquants, equation 12.18 results in contours which have some curvature, although the change in slope is not great. The slope of the isoquants, and their location in the feed plane, is almost identical with the isoquants presented later in Figures 12.3 and 12.4 for equations 12.22 and 12.23. Table 12.9 includes data showing feed combinations, or the milk isoquants derived from equation 12.19, and the marginal rates of feed substitution for three milk isoquants derived from equation 12.20. The substitution rates do not change as rapidly as do those in Table 12.6 for the power function.

However, while the isoquants in Figure 12.2 appear to have but little slope and substitution rates in Table 12.9 do not change as rapidly as those in Table 12.6, a considerable change in substitution rates does occur over the range of observations presented. These data, all derived from equation 12.18, have a logical advantage over the estimates for equations 12.22 and 12.23, since the former allows diminishing returns for hay as well as grain. However, as differences between equations 12.21 and 12.22 show, there is no probability basis for retaining a hay

Table 12.8. Correlation Coefficients and t Values for Equations 12.18, 12.21, 12.22, 12.23, and 12.26

Equation	Value of t in Order of b's in Equation					
	R	b_1	b_2	b_3	b_4	b_5
12.18	.8755*	3.98*	6.16*	5.28*	--	--
12.21	.8770*	.99	3.57*	4.75*	1.42	.38
12.22	.8764*	3.85*	3.89*	4.91*	1.46	--
12.23	.8782*	3.76*	.15	4.94*	1.61	--
12.26	.8672*	3.85*	5.82*	5.27*	--	--

*$p < .01$, based on usual number of degrees of freedom. However, fewer degrees of freedom are probably appropriate for equation 12.18.

Table 12.9. Feed Combinations in Producing Specified Milk Levels and Marginal Rates of Substitution of Grain for Hay Predicted From Modified Quadratic Equation 12.18; Ability Fixed at Mean of 36 Cows for 26-Week Basic Experiment

Feed to Produce 5,300 Pounds Milk			Feed to Produce 6,300 Pounds Milk			Feed to Produce 7,300 Pounds Milk		
Lbs. grain	Lbs. hay	MRS of grain for hay $\delta H/\delta G$	Lbs. grain	Lbs. hay	MRS of grain for hay $\delta H/\delta G$	Lbs. grain	Lbs. hay	MRS of grain for hay $\delta H/\delta G$
1,000	5,560	3.37	1,000	8,256*	3.59	1,000	11,138*	3.86
1,500	3,969	2.99	1,500	6,566	3.17	1,500	9,328*	3.39
2,000	2,558	2.65	2,000	5,074	2.79	2,000	7,738*	2.97
2,500	1,313	2.33	2,500	3,761	2.45	2,500	6,345	2.60
3,000	221*	2.03	3,000	2,612	2.13	3,000	5,130	2.25
--	--	--	3,500	1,616	1.84	3,500	4,080	1.94
--	--	--	4,000	763*	1.56	4,000	3,182	1.64
--	--	--	--	--	--	4,500	2,428	1.36
--	--	--	--	--	--	5,000	1,813	1.09

*Prediction is outside range of observations.

coefficient which results in diminishing returns for this feed category when the 36 cows are used for an aggregate 26-week lactation period.

Equations 12.21, 12.22, and 12.23 represent estimates of the milk production function when particular terms are dropped from the initial quadratic equation 12.4 and the initial square root equation 12.5. In equation 12.21, the A^2 and interaction terms have been dropped from 12.4; in 12.22, H^2 also is dropped. In equation 12.23, $\sqrt{H}$, $\sqrt{A}$ and the interaction term have been dropped from equation 12.5.

$$(12.21)\quad M = .3010\ H + 1.5171\ G + .8978\ A - .000106\ G^2 + .000014\ H^2 - 459.63$$

$$(12.22)\quad M = .4089\ H + 1.4423\ G + .9074\ A - .000091\ G^2 - 573.42$$

$$(12.23)\quad M = .3977\ H - .1028\ G + .8988\ A + 102.9030\sqrt{G} - 2{,}335.45$$

The R and t values for these equations were given in Table 12.8. In equation 12.21, the regression coefficient for H^2 again is nonsignificant and positive, suggesting the unrealistic condition of increasing marginal productivity of hay. In 12.22, the regression coefficient for G^2 is significant at a probability level of less than .20, and the coefficients for the other terms are significant at a probability level of less than .01. An isoquant map for quadratic equation 12.22 is included in Figure 12.3, while one for square root equation 12.23 is included in Figure 12.4. These two sets of isoquant maps are almost identical; milk contours for both equations have only slight curvature. While the location of the milk contours in the feed plane is similar for figures 12.2, 12.3, and 12.4, those in Figure 12.2 have somewhat greater curvature at the lower

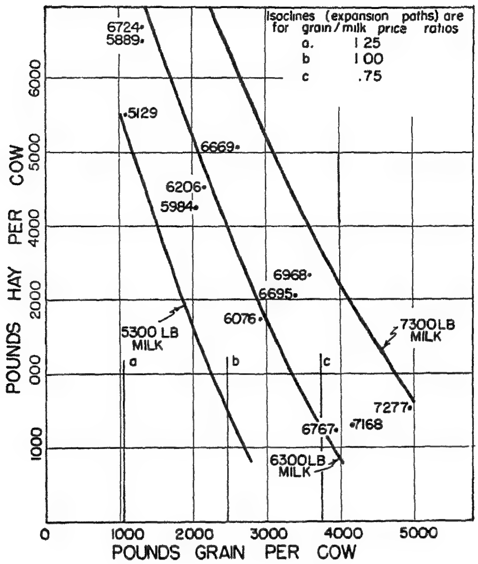

Figure 12.3. Milk isoquants for a 26-week period predicted from equation 12.22 with ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

end. This difference is due to the fact that the marginal rate of substitution of grain for hay is lower as the proportion of grain increases for isoquants based on equation 12.18 than for those based on equations 12.22 and 12.23. The difference is apparent in comparisons of Table 12.9, which includes substitution rates based on equation 12.18, and Table 12.10, which includes substitution rates based on equations 12.22

Table 12.10. Feed Combinations in Producing Specified Milk Levels and Marginal Rates of Substitution of Grain for Hay, Predicted From Equations 12.22 and 12.23; Ability Fixed at Mean of 36 Cows for 26-Week Basic Experiment

Lbs. of Grain	Pounds of Hay Required To Produce						Marginal Rate of Substitution of Grain for Hay at All Milk Levels	
	5,300 lbs. of milk		6,300 lbs. of milk		7,300 lbs. of milk			
	Equation 12.22	Equation 12.23	Equation 12.22	Equation 12.23	Equation 12.22	Equation 12.23	Equation 12.22	Equation 12.23
1,000	5,528	5,642	7,974*	8,157*	10,420*	10,671*	3.08	3.83
1,500	4,043	3,933	6,489	6,447	8,935*	8,961*	2.86	3.08
2,000	2,669	2,512	5,115	5,026	7,561*	7,540*	2.64	2.63
2,500	1,407	1,275	2,853	3,789	6,299	6,304	2.41	2.33
3,000	256*	170*	2,702	2,684	5,148	5,198	2.19	2.10
3,500	--	--	1,663	1,678	4,109	4,192	1.97	1.93
4,000	--	--	735*	750*	3,181	3,264	1.74	1.77
4,500	--	--	--	--	2,365	2,401	1.52	1.67
5,000	--	--	--	--	1,660	1,591	1.30	1.57

*Estimates outside range of observations.

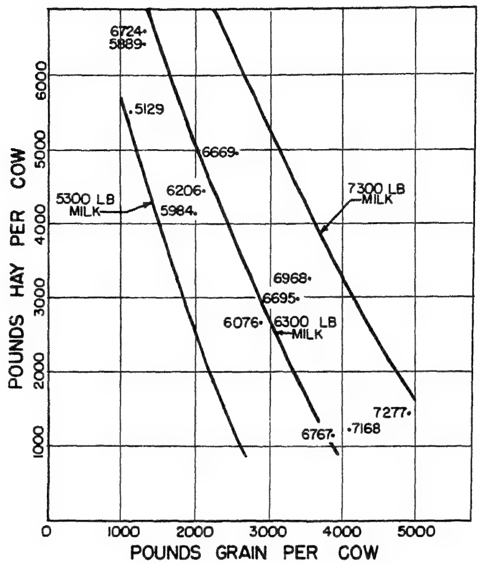

Figure 12.4. Milk isoquants for a 26-week period predicted from equation 12.23 with ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantities from 12 cows at medium level of ability.)

and 12.23. For example, with 3,500 pounds of grain used in combination with hay to produce 7,300 pounds of milk, the substitution rate of grain for hay in Table 12.9 is 1.93; it is 1.97 for equation 12.22 and 1.93 for equation 12.23 in Table 12.10. However, with grain at 5,000 pounds the substitution rate for equation 12.18 in Table 12.9 is only 1.09, while it is 1.30 for equation 12.22 and 1.57 for equation 12.23 in Table 12.10.

In Table 12.10, the rates of substitution at the extremes of the isoquants suggest greater difference between the estimates of equations 12.22 and 12.23 than visual comparison of figures 12.3 and 12.4 would indicate. However, between 2,000 and 4,000 pounds of grain, the two milk production functions give almost identical slopes as indicated by the similarity of substitution ratios. The predicted amount of hay required with a specified amount of grain also is quite similar for equations 12.22 and 12.23 for grain inputs of 2,000 and 4,000 pounds (see Table 12.10). Similarly, the hay quantities in Table 12.9 compare favorably with those of Table 12.10 for grain inputs over the range of 2,000 to 4,000 pounds.

The substitution ratios are the same for a given grain input regardless of the level of milk production in equations 12.22 and 12.23. This condition holds true because equations 12.22 and 12.23 have only a linear term for hay. Therefore, the equations derived from them, which define the marginal ratios of substitution of grain for hay, contain no hay term. A given grain input, regardless of the hay input per cow, will

have the same substitution rate as higher milk levels are attained. Because of this, the isoclines (lines a, b, and c) are vertical and linear, as illustrated in Figure 12.3, for estimates based on equation 12.22 with a squared term for G. (The isoclines for Figure 12.4 are also vertical and linear.)

$$\frac{\delta H}{\delta G} = 3.5270 - .000446\ G \tag{12.24}$$

$$\frac{\delta H}{\delta G} = 129.3668\ G^{-.5} - .2584 \tag{12.25}$$

Since the isoclines in Figure 12.3 are vertical, the optimum ration would be found by determining the quantity of grain which results in a marginal productivity of grain, $\delta M/\delta G$, equal to the price ratio P_g/P_m. Once the quantity of grain has been determined (i.e., points on the grain axis such as those indicated at the bottom of lines a, b, and c), the vertical isocline shows the combinations of grain and hay to be used. Hence, if the grain to milk price ratio were 1.25, line "a" should be followed for any prices of grain and milk giving that ratio. In other words, using equations 12.22 and 12.23 as predictors of the milk production surface would lead to this ration recommendation: Feed the cow an amount of grain consistent with the prices of grain and milk and allow her to eat hay to her stomach capacity.

Lines b and c, Figure 12.3, show estimated optimum amounts of grain for the 26-week period when the grain to milk price ratio is 1.0 and .75. In each case, the cow should be given the specified quantity of grain in the period and allowed to eat hay to stomach capacity. This procedure parallels the common feeding practice or nutrition recommendation where the cow is given grain in relation to her milk production and is allowed to consume forage on a free-choice basis. Hence, it can be said that this conventional practice is consistent with greatest profit only if the milk production surface is characterized by equations such as 12.22 and 12.23; namely, the functions must have a linear term only for hay and must give vertical isoclines.

The ration solutions indicated by the isoclines corresponding to the milk and grain prices shown in Figure 12.3 have been determined by equating the partial derivatives of milk in respect to grain with the appropriate price ratios. Since the derivative for hay is a constant equal to .4089, the cow should be fed hay to the stomach limit after she is fed grain in line with price ratios, as long as the hay to milk price ratio is less than .4089. Grain feeding would, however, be adjusted to changes in price ratios (as indicated by lines a, b, and c in Figure 12.3) if profits are to be maximized. With a hay to milk price ratio greater than .4089, the cow would be fed hay only at the physiological minimum.

The isoquants for equation 12.18 (Figure 12.2) do not have vertical isoclines. However, their slopes are so great that the ration recommendations mentioned above again apply; namely, feed grain in line with the grain to milk price ratio and allow free choice of forage.

Linear Function

A final function fitted to the 26-week data for milk production by the 36 cows of the basic experiment was one with only linear terms for hay, grain, and ability (equation 12.26). As Table 12.8 indicates, the coefficient for each of the three variables is significant at the 1 per cent probability level, a condition which also held true for the power equation (12.3) and the modified quadratic equation (12.18). In equation 12.26, 75.2 per cent of the variance in milk production was explained by the linear terms. This proportion is slightly less than the 76.6 per cent for modified quadratic equation 12.18 and slightly more than for power equation 12.3, although the differences are not significant in a probability sense.

$$M = .4154\,H + .9560\,G + .8570\,A + 50.35 \tag{12.26}$$

$$\frac{\delta M}{\delta H} = .4154 \tag{12.27}$$

$$\frac{\delta M}{\delta G} = .9560 \tag{12.28}$$

$$H = 2.4071\,M - 2.3017\,G - 2.0628\,A - 121.19 \tag{12.29}$$

$$\frac{\delta H}{\delta G} = 2.3017 \tag{12.30}$$

If equation 12.26 were accepted as the estimate of the production surface, the marginal rate of substitution would be predicted by equation 12.30 as 2.30 regardless of the proportions of hay and grain in the ration. Similarly, 1 pound of hay would be predicted by equation 12.27 to yield .415 pound of milk, and 1 pound of grain would be predicted by equation 12.28, to produce .956 pound of milk, regardless of the level of feeding. Constant substitution and transformation rates such as these are assumed in conventional TDN or ENE evaluations of feeds. The substitution ratio predicted from equation 12.26 is considerably higher than that assumed by ENE and TDN transformations. One reason why this greater value may be predicted is that it is based on the entire area of the milk isoquant map included in the study. Hence, the 2.301 figure for the entire surface may be consistent with a smaller substitution value at the convergence of isoclines (the usual point of evaluation), particularly if the production surface is actually nonlinear as illustrated later. Finally, part of the difference in substitution ratios (between the linear estimates of this study as compared to the linear TDN and ENE estimates) may grow out of experimental error in the current data.

The following nutrition recommendations would be followed for the 26-week period if linear equation 12.26 were used as the predicter of the milk production surface: hay alone would be fed to stomach capacity if the grain to hay price ratio were greater than 2.301 and the hay

to milk price ratio were less than .415. This is true since the derivative of milk with respect to hay by equation 12.27 is .415 while the derivative of hay with respect to grain is 2.3017 by equation 12.28. If the grain to hay price ratio were less than 2.301 and the grain to milk price ratio were less than .956, grain alone would be fed to stomach capacity. With a grain to hay price ratio greater than 2.301 and a hay to milk price ratio greater than .415, milk production would not be profitable from the standpoint of feed costs alone, other costs disregarded. Hence, a linear estimate of the production function would always call for extremes in rations: cows should be fed only grain or hay (and never a combination of grain and hay, except in the unique case where the feed substitution ratio equals the feed price ratio) to stomach capacity, if the feed to milk price ratio is less than the milk to feed derivative; they should not be kept in production if the feed to milk price ratio is greater than the milk to feed derivative.

Since the substitution ratios in Table 12.11 are constant at 2.301 (i.e., the value of the derivative in equation 12.30) all isoclines have this same value and can be vertical, horizontal, or positively sloped. There is, in fact, no single line representing an isocline value; rather every point in the feed plane of Figure 12.5 has the hay to grain substitution value of 2.301.

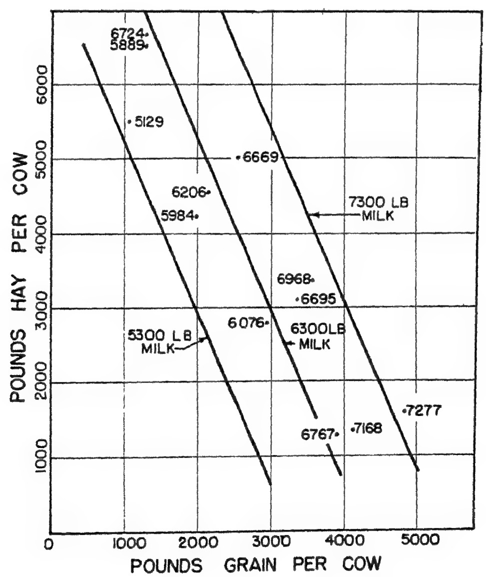

Figure 12.5. Milk isoquants for a 26-week period predicted from linear equation 12.26 with ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

Table 12.11. Feed Combinations in Producing Specified Milk Levels and Marginal Rates of Substitution of Grain for Hay Predicted From Linear Equation 12.26; Ability Fixed at Mean of 36 Cows for 26-Week Basic Experiment

Feed to Produce 5,300 Pounds Milk*			Feed to Produce 6,300 Pounds Milk*			Feed to Produce 7,300 Pounds Milk*		
Lbs. grain	Lbs. hay	Substitution rate $\delta H/\delta G$	Lbs. grain	Lbs. hay	Substitution rate $\delta H/\delta G$	Lbs. grain	Lbs. hay	Substitution rate $\delta H/\delta G$
1,000	5,194	2.3017	1,000	7,602†	2.3017	1,000	10,010†	2.3017
1,500	4,043	2.3017	1,500	6,451	2.3017	1,500	8,859†	2.3017
2,000	2,892	2.3017	2,000	5,300	2.3017	2,000	7,708†	2.3017
2,500	1,741	2.3017	2,500	4,149	2.3017	2,500	6,557	2.3017
3,000	590†	2.3017	3,000	2,998	2.3017	3,000	5,406	2.3017
--	--	--	3,500	1,847	2.3017	3,500	4,255	2.3017
--	--	--	4,000	697†	2.3017	4,000	3,105	2.3017
--	--	--	--	--	--	4,500	1,954	2.3017
--	--	--	--	--	--	5,000†	803	2.3017

*Isoquant predicted from equation 12.29, and substitution rates predicted from equation 12.30.
†Outside range of observation.

TDN and ENE Transformations in Relation to Linear Functions

Procedures which use TDN or ENE transformations in evaluating feeds assume a linear production function with straight-line milk isoquants and, hence, constant (a) hay to grain substitution coefficients and (b) milk to feed grain transformation coefficients. This is true since 1 pound of feed is given the same TDN or ENE value regardless of the ratio of feeds used or the level of feeding. If diminishing rates of substitution were assumed, the TDN or ENE transformation coefficients would need to be changed as the ration changes in proportions of feeds and level of feeding. Using the linear relationships from equation 12.26, a pound of hay is predicted to produce .415 pound of milk and a pound of grain to produce .956 pound of milk, regardless of the ration fed or the level of feeding. A pound of grain is predicted to have a feeding value 2.301 greater than a pound of hay, regardless of the ration fed. If, however, equation 12.18 is used for predictions, a pound of hay is predicted to produce .409 pound of milk, and a pound of grain is predicted to produce .953 pound of milk only at the mean level of feeding used in the experiment. At this feed combination and feeding level, a pound of grain has .9534 to .4089 or 2.3316 times as much feed value, in relation to milk production, as does hay. But in contrast to the linear function, this feed value relationship holds true only for the particular feed combination and level. Turning back to Table 12.9, we note that for a 6,300-pound milk isoquant, grain has a value as great as 3.17 times that of hay when the feed combination includes 1,500 pounds of grain and 6,566 pounds of hay over a 26-week period; it has a value of only 1.84 times the value of hay when the ration includes 3,500 pounds of grain and

1,616 pounds of hay. With feeding at a higher level of milk production, as denoted by the 7,300-pound isoquant, a pound of grain has a feeding value 1.94 times that of hay when the ration includes 3,500 pounds of grain and 4,080 pounds of hay in the 26 weeks.

The ratios of the marginal productivities or the substitution ratios provide a basis for relative evaluation of feeds. If it can be proved that the milk production function is linear, constants such as those assumed in the traditional TDN and ENE evaluation of feeds are appropriate. However, if the function is proved to be nonlinear, constant transformations are not the appropriate basis for ration evaluation. Under curved isoquants, grain must be given a continuously lower value relative to hay for rations which move along a milk isoquant nearer to the grain axis; hay must be given a lower value relative to grain for rations representing movements toward the hay axis. Actually each isocline traces the only path of rations over which feeds can appropriately be given a constant feeding value relative to each other; the constant value which is appropriate will differ for each isocline, except for a milk production function which is linear with straight-line isoquants.

Most nutritionists accept the notion that grain and hay do not substitute at constant rates over the entire milk surface. However, wide use of TDN and ENE transformations has been continued because of lack of other data to indicate the rate at which substitution ratios may change. Also, some believe that the milk isoquant may be "near linear" in the middle, with curvature especially near the ridge isocline. Because of these indications from previous research, grain alone is never recommended and hay alone is seldom recommended.

Equation 12.26 with linear coefficients can be used to test the feeds and results of this study against TDN and ENE transformation ratios which implicitly assume linearity. The constant substitution rate of 2.301 (i.e., the relative feed value of grain and hay) predicted from linear equation 12.26 compares with the constants of 1.7884 predicted from the ENE evaluation and 1.350 for the TDN evaluation, using Morrison's standards, of feeds in the experimental ration. The ENE ratio of 1.7884 is calculated as .7404 therms per pound of grain divided by .4140 therms per pound of hay. The TDN ratio of 1.3550 is calculated as .6863 pound of TDN per pound of grain divided by .5065 pound of TDN per pound of hay.

The question posed is this: "Does the ratio predicted from equation 12.26, indicating the relative feeding value of grain and hay in producing a given amount of milk, correspond to the conventional ratios used for indicating relative values of grain and hay in producing energy or heat?" Using the linear equation for the 36 cows, we examine whether the ratio of marginal productivities of grain and hay (i.e., the substitution ratio of the feeds in producing a given amount of milk) is similar to the ENE and TDN ratio, when sampling or experimental error is considered for the regression coefficients.

Table 12.12 includes the regression coefficients for grain and hay from the linear equation and their 95 per cent fiducial limits. The

Table 12.12. Regression Coefficients From Equation 12.26, 95 Per Cent Confidence Limits and Magnitudes to Give ENE and TDN Ratios

Item	Hay	Grain
1. Upper confidence limit	.64	1.29
2. Regression coefficient	.42	.96
3. Lower confidence limit	.20	.62
4. Magnitude of regression coefficient to give 1.7884 ENE ratio	.53*	.74†
5. Magnitude of regression coefficient to give 1.3530 TDN ratio	.71*	.56†
6. Value of t in testing regression coefficient (2) against coefficient needed (4) to give ENE ratio	.58	1.01‡
7. Value of t in testing regression coefficient (2) against coefficient needed (5) to give TDN ratio	1.40§	1.86‖

* If regression coefficient for grain of .956 is accepted as a parameter.
† If regression coefficient for hay of .4154 is accepted as a parameter.
‡ $p < .30$
§ $p < .20$
‖ $p < .10$

computed values of regression coefficients which are necessary to give ENE and TDN ratios of feeds in the study, based on Morrison, have then been entered on lines 4 and 5, respectively. These computed values fall within the 95 per cent confidence limits for the ENE evaluation but not for the TDN evaluation. Similarly, the t values testing the differences between the regression coefficients and coefficient values necessary to give the TDN ratio are at a higher probability level than those for the ENE evaluation. Hence, for the feeds, cows, and data of this study, it appears that the TDN ratio expressing the value of grain and hay in producing a given amount of energy or number of therms is not appropriate for comparison of these feeds in producing a given amount of milk.[12]

Additional research is needed to study further the appropriateness of the ENE energy evaluation of feeds. A larger study with lower standard errors also might prove differences to be significant for ENE evaluations. While the comparisons above rest on the linear regression equation, it is unlikely that the milk production function is of this empirical nature. The above analysis was made to examine differences or similarities of conclusions if the purely linear relationships (a condition assumed by ENE and TDN evaluations) were accepted. The functions derived with variable substitution rates in other sections of this chapter are likely more appropriate for a fundamental analysis of feed values. Inclusion of time as a variable in later sections of this chapter undoubtedly improves the estimates of feed substitution rates.

[12] The test was made with the regression coefficients, but, since in the linear equation these form the substitution coefficient for producing a given amount of milk, the statement holds true for a linear production function.

PRODUCTION FUNCTIONS INCLUDING AUXILIARY EXPERIMENT FOR 51 COWS IN AN OVER-ALL PERIOD

Production functions paralleling those for the 36-cow, 26-week basic experiment were derived when data from this experiment were pooled with the 15-cow auxiliary experiment explained previously. The auxiliary experiment included only 16 weeks and the data for the 36-cow basic experiment were transformed similarly. Milk output and feed input include data for the first 16 weeks of the experimental period only for the 36-cow basic experiment. Except that measurements refer to 16 weeks and 51 cows, the variables in the function explained below are the same as those outlined earlier for the basic experiment. Time is not included as a variable.

Quadratic-Type Functions

Since functions with squared terms and square root terms gave results which did not differ significantly for the 36-cow experiment, only the former type of equation has been used for the 51-cow data over the 16-week period.[13] The equations are listed below in the following sequence: 12.31, 12.34, and 12.37 refer to the basic production function of a particular algebraic form; 12.32, 12.35, and 12.38 are the isoquant equations derived from the production function with ability set at the mean; 12.33, 12.36, and 12.39 are the marginal rate of substitution equations derived from the isoquant equation.[14]

$$M = .3449\,H + 2.2711\,G + .8745\,A - .000264\,G^2 + .000072\,H^2 - 2{,}264.61 \tag{12.31}$$

$$H = -2398.0 \pm (6954.0)\sqrt{.1610 - .000653\,G + .000000076\,G^2 + .000288\,M} \tag{12.32}$$

$$\frac{\delta H}{\delta G} = \frac{2.2711 - .000527\,G}{.3449 + .000144\,H} \tag{12.33}$$

$$M = .8076\,(H - .00005\,H^2) + 1.9895\,(G - .0001\,G^2) + .9220\,A - 2{,}472.35 \tag{12.34}$$

$$H = 10{,}000 \pm (-12{,}383.0)\sqrt{.6130 + .000321\,G - .000000032\,G^2 - .000162\,M} \tag{12.35}$$

$$\frac{\delta H}{\delta G} = \frac{1.9895 - .000398\,G}{.8076 - .000081\,H} \tag{12.36}$$

$$M = .6650\,H + 2.0436\,G + .8860\,A - .000199\,G^2 - 2{,}404.71 \tag{12.37}$$

---

[13] One exception was prediction of a production function with a square root term for G. The function, $M = .6601\,H + .0589\,G + .8763\,A + 101{,}9959\sqrt{G} - 3{,}776.83$, has an R value of .9308 and t's for regression coefficients of 6.35, .17, 7.08, and 3.92, respectively. The relation of this function to equation 12.37 in the text is similar to the relation of equation 12.23 to equation 12.22 for 36 cows.

[14] In each isoquant equation, ability has been set at the mean preliminary period milk production for 51 cows. The 2-week preliminary period data for the 15 auxiliary cows were extrapolated to get an estimate comparable to the basic experiment.

(12.38) $H = -3.0731\ G + .000299\ G^2 + 1.5038\ M + 393.6412$

(12.39) $\frac{\delta H}{\delta G} = 3.0731 - .000599\ G$

Related statistics for the production functions are given in Table 12.13. All functions account for a greater proportion of the variance in milk production than the parallel functions for 36 cows presented earlier. For example, equation 12.34 accounts for 86.7 per cent of the variance in milk production while equation 12.22 accounts for only 75.7 per cent. The t values for regression coefficients are all significant at a probability level of .05 or lower, except those for H and H^2 in equation 12.31. However, this latter equation again appears unrealistic because the coefficient for H^2 is positive, indicating an increasing return to hay as more of this feed is consumed. In equation 12.34, the coefficients for H^2 and G^2 were estimated prior to prediction of the regression equation. (Coefficients were predicted only for the terms $H - .00005\ H^2$, $G - .0001\ G^2$, and A.) Accepting this form for the production function would allow diminishing returns to both feeds and diminishing substitution rates between them. The milk isoquants for equation 12.34 do not have a sharp curvature. The same statement holds true for equation 12.37 which has a power term only for grain (see isoquants in Figure 12.6 for equation 12.37). Again, the isoclines for equation 12.37 are vertical with the implications for hay feeding mentioned in respect to equation 12.22 and Figure 12.3. Too, the substitution ratios are always the same for a given amount of grain, regardless of the amount of hay or the level of milk production, because the equation of substitution rates (12.39) has only a linear grain term and does not include a term for hay. Table 12.14 allows comparisons of rations for three levels of milk production predicted by equations 12.34 and 12.37. For both functions, the relative feeding value of hay and grain would change with the proportions fed for a given milk isoquant. Since constants for G^2 and H^2 have been "forced" into equation 12.34, it is likely that the substitution

Table 12.13. Values of R and t for 51-Cow Data Pooled for Basic and Auxiliary Experiments for 16-Week Period (t Values Refer to Regression Coefficients in Order Presented in Equations 12.31, 12.34, and 12.37)

Equation	R	Value of t for b_1	b_2	b_3	b_4	b_5
12.31	.9338*	1.25‡	5.92*	7.10*	2.67*	1.25‡
12.34	.9275*	6.35*	8.78*	7.66*	--	--
12.37	.9314*	6.43*	6.01*	7.17*	2.28†	--
12.40	.9233*	6.23*	8.40*	6.64*	--	--

*$p < .01$
†$p < .05$
‡$p < .30$

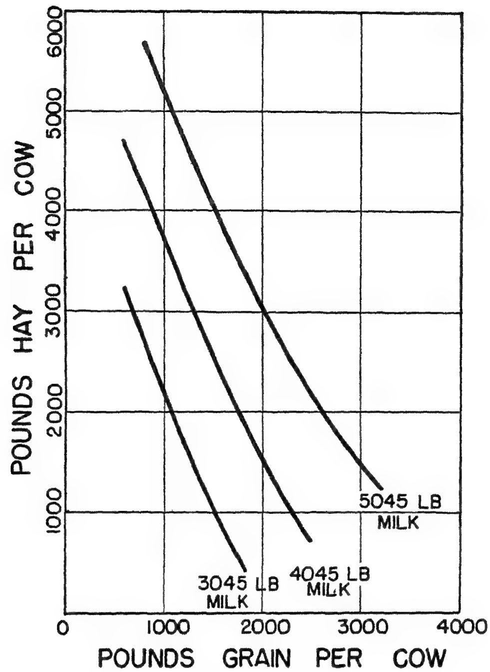

Figure 12.6. Milk isoquants for a 16-week period predicted from equation 12.37 with ability fixed at mean for 51 cows

ratios for equation 12.37 in Table 12.14 are most appropriate for the 51 cows.

Linear Function

Equation 12.40 is a linear function fitted to the 16-week period for 51 cows. For this equation, an additional pound of hay would always produce .6717 pound of milk, regardless of the grain to hay ratio or the level of feeding; an additional pound of grain would always produce 1.353 pounds of milk. Grain would

(12.40) $$M = .6717\ H + 1.3527\ G + .8500\ A - 1{,}841.40$$

(12.41) $$H = 1.4888\ M - 2.0138\ G - 1.2655\ A + 2{,}741.40$$

(12.42) $$\frac{\delta H}{\delta G} = 2.0137$$

replace 2.014 pounds of hay in producing a given amount of milk,

regardless of the ration. Hay fed to the stomach limit line (or all hay) would give the lowest cost ration for any grain to hay price ratio of greater than 2.014; grain fed to the physiological limit (or all grain) would give the lowest cost ration for a grain to hay price ratio less than 2.014. The ratio, 2.014, expressing the relative feeding value of grain and hay is less than the 2.302 ratio computed from equation 12.26 for the 36-cow basic experiment.

A comparison of (a) the substitution ratio of grain and hay in producing a given amount of milk for the 51-cow linear function with (b) the ENE and TDN ratios for producing a given amount of energy, had the same results as the same test for the 36-cow linear function; the substitution ratio could be said to differ significantly from the TDN ratio but not the ENE ratio, supposing that a linear production function could be accepted in predicting the milk production surface.

PRODUCTION FUNCTIONS WITH TIME AS VARIABLE FOR 36-COW BASIC EXPERIMENT

Since time, or point of time in the lactation period, is an important variable affecting milk production, regression equations have been computed with time included and are based on the variables following. Too, as mentioned earlier, changes in cow weights approached an equilibrium after the outset of the experiment and allowed better prediction after this change had been taken into account.

M is production of 4 per cent, fat-corrected milk in each 4-week period, measured in pounds.

G is quantity of grain mix (see earlier explanation), measured in pounds, consumed in each 4-week period.

H is quantity of hay, measured in pounds, consumed in each 4-week period.

A is ability measured as pounds of milk produced in the preliminary period.

T is time in 4-week periods. Six such periods were included and were measured as the cardinal numbers, 1 through 6. The last 2 weeks of the 26-week experimental period were not included.

Four production functions have been derived with variables defined as above. These are:

$$M = 4.1937\ H^{.1506}\ G^{.3082}\ A^{.3716}\ T^{-.1973} \tag{12.43}$$

$$M = 1.6302\ H + 3.1309\ G + .1497\ A + 14.2243\ T - .000388\ H^2 - .001192\ G^2 + 4.3792\ T^2 - .001056\ HG - .1570\ GT - .0865\ HT - 731.76 \tag{12.44}$$

$$M = .5513\ H + 1.3285\ G + .1488\ A - 110.2640\ T - .000081\ H^2 - .000360\ G^2 + 6.0320\ T^2 + 151.36 \tag{12.45}$$

Table 12.14. Feed Combinations in Producing Three Milk Levels Predicted From Equations 12.34 and 12.37 (for 16-Week Period With Ability Fixed at Mean of 51 Cows)

Feed Combinations for 3,045 Pounds Milk				Feed Combinations for 4,045 Pounds Milk				Feed Combinations for 5,045 Pounds Milk				Marginal Rate of Substitution of Grain for Hay for 4,045 Pounds Milk	
Equation 12.34		Equation 12.37		Equation 12.34		Equation 12.37		Equation 12.34		Equation 12.37			
Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Lbs. grain	Lbs. hay	Equation 12.34	Equation 12.37
600	3,190	600	3,236	600	5,351*	600	4,740*	--	--	--	--	4.66	2.71
1,000	2,067	1,000	2,199	1,000	3,822	1,000	3,702	1,000	6,339*	1,000	5,206*	3.19	2.47
1,400	1,173	1,400	1,256	1,400	2,709	1,400	2,760	1,400	4,677*	1,400	4,264	2.43	2.23
1,800	443*	1,800	410*	1,800	1,841	1,800	1,914	1,800	3,535	1,800	3,418	1.93	2.00
--	--	--	--	2,200	1,146	2,200	1,165	2,200	2,677	2,200	2,669	1.56	1.76
--	--	--	--	2,600	584	2,600	510*	2,600	2,007	2,600	2,014	1.26	1.52
--	--	--	--	--	--	--	--	3,000	1,481	3,000	1,455	--	--
--	--	--	--	--	--	--	--	3,200	1,264	3,200	1,212	--	--

*Estimates outside range of observation.

(12.46) $$M = .6601\ H + 1.4276\ G + .1553\ A - .000054\ H^2 T - .000152\ G^2 T - 2.0752\ T^2 - 157.24$$

While all of the regression coefficients of equation 12.43 are acceptable at a probability level of less than 1 per cent, this equation explains a somewhat smaller proportion of the variation in milk production than do the other three equations. The percentage of variation in milk production explained by the regression equations is 74.9, 81.3, 80.2, and 80.3, respectively, for the four functions. Equation 12.44 appears to have the greatest logical basis since (a) it allows definition of a single ration with a single milk level maximum per cow, and (b) it allows a more complete specification of interaction between variables than do equations 12.45 and 12.46. Also, as indicated later, it results in an isocline map which seems more consistent with nutrition logic. In equation 12.44, the crossproduct terms reduce the sum of squares of deviation from regression by an amount acceptable in probability terms, as compared with equation 12.45. Equation 12.44 gives estimates quite similar to equation 12.46, which has t values acceptable at lower levels of probability. Because of these several reasons, the writers believe that equation 12.44 is as appropriate, or more appropriate, than any of the equations in estimating the milk production surface, feed substitution rates, and profit maximizing rations. It may, however, tend to overestimate substitution ratios near the end of the lactation period. It is likely that, in a larger experiment with a smaller proportion of the degrees of freedom used in estimating regression coefficients, values would fall at probability levels as low as those for equations 12.43 and 12.46.

In evaluating the t values for the 216 observations (36 cows for 6 months) in Table 12.15, it must be remembered that the six observations for each cow are not independent. If, however, the number of degrees of freedom is considered to be as low as 25 (36 minus 11) for equation 12.44, rather than 205 (216 minus 11), the probability statements indicated in the footnotes of Table 12.15 still hold true. To obtain complete independence with 205 degrees of freedom for equation 12.44,

Table 12.15. Correlation Coefficients and t Values for Equations 12.43 Through 12.46 (36 Cows With Time as Variable)

Equation	R	Values of t in Order of b's in Equation									
		b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}
12.43	.8654	6.69*	12.04*	9.27*	15.61*	--	--	--	--	--	--
12.44	.9016	1.94‡	2.60*	9.86*	.34	1.14	1.67‖	1.62‖	1.10	3.49*	2.82*
12.45	.8953	3.77*	6.58*	9.60*	5.74*	.75	1.56‖	2.24†	--	--	--
12.46	.8960	8.70*	13.72*	10.22*	4.54*	5.92*	1.65‖	--	--	--	--

*$p < .01$
†$p < .05$
‡$p < .10$
‖$p < .20$

Table 12.16. Regression Coefficients for Quadratic Equations Estimated for Each Month of Experimental Period

Month (Equation)		Constant	Regression Coefficient for					
			H	G	A	H^2	G^2	HG
1.	12.47	- 506.0	.7495	1.7177	.3075	-.000065	-.000545	-.000392
2.	12.48	- 764.0	1.1904	2.6425	.2014	-.000228	-.001096	-.000682
3.	12.49	1,851.0	-2.5569	-3.2638	.1257	.001103	.002296	.003868
4.	12.50	-1,140.0	2.9770	4.5732	.1085	-.001188	-.002368	-.003026
5.	12.51	- 993.0	1.9343	4.1901	.1038	-.000577	-.002542	-.002402
6.	12.52	-2,313.0	4.4090	7.8840	.0591	-.001644	-.005090	-.005749

for example, it would be necessary to have 216 cows, with each one used for a single observation.

Functions by Months

In addition to estimating production functions with time (months) as a variable, functions of the form of equation 12.44 were estimated for each month separately. In these six separate equations, variables included feed input and milk output measured over the particular month. (Time was not included as a variable, and there were only 36 observations for each month.) The regression coefficients for these monthly observations are given in Table 12.16. Here equation 12.47 is for the first month in the experimental period, equation 12.48 is for the second month, etc. Table 12.17 includes the R and t values for the six monthly functions. Because of the large proportion of degrees of freedom used in the regression estimates and the relative within month variability of milk production, the t values correspond to quite high probability levels.

Table 12.17. Correlation Coefficients and t Values for Equations 12.47 Through 12.52 (Presented in Table 12.16)

Equation	R	Value of t in Order of b's in Equation					
		b_1	b_2	b_3	b_4	b_5	b_6
12.47	.9614	.72	1.28	11.37	.14	.68	.34
12.48	.9464	.66	1.10	7.64	.30	.78	.33
12.49	.8396	.89	.76	3.49	.94	.88	1.14
12.50	.7935	1.30	1.30	3.04	1.26	1.06	1.07
12.51	.7379	.76	1.06	2.57	.56	.98	.72
12.52	.6720	1.64	1.92	1.32	1.40	1.80	1.60

Predictions From Functions With Time (Month of Lactation) as a Variable

In this section results are shown using equations 12.44 and 12.46, with time as a variable, in predicting feed combinations possible in attaining specified milk levels (isoquants), and in estimating the rate of substitution between hay and grain in the various months. Equation 12.53 is the milk isoquant, based on production function equation 12.44, where T has been set at 1 and A has been set at 2,492 (the mean of the 36 cows in the experimental period). While equation 12.53 provides estimates for the first month (T = 1), similar estimates can be provided other months by assigning a particular value to T. Equation 12.54 provides estimates of substitution rates for equation 12.53. Equations 12.55 and 12.56 are, respectively, the isoquant and substitution functions for function 12.46, with the values for T and A mentioned above.

$$(12.53) \qquad H = \frac{1{,}989.36 - 1.3608\ G \pm (-1{,}288.66)\sqrt{1.8553 + .001355\ G - .00000073\ G^2 - .001552\ M}}{}$$

$$(12.54) \qquad \frac{\delta H}{\delta G} = \frac{2.9740 - .002384\ G - .001056\ H}{1.5437 - .000776\ H - .001056\ G}$$

$$(12.55) \qquad H = 6{,}106.28 \pm (-9{,}250.69)\sqrt{.4850 + .000309\ G - .000000033\ G^2 - .000216\ M}$$

$$(12.56) \qquad \frac{\delta H}{\delta G} = \frac{1.4276 - .000304\ G}{.6601 - .000108\ H}$$

Functions with time as a variable also allow estimates of the rate of change in milk production as the lactation period progresses. Equations 12.57 and 12.58, based respectively on equations 12.44 and 12.46, are marginal time-yield equations. They indicate the decline in milk production associated with each unit progress of time beyond the beginning of the experimental period and are used for estimates in a following section.

$$(12.57) \qquad \frac{\delta M}{\delta T} = 14.22 + 8.7584\ T - .1570\ G - .0865\ H$$

$$(12.58) \qquad \frac{\delta M}{\delta T} = -.000054\ H^2 - .000152\ G^2 - 4.1504\ T$$

The milk production surfaces derived from equations 12.44 and 12.46 are presented in figures 12.7 and 12.8, respectively, for the "mean" month of the experiment (i.e., T = 3.5).[15] The surfaces are quite similar with respect to the milk contours. Feed quantities for

[15] For the "mean" month, T has been set at 3.5 to include the last half of the third month and the first half of the fourth month.

Table 12.18. Feed Combinations in Producing Specified Milk Levels in 28 Days and Marginal Rates of Substitution of Grain for Hay Based on Equation 12.44; Ability Fixed at Mean of Cows in Experiment — Mean Month of Experimental Period (T = 3.5)

Level of Grain (Pounds)	Pounds Hay Required to Maintain Milk Output of					Marginal Rates of Substitution: Pounds Hay Replaced by 1 Additional Pound of Grain Along Indicated Milk Isoquants				
	800 lbs.	900 lbs.	1,000 lbs.	1,110 lbs.	1,190 lbs.	800 lbs.	900 lbs.	1,000 lbs.	1,110 lbs.	1,190 lbs.
150	815	1,036	--	--	--	2.54	3.09	--	--	--
200	692	891	--	--	--	2.37	2.74	--	--	--
250	577	760	1,032	--	--	2.24	2.50	3.41	--	--
300	468	640	877	--	--	2.12	2.32	2.85	--	--
350	365	528	744	--	--	2.02	2.17	2.53	--	--
400	266	423	623	973	--	1.93	2.05	2.31	4.00	--
450	172	323	513	806	--	1.85	1.94	2.13	2.90	--
500	--	229	410	674	--	--	1.85	1.99	2.45	--
550	--	139	314	559	--	--	1.76	1.87	2.17	--
600	--	--	223	455	--	--	--	1.76	1.97	--
650	--	--	138	361	--	--	--	1.66	1.80	--
700	--	--	--	275	716	--	--	--	1.66	4.79
750	--	--	--	194	575	--	--	--	1.53	2.09
800	--	--	--	122	486	--	--	--	1.41	1.53

Table 12.19. Feed Combinations in Producing Specified Milk Levels in 28 Days and Marginal Rates of Substitution of Grain for Hay, Based on Equation 12.46; Ability Fixed at Mean of Cows in Experiment — Mean Month of Experimental Period (T = 3.5)

Level of Grain (Pounds)	Pounds Hay Required to Maintain Milk Output of					Marginal Rates of Substitution: Pounds Hay Replaced by 1 Additional Pound of Grain Along Indicated Milk Isoquants				
	800 lbs.	900 lbs.	1,000 lbs.	1,110 lbs.	1,200 lbs.	800 lbs.	900 lbs.	1,000 lbs.	1,110 lbs.	1,200 lbs.
150	763	1,084	--	--	--	3.41	5.07	--	--	--
200	608	870	--	--	--	2.82	3.67	--	--	--
250	477	706	1,003	--	--	2.42	2.96	4.14	--	--
300	363	570	822	--	--	2.12	2.49	3.18	--	--
350	263	454	678	965	--	1.88	2.16	2.62	3.58	--
400	174	352	557	806	--	1.69	1.90	2.23	2.82	--
450	--	262	453	677	963	--	1.69	1.94	2.35	3.21
500	--	182	362	568	820	--	1.52	1.71	2.01	2.56
550	--	--	281	474	702	--	--	1.52	1.75	2.14
600	--	--	208	391	603	--	--	1.36	1.54	1.83
650	--	--	144	318	522	--	--	1.13	1.36	1.59
700	--	--	--	254	443	--	--	--	1.21	1.48
750	--	--	--	196	377	--	--	--	1.07	1.21
800	--	--	--	145	320	--	--	--	.95	1.07

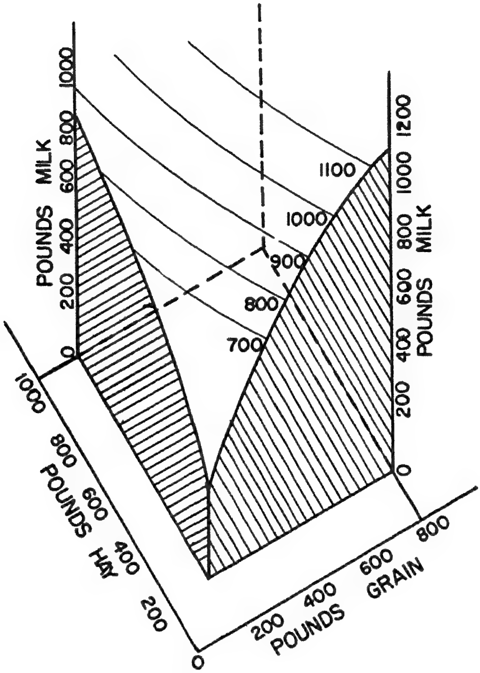

Figure 12.7. Milk production surface and milk isoquants estimated from equation 12.44 for mean month of experiment, ability at mean for 36 cows

producing stated milk production levels and marginal rates of substitution derived from equation 12.44 are presented in Table 12.18 for the "mean" month of the experimental period. Parallel quantities are provided in Table 12.19 for the mean month predictions based on equation 12.46. As data of the tables indicate, increasing inputs of hay are predicted for higher milk levels, for a given grain level, because of diminishing productivity of feed. Also, the marginal rate of substitution of grain for hay increases (a) as the ration includes a greater proportion of hay for given milk level and (b) as higher levels of milk are attained with greater hay inputs, grain remaining constant. While not directly apparent from parallel tables, the marginal rates of substitution between grain and hay for the two equations, for a given combination of the two feeds and time and ability fixed at the same level, are similar in early parts of the period. However, the rate of substitution tends to widen between the two functions as time increases. Substitution ratios for equation 12.44 differ more from customary evaluation standards than do those for equation 12.46.

Figures 12.9, 12.10, and 12.11 include milk isoquants for the first, the "mean", and the sixth month of the experimental period, based on equations 12.44 and 12.46. The fact that the slopes of the isoquants

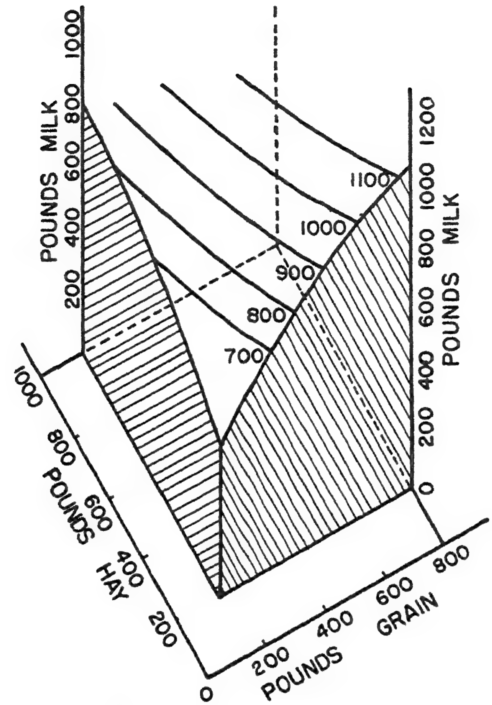

Figure 12.8. Milk production surface and milk isoquants from equation 12.46 for mean month of experiment, ability at mean for 36 cows

are similar emphasizes the point mentioned above — namely, that estimates of the feed substitution rates based on equations 12.44 and 12.46 do not differ significantly. However, one condition is apparent from the isoquant figures. The milk contours in figures 12.9, 12.10, and 12.11 take on greater curvature for higher milk levels for both equations 12.44 and 12.46 indicating that as feeding levels become greater (1) relatively small variations in feed combinations tend to cause larger variations in the substitution rates or (2) smaller ranges of feed combinations will allow a specified level of milk production. Only one ration or ratio of grain and hay would allow maximum production per cow (i.e., the point of isocline convergence). Similarly as time progresses from Figure 12.9 to Figure 12.10 to Figure 12.11 the milk isoquants tend to take on greater curvature, indicating a more rapid change in substitution rates as feed proportions are varied in attaining a particular milk level. The tendency of the milk isoquants to increase in curvature with progress of the lactation period also is emphasized in Figure 12.12. In this figure, 1,000-pound milk isoquants based on the two functions are presented for each of the 6 months of the experimental period. The isoquants for the

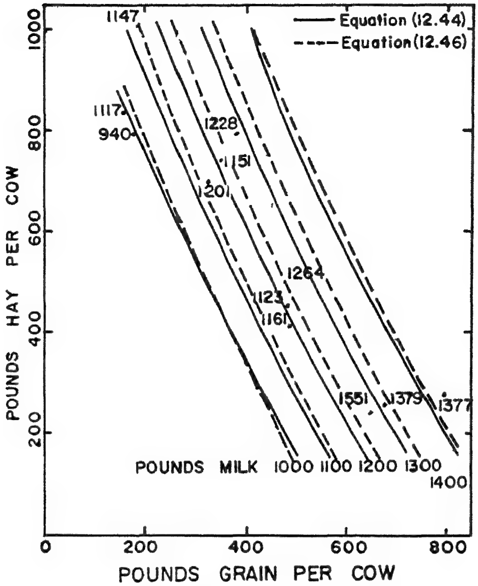

Figure 12.9. Milk isoquants for the first month of the experiment, based on equations 12.44 and 12.46, ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

two functions tend to "spread apart" with greater amounts of time because the coefficients for time, or the product of time and feeds, is relatively greater in equation 12.46 than in equation 12.44.

Isoclines Predicted From Functions With Time Variable

Use of the functions with time as a variable would indicate that standard ratios, such as ENE and TDN constants, are not appropriate for evaluating feeds in relation to milk production. The basis for this statement is the changing nature of substitution rates in tables 12.18 through 12.19. (These substitution rates have been determined relative to a given milk level.) Since the hay to grain substitution ratio declines with an increased proportion of grain and increases with a decreased proportion of grain in the ration, the coefficient for evaluating feeds should change accordingly if evaluation is relative to milk production. The marginal substitution rate, equations 12.54 and 12.56, are the basis for coefficients which can serve as the basis for feed evaluation when the entire milk production surface is considered in the evaluation.

The isoclines explained earlier also provide a basis for feed evaluation and recommendations: Feeds have a constant value relative to each other only along an isocline. Isoclines, along with milk isoquants,

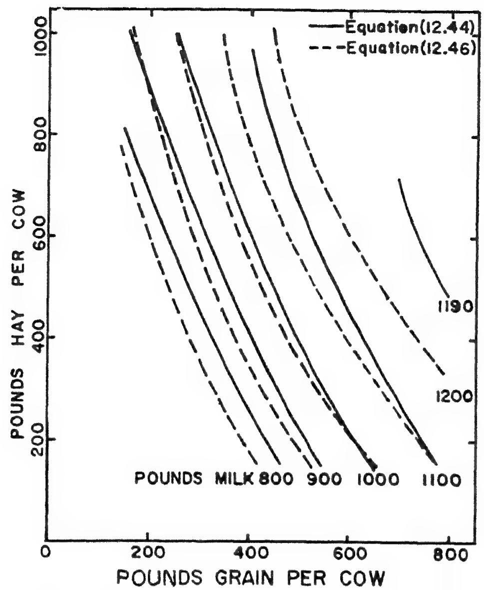

Figure 12.10. Milk isoquants for the mean month of experiment, based on equations 12.44 and 12.46, ability fixed at mean for 36 cows

are presented in Figure 12.13 for milk function 12.44 and in Figure 12.14 for function 12.46. These isoclines again trace out the path of feed combinations which result in a given substitution rate between hay and grain as milk is taken to higher levels (i.e., is denoted by isoquants higher in the feed plane). In other words, 1 pound of grain substitutes for 3.00 pounds of hay for the feed combinations traced out by isocline A. If the grain to hay price ratio were also 3, isocline A traces the least-cost combination of feeds for the various milk levels. Similarly, isocline C traces out the least-cost ration when the grain to hay price ratio is 2 if equation 12.44 is used for predicting the milk production surface. Similarly, isoclines A, B, and C in Figure 12.14 indicate the paths of least-cost rations if equation 12.46 is used for the estimates.

While the regression coefficients for equation 12.46 have lower standard errors than do the coefficients for equation 12.44, the slopes of the isoclines in Figure 12.13 appear more in line with physiological characteristics of milk production. They converge more rapidly, suggesting a maximum possible level of milk production. The point at which the isoclines converge defines a single ration consistent with maximum milk production per cow. Convergence and definition of a maximum milk level is not so apparent in Figure 12.14 for equation 12.46. However, the slopes of the isoclines in Figure 12.14 do approach those in Figure 12.3 which are closer to the historic basis for recommendations on rations. However, the historic basis is appropriate only if the

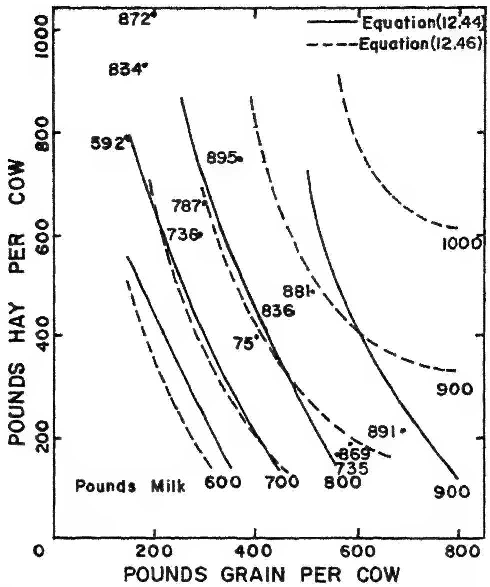

Figure 12.11. Milk isoquants for the sixth month of the experiment, based on equations 12.44 and 12.46, ability fixed at mean for 36 cows (Dots show feed inputs yielding indicated milk quantity from 12 cows at medium level of ability.)

marginal rate of substitution equation includes only one feed variable with a linear coefficient, an unlikely situation unless it can be proved that the other feed has constant marginal productivity.

Surfaces With Time and Ability Varied

The isoquant and isocline maps in figures 12.13 and 12.14 are for the mean month (T = 3.5) of the experimental period and the mean level of ability of the 36 cows in the experiment. The basic production functions can be used to predict feed combinations, substitution rates, and milk levels for other points of time in the lactation period and other ability levels. Figure 12.15 includes milk production surfaces estimated from equation 12.44 when ability is fixed at the mean of the 36 cows, for the first, the mean, and the sixth month of the experimental period. Parallel estimates for equation 12.46 are provided in Figure 12.16. Because of the time coefficients in the milk production functions, the milk level declines at a decreasing rate with time. Also, the slope of the surface declines, indicating a lower marginal productivity of either feed as the lactation period progresses.

Figure 12.17 provides production surfaces estimated from equation 12.44 for cow ability at three different levels with time fixed at the

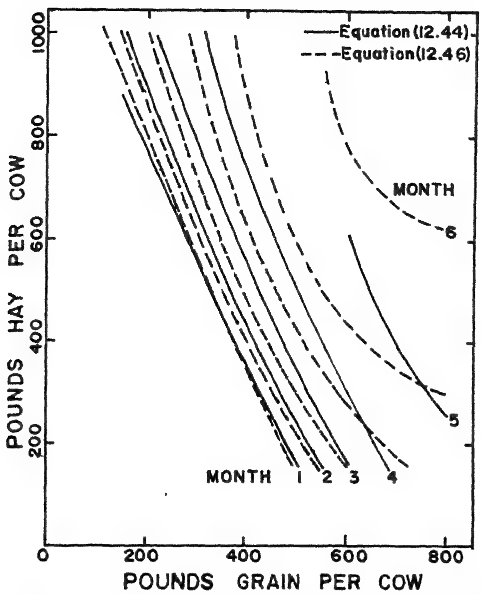

Figure 12.12. Feed combinations to produce 1,000 pounds milk in each month of the experiment, ability set at mean for 36 cows (All isoquants represent a 1,000-pound milk level. Equation 12.44 does not permit a 1,000 pound estimate in the sixth month within the grain input range shown.)

mean month. The slopes of these surfaces are identical, the only difference between them is the height of milk level at which the surface slopes begin. In other words, the entire surface is moved upward as the level of ability increases because ability is included as a linear term only in the equation used. However, the rates of hay to grain substitution differ between ability levels for a given milk level. This is true because a milk contour, such as 1,000 pounds, falls lower on the sloping portion of the surface as the level of ability increases. As is illustrated in figures 12.9 and 12.11, the curvature of the isoquants (i.e., the marginal rates of substitution) changes for milk contours spaced further over the feed plane. (While surfaces from equation 12.46 have not been provided for different ability levels, they have the same basic differences illustrated in Figure 12.17.)

SPECIFICATIONS OF ECONOMIC OPTIMA IN RATIONS

Prediction of the milk production function or surface allows specification of the ration which will maximize returns above feed costs. The basic conditions for profit maximization were indicated in Chapter 2. Using production function 12.44 for predictions, equation 12.54 can now be set to equal any grain to hay price ratio with time and ability at

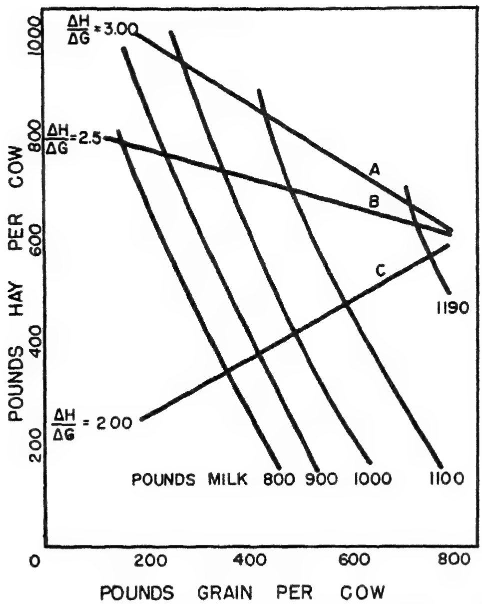

Figure 12.13. Isoquants and isoclines for mean month of experiment and for given substitution rates, estimated from equation 12.44, ability at mean for 36 cows

previously stated levels, as indicated in equation 12.59. (This equation is a particular use of equation 12.54.)

$$\frac{2.9740 - .002384\ G - .001056\ H}{1.5437 - .000776\ H - .001056\ G} = \frac{P_g}{P_h} \tag{12.59}$$

In this case, with equation 12.59 serving as the basis for predictions, ability is set at the mean of cows in the experiment, and time is set at the first month of the experimental period. By reference to Table 12.18, it may be seen that for any grain to hay price ratio equal to 2.5, the least-cost combination to produce 900 pounds of milk in 28 days with a cow of mean ability is 250 pounds of grain and 760 pounds of hay. This is true for any level of grain and hay prices (per pound) which yields a ratio of 2.5 for 900 pounds of output and makes no reference to the most profitable level of production. The substitution rates in tables such as 12.18 and 12.19 allow prediction of least-cost rations for particular milk levels in any month of the experiment (months 3 to 8 of the lactation). For example, when the price of grain is 2.45 cents per pound and the price of hay is 1 cent per pound, the grain to hay price ratio is 2.45. In Table 12.18, when the marginal rate of substitution of grain for hay is 2.45, the minimum cost feed combination to produce 1,100 pounds of

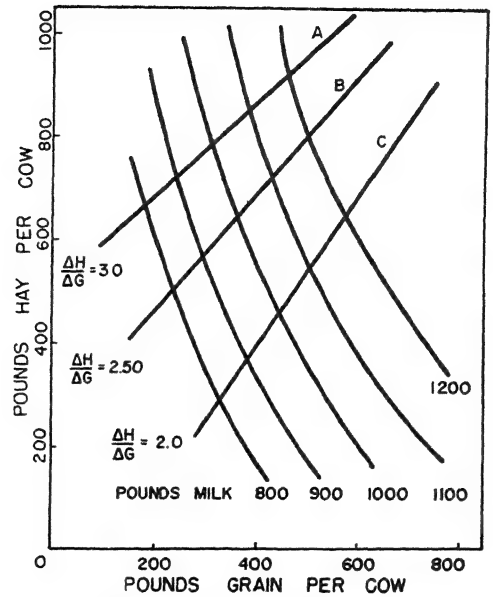

Figure 12.14. Isoquants and isoclines for mean month of experimental period and for given substitution rates, estimated from equation 12.46, ability at mean for 36 cows

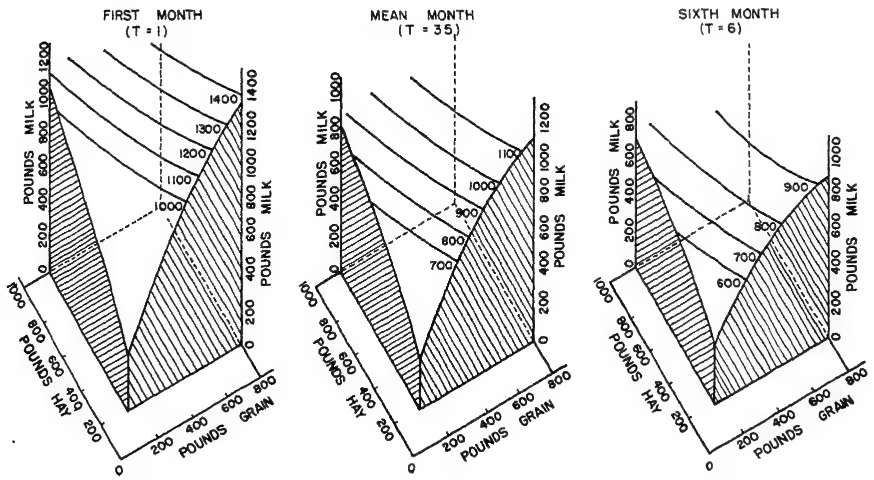

Figure 12.15. Milk production surfaces estimated from equation 12.44 for first, mean, and sixth month (28 days) of experiment, ability at mean

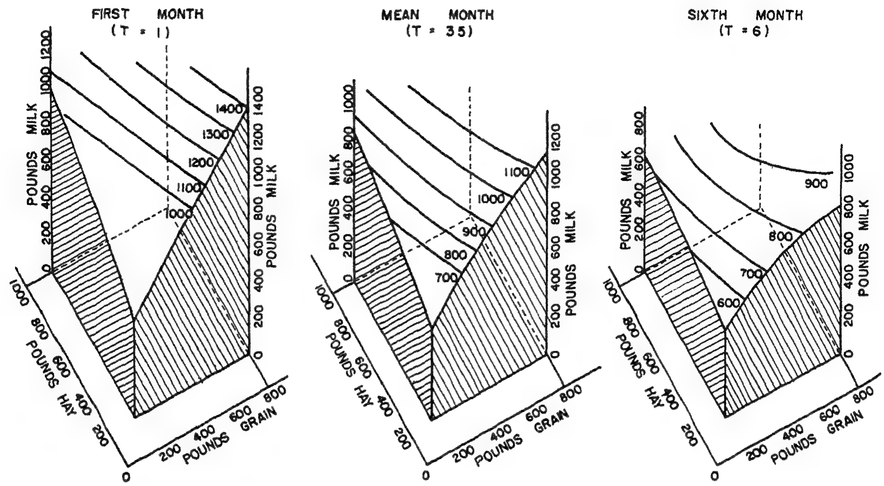

Figure 12.16. Milk production surfaces estimated from equation 12.46 for first, mean, and sixth month (28 days) of experiment, ability at mean

milk in the mean month of the experiment with a cow of mean ability is 500 pounds of grain and 674 pounds of hay.

Other price ratios can be figured similarly with interpolations made between feed combinations. Increases in costs or sacrifices in profit are not great for small deviations away from the feed combination where the substitution rate is equal to the price ratio. In the case above, for example, the cost of the optimum ration for 1,100 pounds of

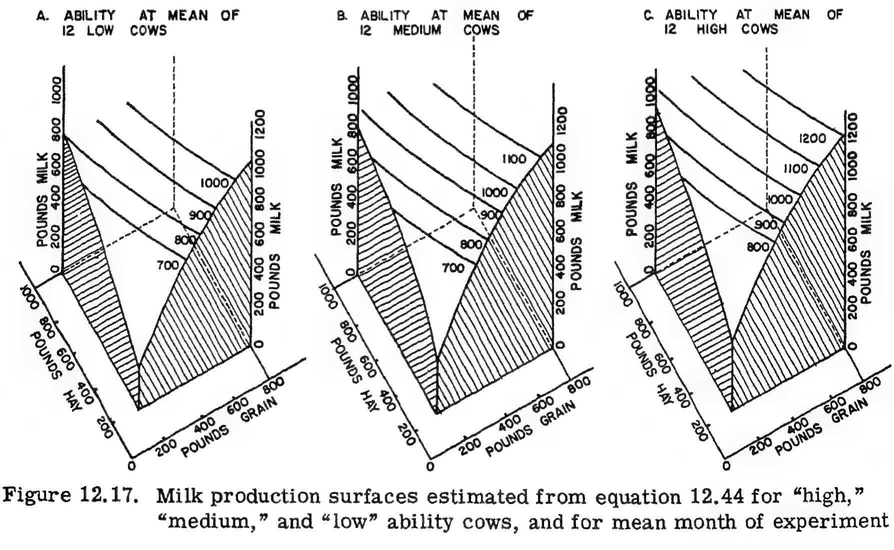

Figure 12.17. Milk production surfaces estimated from equation 12.44 for "high," "medium," and "low" ability cows, and for mean month of experiment

milk is \$18.99. If a ration of 600 pounds of grain and 455 pounds of hay were used, the feed cost would be \$19.25; for 800 pounds of grain and 122 pounds of hay, the feed cost would be \$20.82. These differences in feed costs are perhaps not great enough to offset the added labor costs for feeding particular hay rations. When labor costs are figured, the least-cost milk production may be obtained when the optimum level of grain feeding is determined in relation to the grain to milk price ratio, with self-feeding of hay to stomach capacity as was explained in respect to Figure 12.3.

The gain from feeding one ration rather than others along a milk contour increases with greater curvature of the isoquant. Since the curvature of the milk isoquants for equations such as 12.44 or 12.46 tends to increase with level of milk production, gains in feeding the unique optimum ration are greater as the level of milk production increases (i.e., the grain to milk price ratio decreases) or as time in the lactation period increases. While the gains from feeding unique rations are relatively low from the predictions of this study, the final advantages of particular rations for given milk levels can be determined only as the nature of the milk surface and its isoquant family are better established.

Simultaneous Specification of Ration and Milk Level

Equation 12.44 now can be used to compute the partial derivatives. In equation 12.60, based on equation 12.44, the partial derivative for

$$(12.60) \qquad \frac{\delta M}{\delta G} = 2.9740 - .002384\ C - .001056\ H = \frac{\$3.00}{\$4.00}$$

grain is equated to the grain to milk price ratio when milk is 4 cents per pound and grain is 3 cents per pound, in the first month of the experiment for a cow of mean ability; in equation 12.61 (also derived from equation 12.44), the partial derivative for hay is equated to the hay to milk price ratio when hay is 1.25 cents per pound (\$25 per ton). By simultaneous solution of equations 12.60 and 12.61, it is estimated that the ration which will maximize return above feed costs should include 799 pounds of hay and 579 pounds of grain fed over 28 days (28 pounds of hay and 21 pounds of grain per day) to produce 1,479 pounds of milk and a return of \$31.80 above feed costs. If a radically different ration such as 275 pounds hay and 707 pounds grain had been fed in the first month under these price relationships, the return above feed costs is estimated at \$29.63. With feeds remaining at the above prices and the milk price rising \$5 per hundred pounds, the optimum ration would include 786 pounds of hay and 648 pounds of grain, with milk production at 1,522 pounds. If the milk price were \$3 per hundred pounds, the optimum ration would include 820 pounds of hay and 465 pounds of grain, with milk production at 1,387 pounds.

$$(12.61) \qquad \frac{\delta M}{\delta H} = 1.5437 - .000776\ H - .001056\ G = \frac{\$1.25}{\$4.00}$$

If the milk price remains at $4 per hundred pounds and hay increases to 1.75 cents per pound ($35 a ton), an increase in grain price to 4 cents per pound has this effect on the optimum ration: The feed combination should include 756 pounds of hay and 493 pounds of grain to produce 1,388 pounds of milk in the first month from a cow of medium ability. With these same prices in the sixth month, the ration would include only 480 pounds of hay and 286 pounds of grain, producing an estimated 712 pounds of milk.

Feed and milk quantities in Table 12.20 represent optimum rations and milk production levels derived from equation 12.44 for certain feed and milk price situations. Estimates in Table 12.21 are based on equation 12.46. However, since the production surface for equation 12.46 is nearly linear with time fixed at the first month (see Figure 12.16), the feed quantities suggested for price variations in this month generally exceed the cow's stomach capacity. The estimates from equation 12.46 for the mean and sixth months are more similar to the estimates from equation 12.44 in Table 12.20. The wide differences for the first month, and the fact that equation 12.46 provides estimates far outside the range

Table 12.20. Estimated Optimum Feed Quantities and Milk Production in the First Month, Mean Month, and Sixth Month of Experimental Period, for Various Price Ratios; Estimates From Equation 12.44 With Ability at Mean*

Feed Prices		Price Ratio: Grain to Hay †	Hay, Grain, and Milk Quantities (Pounds) With Milk Prices Per Hundredweight at								
			$3.00			$4.00			$5.00		
Grain per cwt.	Hay per ton		Hay	Grain	Milk	Hay	Grain	Milk	Hay	Grain	Milk
						Month 1					
$2	$15	2.67	882	577	1,501	846	663	1,544	843	715	1,563
3	15	4.00	1,361	225	1,328	1,204	400	1,446	1,110	504	1,500
3	25	2.40	820	465	1,387	799	579	1,479	786	648	1,522
3	35	1.71	275	707	1,357	390	761	1,462	458	793	1,511
4	35	2.29	762	350	1,224	756	493	1,388	752	579	1,464
						Mean Month					
2	15	2.67	745	473	1,100	710	560	1,143	686	612	1,163
3	15	4.00	1,223	121	926	1,066	296	1,045	972	400	1,100
3	25	2.40	682	361	986	662	476	1,080	649	544	1,121
3	35	1.71	-- ‡	503	--	252	657	1,062	322	690	1,111
4	35	2.29	624	247	824	618	390	988	613	476	1,064
						Month 6					
2	15	2.67	608	369	825	570	456	867	548	508	887
3	15	4.00	1,085	18	651	928	193	770	834	297	824
3	25	2.40	545	257	710	524	372	803	511	440	845
3	35	1.71	-- ‡	499	--	-- ‡	553	--	184	586	835
4	35	2.29	486	144	549	480	286	712	475	372	787

*Figures show the most profitable ration and milk level for each combination of hay, grain, and milk prices. For example, with hay at $25.00, grain at $3.00 and milk at $3.00, the most profitable ration includes 465 pounds of grain, 820 pounds of hay and produces 1,387 pounds of milk in the first month.

† Price per pound of grain divided by price per pound of hay.

‡ Physiological minimum hay quantity.

Table 12.21. Estimated Optimum Feed Quantities and Milk Production in the First Month, Mean Month, and Sixth Month of Experimental Period, for Various Price Ratios; Estimates From Equation 12.46 With Ability at Mean

Feed Prices		Price Ratio: Grain to Hay*	Hay, Grain, and Milk Quantities (Pounds) With Milk Prices per Hundredweight at †								
Grain per cwt.	Hay per ton		$3.00			$4.00			$5.00		
			Hay	Grain	Milk	Hay	Grain	Milk	Hay	Grain	Milk
						Month 1					
$2	$15	2.67	3,797	2,492	--	4,353	3,051	--	4,723	3,380	--
3	15	4.00	3,797	1,407	--	4,353	2,229	--	4,723	2,722	--
3	25	2.40	2,223	1,407	--	3,242	2,229	--	3,797	2,722	--
3	35	1.71	742	1,407	--	2,038	2,229	--	2,871	2,722	--
4	35	2.29	742	321	--	2,038	1,407	--	2,871	2,064	--
						Mean Month					
2	15	2.67	1,085	712	1,444	1,244	872	1,573	1,349	966	1,633
3	15	4.00	1,085	402	1,186	1,244	637	1,427	1,349	778	1,539
3	25	2.40	635	402	1,035	926	637	1,347	1,085	778	1,486
3	35	1.71	212	402	823	582	637	1,218	820	778	1,407
4	35	2.29	212	92	462	582	402	1,012	820	590	1,275
						Month 6					
2	15	2.67	633	415	878	726	508	953	787	563	988
3	15	4.00	633	234	727	726	372	868	787	454	934
3	25	2.40	370	234	639	540	372	822	633	454	903
3	35	1.71	124	234	516	340	372	747	478	454	857
4	35	2.29	124	54	306	340	234	626	478	344	780

*Price per pound of grain divided by price per pound of hay.
†Pounds milk not computed for feed quantities in first month because of obvious inability of animal to consume such quantities.

of stomach capacity, again suggests that equation 12.44 is preferable to equation 12.46, even though the latter has lower standard errors than the former.

It is of interest to compare the optimum rations indicated in Table 12.20 with those based on Jensen *et al.*[16] Estimates based on the Jensen study indicate that for prices of $25 per ton for hay and $3 and $4 per hundred pounds respectively, for grain and milk, the optimum ration would include about 350 pounds of grain, leaving capacity for 800-900 pounds of hay taken free choice. This estimate is for a cow producing 40 pounds of milk per day (1,120 pounds in 28 days). Estimates for the same prices, based on equation 12.44 of this study (Table 12.20), indicate 476 pounds of grain and 662 pounds of hay to produce 1,080 pounds of milk, for a cow of mean ability in the mean month of the experimental period. With prices of $35 per ton for hay, $4 for grain, and $5 for milk, the estimates based on the Jensen study include about 330 pounds of grain and a residual of 800-900 pounds of hay; the estimates of the current study include 476 pounds of grain and 613 pounds of hay to produce 1,064 pounds of milk in 28 days.

Predictions of the daily rate of grain and hay feeding to give maximum return above feed costs in the first, mean, and sixth months of the

[16] Jensen, E. *et al.* Input-output relationships in milk production. Tech. Bul. 815. USDA, Washington, D.C. 1942.

experimental period can be made by dividing the quantities of Table 12.20 by 28 days. These estimates are for a cow of mean ability and are based on equation 12.44. Estimates for other ability levels can be predicted by adding to or subtracting from these quantities an amount based on the coefficient for A in equation 12.44.

STOMACH CAPACITY

Dairy animal stomach capacity depends partly upon the size of the cow and, hence, upon the breed of cow. For the large dairy breeds, estimates may be made from data of this and other studies. While animals in this study were not fed rations of hay or grain alone, other studies have included such rations. The United States Department of Agriculture conducted hay feeding trials in which animals were full-fed alfalfa for 365 days.[17] Consumption averaged 14,352 pounds per cow or 39.3 pounds per day. Maximum consumption was 47.1 pounds per day, while the minimum taken ad lib was 30.4 pounds. Other agricultural experiment stations reporting such experiments include Kansas[18], Nevada[19], Oregon[20], and California.[21] Missouri also has reported feeding only a concentrate mixture for 15 entire lactations without materially lowering production.[22] However, other trials suggest a minimum hay requirement of 5 to 6 pounds per day.[23]

Table 12.22. Estimated Maximum Grain and Hay Consumption by Large Breed Dairy Cows in Pounds per Day, per 28 Days, and per 182 Days

Pounds per Day		Pounds per 28 Days		Pounds per 182 Days	
Grain	Hay	Grain	Hay	Grain	Hay
0	40	0	1,120	0	7,280
5	37	140	1,036	910	6,734
10	32	280	896	1,820	5,824
15	27	420	756	2,730	4,914
20	21	560	588	3,640	3,822
25	13	700	364	4,550	2,366
29	5	812	140	5,278	910

[17] Groves, R. R., *et al*. Feeding dairy cows on alfalfa hay alone. Tech. Bul. 610. USDA, Washington, D.C. 1938.

[18] Reed, O. E., *et al*. The relation of feeding and age of calving to development of dairy heifers. Kan. Agr. Exp. Sta. Bul. 233. Manhattan. 1924.

[19] Headley, F. B. Feeding experiment with dairy cows. Nev. Agr. Exp. Sta. Bul. 119. Reno. 1930.

[20] Oregon Agr. Exp. Sta. Studies with alfalfa hay. Director's Biennial Report. P. 41-43. Corvallis. 1924-26.

[21] Wall, F. W. Alfalfa as the sole feed for dairy cows. Jour. Dairy Sci. 1: 447-61. 1918.

[22] Missouri Agr. Exp. Sta. Bul. 444. Agricultural investigations. Columbia. 1942.

[23] Loosli, J. K., Lucas, H. L., and Maynard, L. A. The effect of roughage intake upon the

Table 12.22 is derived from data of this study and others mentioned. It suggests maximum daily, 28-day, and 182-day intakes of several ration combinations and is presented only as a guide for use with large breed dairy animals. Figure 12.18 shows the estimates of Table 12.22 for a 28-day period, with line ab defining the estimated maximum feed intake of various ration combinations.

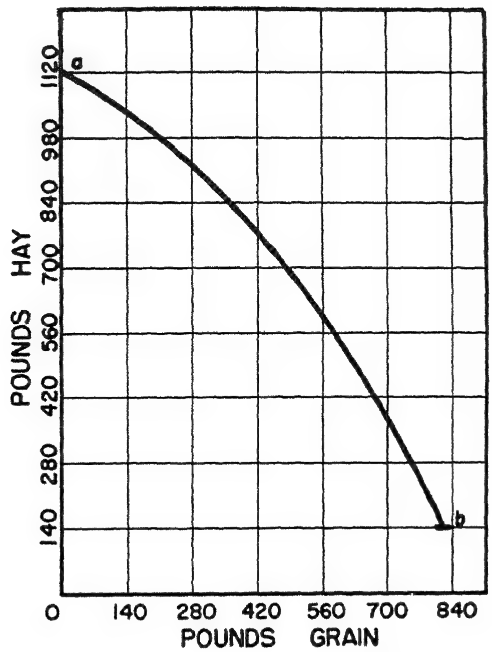

Figure 12.18. Estimates of stomach capacity of large breed dairy cows in 28-day period

fat content of milk. Jour. Dairy Sci. 28: 147-53. 1945; and Monroe, C. F. and Krauss, W. E. Relationship between fat content of dairy grain mixtures and milk and butterfat production. Ohio Agr. Exp. Sta. Bul. 644. Wooster. 1943.

CHAPTER 13

Production Functions and Substitution Coefficients for Beef

Earl O. Heady
Harold O. Carter
C. C. Culbertson

THIS STUDY deals with prediction of production functions, isoquants, isoclines, marginal substitution rates, and related economic quantities in the rations of feeder cattle. These basic data are important to managerial decisions by farmers since feed represents around 75 per cent of the costs involved in feeding cattle. Information on feed replacement rates also is needed as a basis for analyzing the effects of government production control programs as well as for individual farmer decisions. It has been shown previously that whether acreage allotments, allowing shifts of land from grain to forage, will reduce or increase total output depends on two things: (1) the extent and rates at which crops substitute for each other in the crop rotation, (2) the extent and rates at which these same crops as feed substitute for each other in the livestock ration.[1] The optimum soil conservation plan also depends on these same substitution data.

OBJECTIVES AND DATA

Because little knowledge is available on feed substitution quantities for beef cattle, one objective of this study is to estimate livestock-feed production functions and the feed substitution relationships which can be derived from them. Production functions are derived for steer calves where the feeds include corn, legume hay, and protein supplement. Time functions are derived to indicate the amount of time required for attainment of a given weight when different rations are fed. Given the production functions and the time functions, comparisons can be made of the monetary gain from feeding the least-cost rations with that from feeding to produce a specific quality of beef in a minimum amount of time.

The data upon which this study is based are drawn from experiments conducted by the Animal Husbandry Department at Iowa State

[1] Cf. Heady, Earl O. Resource and revenue relationships in agricultural production control programs. Review of Economics and Statistics, 33(3): 228-40. 1951.

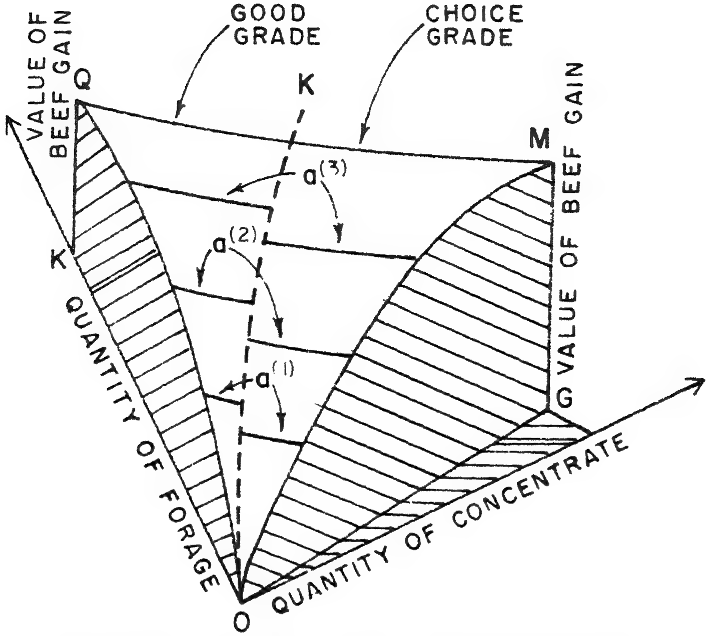

Figure 13.1. Production surface illustrating beef grades

University over a period of 25 years.[2] Data were used from experiments employing 272 choice feeder calves and including approximately the same quality and kinds of feeds in each year. Experiments used for estimating production functions in this study were those including only good quality legume hay and corn. In the experiments of the 25 years, rations used included a fairly wide range of corn, hay, and protein supplement ratios. The main supplement used was linseed and soybean oilmeal. To reduce the number of feed variables to three (corn, hay, and protein supplement), all supplements were converted in terms of digestible protein content to a linseed oilmeal (L.O.M.) basis.

Since the feeds from the different years are uniform generally in respect to kind and quality, it appears that the data lend themselves to production function analyses. Also, the quality of livestock appears sufficiently homogeneous over the time period concerned. Generally, a feeder calf grading choice in one year has been comparable to a calf of the same grade in another year. Beef cattle production has not experienced breeding revolutions comparable to those for hogs and poultry. Too, given grades of corn and hay changed little if any over the years from which experimental data have been drawn.

[2] The experiments included are those in the Iowa State University, Animal Husbandry Department numbered: 272 X; 287 XI; 298 III; 314 V, VI, VII, VIII, IX, X, XI, XII; 340 I; 350 I, II, III, V, VI, VII, VIII; 352 I; 360 I, II, IV, V, VII, VIII; 378 I, IV, V, VII, VIII. The initial weight of calves was approximately 400 pounds.

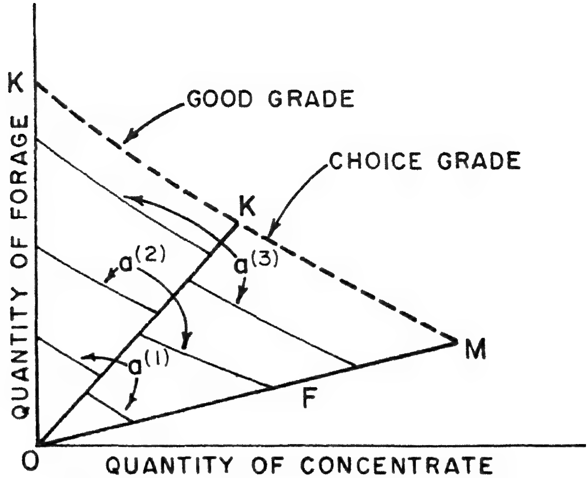

Figure 13.2. Iso-value map for Figure 13.1.

This study is made in an attempt to utilize existing data in deriving production functions and substitution coefficients for beef cattle. It follows the recommendations provided by a joint meeting of animal nutritionists and farm management workers in March, 1952. This meeting was sponsored, through the facilities of the Farm Foundation, by the North Central Farm Management Committee to outline experiments and methods for obtaining information on feed substitution rates. The animal nutritionists present expressed the belief that existing experimental data provided a sufficient basis for these estimates. The current study was undertaken accordingly. It is, however, the belief of the authors that existing data, while useful in obtaining information on substitution coefficients, is not ideal for this purpose. A 3-year experiment designed for prediction of beef production functions has now been initiated at Iowa State University.

The procedure generally outlined for simultaneously specifying the optimum marketing weight and the optimum ration must be modified in beef production, because different rations result in different qualities of beef. Thus P_g, the price of gain, will change with the ration fed over a limited range of forage to grain ratios. Thus, the models of beef gain surfaces and isoquants may need to be modified as illustrated in Figure 13.1. The gain surface is identical to conventional ones as in Chapter 9 except iso-value lines are substituted in place of iso-gain lines and a physiological limit line is introduced. Iso-value lines represent, for example, all possible combinations of concentrate and forage which will produce a given value of beef gain. Line OK divides the hypothetical production surface into grades "good" and "choice" according to the effects of the ration fed. Figure 13.2, the two-dimensional counterpart of Figure 13.1, shows iso-value lines $a^{(1)}$, $a^{(2)}$, and $a^{(3)}$. Iso-value lines are discontinuous since two separate products or grades are produced.

For example, if "good" beef has a price of $15 per hundredweight while "choice" has a price of $20, then $a^{(1)}$ may represent a $40 iso-value line following the 266-pound gain contour through the "good" area of the surface and then follow the 200-pound gain contour through the choice area.[3]

The principle which specifies the least-cost method of producing a given gain, regardless of the grade, is the same as that for determining the least-cost method of producing a given value of beef when grade differentials are considered. In application to Figure 13.1, for example, usual procedures might be employed to specify the least-cost method of producing a value of beef represented as $a^{(1)}$, $a^{(2)}$, or $a^{(3)}$. Difficulty arises in simple applications of the principle, however, because a continuous value function currently cannot be derived. In most cattle feeding operations, it is more important, at some level of gain, to feed a ration which increases the grade but does not minimize costs of producing a physical quantity of gain. However, within a grade range, knowledge of substitution coefficients is important for specifying profitable rations.

EMPIRICAL RESULTS

As a basis for predicting the beef production surface from available data, several algebraic forms of equations were employed. Of the functions fitted, the one presented in equation 13.1 appeared logically and statistically most appropriate.

$$\begin{aligned}(13.1)\quad Y = {} & .157402C + .361070P + .112332F - .00001612C^2 \\ & - .00090147P^2 - .00005958F^2 - .00003612CP - .00000637CF \\ & + .00054144PF + 3.58753\end{aligned}$$

The variables in this equation are defined and measured as follows:

Y is total gain in pounds per steer measured from the beginning of the feeding period to weighing date. Average starting weight of all groups of steers included in the analysis was 400 pounds. Consecutive gain observations were taken at 28-day intervals on a given ration line; with each observation being the accumulated sum of previous observations on that ration line. In most cases, the average performance of groups of 8 or 10 steers to a lot were obtained.

C is total intake of corn in pounds from the time grain feeding is started to the particular weighing dates. Hence, the first observation of corn is from beginning of the experiment to the first weighing date, the second observation is from the beginning of the experiment to the second weighing date, etc. Corn silage was converted to corn equivalent.

[3] Exaggerated price differentials were used to illustrate the point.

P is total intake of protein supplement measured in pounds. Observations are measured as outlined above for corn. Protein supplement is in terms of pounds of linseed oilmeal. All other proteins used in the experiment were converted to linseed oilmeal equivalent (L.O.M.) on the basis of per cent digestible protein.

F is total intake of good quality legume hay measured in pounds. Observations are measured as outlined for corn. Grass silage, used in small amounts in some experiments, was converted to hay equivalent.

The coefficient of determination for equation 13.1 is .87, indicating that the major portion of variance in beef gains on choice feeder calves is explained by the three feed categories employed. Standard errors for the regression coefficients are presented in Table 13.1. Variables were left in the estimating equation where they appeared consistent with production logic; even if they were not significant at conventional probability levels. However, as is pointed out later, the coefficients for certain nonlinear variables are small and have only slight effect on substitution rates. Certain equations were fitted where all coefficients could be accepted at probability levels of .99 per cent. However, these appeared, by inspection of scatter diagrams, and in terms of previous nutrition and production logic, to be less in accord with available data.

Table 13.1. Standard Errors for Regression Coefficients in Equation 13.1

Regression Coefficient for	Standard Error
C	.03858
P	.17660
F	.07747
C^2	.0002
P^2	.00043
F^2	.00007
CP	.00016
CF	.00006
FP	.00023

Autocorrelation Problems

When fitting the production function with the classical method of least squares, the assumption of independence for each observation is explicitly made. In these livestock experiments, each observation is the average accumulated sum of feed and gain for each pen of animals. For example, feed and gain observations taken at the end of each month are accumulated sums of observations taken in previous months. Hence, the second observation is related to the first, and the third observation in a ration is related to the first and second observation, etc. Although the series of observations taken on a pen is correlated, it is independent of series of observations taken on other pens or rations.

Equations Derived for Estimates

The basic production function of equation 13.1 allows derivation of equations for isoquants, isoclines, and input-output relationships. Since corn and hay represent the major portion of feed costs in beef fattening and because major land use problems revolve around these crops, most of the analysis which follows is in terms of subfunctions defining relationships between corn and forage. Accordingly, linseed oilmeal equivalent, hereafter indicated as L.O.M., is held constant at different ratios in respect to corn and at different absolute levels, and feed relationships are derived accordingly. More specifically, for part of the following analysis L.O.M. is held constant at (a) 25 per cent of the quantity of corn in the ration (i.e., corn and L.O.M. are always combined in a fixed ratio of 4 to 1), (b) 15 per cent of the quantity of corn in the ration (i.e., corn and L.O.M. are always combined in a fixed ratio of approximately 7 to 1), and (c) a constant amount of 175 pounds (i.e., the ratio of corn to L.O.M. changes as the quantity of corn in the ration is varied) is used. A different production surface, with corn and hay as variables, exists for each ratio or absolute quantity of L.O.M. The prediction equations for a, b, and c above in terms of corn and hay become, respectively, those in equations 13.2, 13.3, and 13.4. Variables in the following equations are measured as defined previously.

(13.2) L.O.M. held constant at 25 per cent of corn.

$$Y = .2476C - .000081C^2 + .1123F - .000059F^2 + .000129CF + 3.59$$

(13.3) L.O.M. held constant at 15 per cent of corn.

$$Y = .2116C - .000042C^2 + .1123F - .000059F^2 + .000075CF + 3.59$$

(13.4) L.O.M. held constant at 175 pounds.

$$Y = .1511C - .000016C^2 + .2071F - .000059F^2 - .000006CF + 39.17$$

Equations 13.2 to 13.4 can be used to predict a beef production function when L.O.M. is either held in a constant ratio to corn or is used in a fixed absolute amount. Equations of gain isoquants then can be derived from these production function equations. Corresponding to the three production function equations above, the isoquant equations are:

(13.5) L.O.M. equal to 25 per cent of corn (i.e., held in constant ratio of 4 parts corn and 1 part L.O.M.).

$$F = 942.699 + 1.0825C \pm 8{,}392.08(.01347346 + .000088C - .0000000028C^2 - .00023832Y)^{.5}$$

(13.6) L.O.M. equal to 15 per cent of corn (i.e., held in constant ratio of approximately 7 parts corn and 1 part L.O.M.).

$$F = 942.699 + .6281C \pm 8{,}392.08(.01347346 + .00006724C - .0000000044C^2 - .00023832)^{.5}$$

(13.7) L.O.M. held constant at 175 pounds (i.e., the proportion of corn to L.O.M. changes as the quantity of corn varies).

$$F = 1{,}737.865 - .0535C \pm 8{,}392.08(.05221811 + .00003336C - .0000000039C^2 - .00023832Y)^{.5}$$

In these isoquant equations, amounts of hay (F) required to produce specified gains (Y) are expressed as a function of corn intake per animal. These equations can be used to derive all possible combinations of corn and hay which will produce gains of specified amounts.

Equations defining marginal rates of substitution can be derived from the isoquant equations 13.5 to 13.7. For example, with L.O.M. specified as above, the marginal rates of substitution become the derivatives of forage (F) in respect to corn (C) as shown below for equation 13.5.

(13.8) L.O.M. equal to 25 per cent of corn (i.e., held in a constant ratio of 4 parts corn and 1 part L.O.M.).

$$\frac{\delta F}{\delta C} = \frac{.000163C - .000129F - .247669}{.112332 - .000119F + .000129C}$$

Equations such as 13.8 above can be used to predict the marginal rate of substitution of corn for hay — at a given L.O.M. level — for the various combinations of the two feeds which will produce a specified gain.[4] While the equations are expressed in terms of substitution of corn for hay, marginal rates of substitution of hay for corn can be derived as the reciprocal of 13.8.

GAIN ISOQUANTS AND SUBSTITUTION RATES

Beef gain isoquant schedules and marginal rates of substitution can be derived from the above equations. They are presented in the tables that follow. Isoquant equation 13.5 and substitution rate equation 13.8,

[4] The marginal rate of substitution of corn for hay is empirically derived from the ratio:

$$MRS = -\frac{\delta F}{\delta C} = -\frac{\delta Y}{\delta C}\frac{\delta F}{\delta Y}$$

It is important to note, however, that with respect to the two cases where protein is fixed as a percentage of the quantity of corn in the ration $\partial Y/\partial C$ is slightly modified. That is, an additional side condition is made that $P = f(C)$. Thus, in general terms, equation 13.8 is written:

$$\frac{\delta F}{\delta C} = \frac{\frac{\delta Y}{\delta C} + \frac{\delta Y}{\delta P} \cdot \frac{dP}{dC}}{\frac{\delta Y}{\delta F}}.$$

For equation 13.8, $\frac{\delta P}{\delta C} = 0.25$. Thus equation 13.8 provides estimates of the marginal rate of substitution of 1 pound of corn and 0.25 pounds of L.O.M. for hay.

Table 13.2. Isoquant Schedules Showing Possible Feed Combinations and Marginal Rates of Substitution of Corn for Forage at Four Gain Levels, With Good-to-Choice Calves Weighing 400 Pounds at the Outset (L.O.M. Held Constant at 25 Per Cent of the Quantity of Corn)

Total Pounds of Feed		Pounds of Feed per Pound of Gain		Ratio of Forage to Corn	$\frac{\delta F}{\delta C}$‡	$\frac{\delta C}{\delta F}$§
Corn*	Forage*	Corn†	Forage			
		200 pounds gain				
500	683	2.50	3.41	1.37	2.66	0.38
600	479	3.00	2.39	0.80	1.59	0.63
700	346	3.50	1.73	0.49	1.10	0.91
800	252	4.00	1.26	0.31	0.81	1.23
		300 pounds gain				
700	1,306	2.33	4.35	1.87	6.40	0.16
800	945	2.67	3.15	1.18	2.32	0.43
900	763	3.00	2.54	0.85	1.45	0.69
1,000	642	3.33	2.14	0.64	1.02	0.98
1,100	555	3.67	1.85	0.50	0.74	1.35
1,200	490	4.00	1.63	0.41	0.55	1.82
		400 pounds gain				
1,024	1,439	2.56	3.60	1.40	3.64	0.27
1,100	1,231	2.75	3.08	1.12	2.11	0.47
1,200	1,063	3.00	2.66	0.89	1.35	0.74
1,300	950	3.25	2.38	0.73	0.95	1.05
1,400	869	3.50	2.17	0.62	0.69	1.45
1,500	809	3.75	2.02	0.54	0.51	1.96
1,600	765	4.00	1.91	0.48	0.38	2.63
1,700	733	4.25	1.83	0.43	0.27	3.70

*Derived from isoquant equation 13.5.
†An amount equal to 25 per cent of these amounts would be needed to give total concentrates per pound of gain.
‡Derived from substitution equation 13.8.
§Column 7 is the reciprocal of column 6.

both derived from production function 13.1, provide the basis for the estimates in Table 13.2. For the estimates in Table 13.2, L.O.M. is held constant at 25 per cent of the quantity of corn in the ration. Equation 13.6 and its substitution equation provide the basis of the estimates in Table 13.3 where protein (pounds of L.O.M. equivalent) is held constant at 15 per cent of the quantity of corn in the ration.

For each gain level (i.e., 200, 300, 400, and 500 pounds) and each L.O.M. level, the derived marginal rates of substitution between corn and forage are at a diminishing rate.[5] In other words, based on the

[5]It should be pointed out that marginal rates of substitution between feeds apply only to a specific gain contour, i.e., 200, 300, 400 or 500 pounds of gain, rather than the gain intervals, i.e., the portion of the production surface between the initial feeder calf weight of 400 pounds and a specific gain contour. However, the feed quantities shown refer to the gain interval because they are derived as consecutive observations from the outset of the feeding experiment.

Table 13.3. Isoquant Schedules Showing Possible Feed Combinations and Marginal Rates of Substitution of Corn for Forage at Four Gain Levels, With Good-to-Choice Calves Weighing 400 Pounds at the Outset (L.O.M. Held Constant at 15 Per Cent of the Quantity of Corn)

Total Pounds of Feed		Pounds of Feed per Pound of Gain		Ratio of Forage to Corn	$\frac{\delta F}{\delta C}$‡	$\frac{\delta C}{\delta F}$§
Corn*	Forage*	Corn†	Forage			
			200 pounds gain			
600	752	3.00	3.76	1.25	3.21	0.31
700	513	3.50	2.57	0.73	1.84	0.54
800	358	4.00	1.79	0.45	1.32	0.76
900	242	4.50	1.21	0.27	1.02	0.98
			300 pounds gain			
950	1,175					
1,000	988	3.33	3.29	0.99	2.89	0.35
1,100	769	3.67	2.56	0.70	1.71	0.58
1,200	624	4.00	2.08	0.52	1.23	0.81
1,300	516	4.33	1.72	0.40	0.95	1.05
1,400	431	4.67	1.44	0.31	0.76	1.32
			400 pounds gain			
1,400	1,315	3.50	3.29	0.94	3.17	0.31
1,500	1,084	3.75	2.71	0.72	1.74	0.57
1,600	938	4.00	2.35	0.59	1.22	0.82
1,700	832	4.25	2.08	0.49	0.93	1.08
1,800	749	4.50	1.87	0.42	0.74	1.35

*Derived from isoquant equation 13.6.
†An amount equal to 15 per cent of these amounts would be needed to give total concentrates per pound of gain.
‡Derived from substitution equation corresponding to 13.6.
§Column 7 is the reciprocal of column 6.

equations used, the rate in which one feed substitutes for the other feed declines as the ration is changed to include a greater proportion of the first. For example, in Table 13.2, with 500 pounds of corn and 683 pounds of forage combined in the ration to produce 200 pounds of gain, the marginal rate of substitution of corn (plus 0.25 pounds of L.O.M.) for forage is 2.66; i.e., at a 600-pound weight level per animal and with feed combinations above, one additional pound of corn and 0.25 pounds of L.O.M. is predicted to replace 2.66 pounds of forage. At a point on the same 200-pound gain contour with 700 pounds of corn and 346 pounds of forage, one additional pound of corn and 0.25 pounds of L.O.M. is predicted to replace only 1.10 pounds of forage.

Gain isoquants corresponding to the data in columns 1 and 2 in tables 13.2 and 13.3 and derived from equation 13.6 are presented in figures 13.3 and 13.4, respectively. As mentioned previously, these isoquants specify the various feed combinations which are predicted to produce the gain levels specified on 400-pound choice feeder calves. Similarly, the slope at a given point on an isoquant indicates the rate at

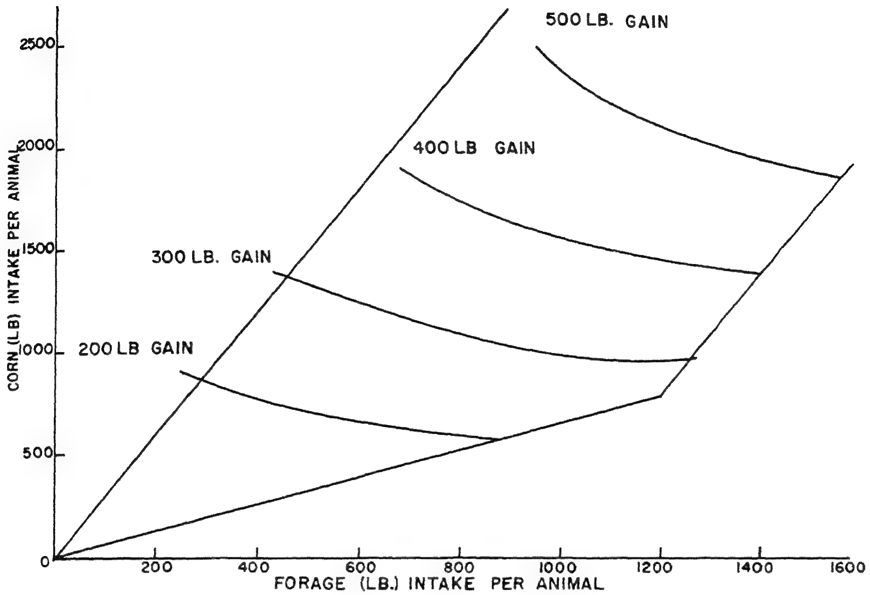

Figure 13.3. Gain isoquants for 400 lb.-calves (good to choice), protein (L.O.M.) constant at 15 per cent level of corn inputs

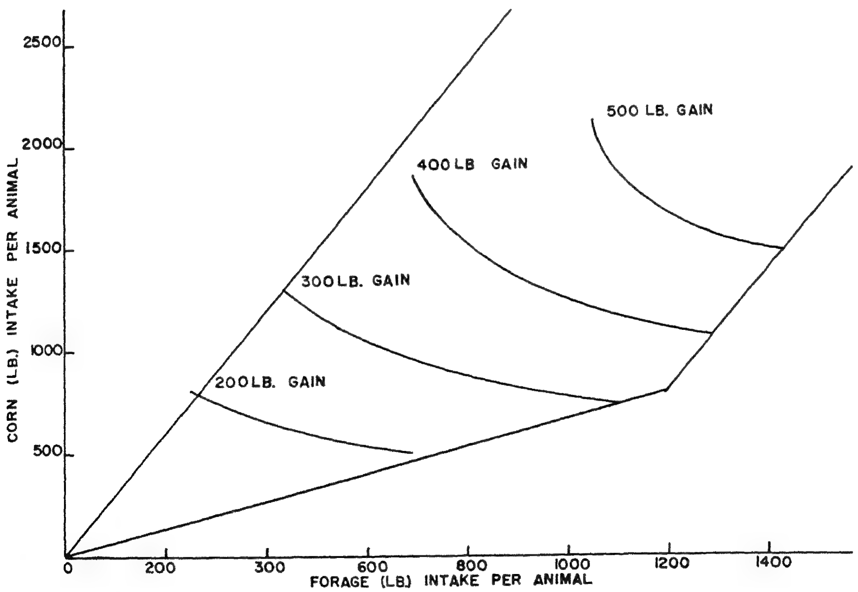

Figure 13.4. Gain isoquants for 400 lb.-calves (good to choice), protein (L.O.M.) constant at 25 per cent level of corn inputs

which forage substitutes for corn, or the rate at which corn substitutes for forage. As indicated in the figures, the curvature of the isoquants changes only very gradually, suggesting that substitution ratios may not depart greatly from constant rates for animals of this size, age, and grade. It is possible that for small animals in early rapid growth stages, or heavier and older animals in final fattening stages, the substitution rates may change more rapidly.[6] Too, greater curvature of the isoquants might have been predicted had the range of experimental rations been greater, and had the experiments been designed for the particular purpose. However, from the data available in the experiments, it appears that cattle in the weight range 400-1,000 pounds have relatively great physiological ability to substitute forage and corn, or vice versa, without a rapid decline in marginal substitution rates.

Corn-Forage Substitution Rates on Selected Ration Lines

Of further interest in decision on livestock feeding is the relative productivity of various feeds in rations for cattle at different weight levels. Consequently, marginal rates of substitution between corn and forage were derived for selected forage to corn ratios (i.e., the quantity of corn in the ration is held in a constant proportion to the quantity of forage, at different gain levels). Isoquant equation 13.6 and its corresponding substitution equation provides the basis for the corresponding estimates in Table 13.4. Forage to corn ratios of 0.33, 0.50, 0.67, 0.83, and 1.00 were selected as being representative of feed combinations observed in the experimental data used.[7]

Table 13.4 shows that with a forage to corn ratio of 0.50 (i.e., 2 pounds of corn for each 1 pound of forage), the predicted marginal rate of substitution of corn for forage is 1.63 at the 200-pound gain contour, 1.17 at the 300-pound gain contour, and 0.96 at the 500-pound gain contour. These data would indicate that corn is less important in the ration relative to forage (given a high protein level) as the feeder calf acquires additional gain. This result appears unrealistic in view of nutritional logic and research which indicates that energy feeds (e.g., corn) are relatively more important than forage in the ration as the weight of the animal increases and the fattening process takes place. However, the "high" protein (pounds of L.O.M.) content in the ration probably substitutes in part, on an energy basis, for corn. At high forage ratios, such as 1.0, the rate of substitution of corn for hay is predicted to increase with greater steer weight.

[6] The linear, positively-sloped lines are included to suggest limits in rations which might be expected to provide satisfactory growing and fattening rations. Few of the experiments provided data outside of these ranges.

[7] The reader is again reminded to refer to a previous footnote. Briefly, the marginal rates of substitution in Table 13.4 refer, respectively, to 1 pound of corn and 0.25 pounds of L.O.M. for hay.

Table 13.4. Marginal Rates of Substitution and Feed Combinations on Different Forage to Corn Ration Lines at Four Levels of Gain With 400-Pound Good-to-Choice Calves (L.O.M. Equal to 15 Per Cent of the Quantity of Corn)

Ratio of Forage to Corn	200-Pound gain			300-Pound gain			400-Pound gain		
	Corn	Forage	$\frac{\delta F}{\delta C}$*	Corn	Forage	$\frac{\delta F}{\delta C}$*	Corn	Forage	$\frac{\delta F}{\delta C}$*
0.33	860	286	1.12	1,350	450	0.82	1,950	648	0.53
0.50	775	387	1.63	1,225	610	1.17	1,690	845	0.96
0.67	727	475	1.69	1,125	740	1.59	1,550	1,015	1.46
0.83	740	455	1.61	1,150	710	1.47	1,575	980	1.84
1.00	630	630	2.43	1,000	1,000	2.96	1,400	1,400	3.92

*Derived from equation 13.6.

Corn-Protein Substitution Relationships at Various Gain Levels

Corn-forage substitution relationships, where protein is held at constant ratios and absolute amounts were discussed in the previous section. Of similar interest are corn-protein substitution relationships when forage is held constant at specified levels. Estimates of this type are presented in Table 13.5 for a 200-pound gain. It shows the pounds of corn necessary to produce 200 pounds of gain when the L.O.M. levels are allowed to vary to 100, 150, and 200 pounds. Simultaneously, the forage levels are varied between 300, 500, and 700 pounds.

The feed inputs for the various gain levels are derived from iso-quant equations based on function 13.1. The marginal rates of substitution, $\frac{\delta C}{\delta P}$, are derived from the equations below, also based on equation 13.1 for forage held constant at the levels indicated:

Forage constant at 300 pounds.

$$\frac{\delta C}{\delta P} = \frac{0.523502 - 0.001802P - 0.000036C}{0.155491 - 0.000032C - 0.000036P} \tag{13.9}$$

Forage constant at 500 pounds.

$$\frac{\delta C}{\delta P} = \frac{0.631791 - 0.001802P - 0.000036C}{0.154217 - 0.000032C - 0.000036P} \tag{13.10}$$

Forage constant at 700 pounds.

$$\frac{\delta C}{\delta P} = \frac{0.740078 - 0.001802P - 0.000036C}{0.152943 - 0.000032C - 0.000036P} \tag{13.11}$$

At all gain levels the marginal rate of substitution of protein for corn is predicted to decrease as the quantity of protein is increased,

Table 13.5. Feed Combinations and Marginal Rates of Corn-Protein Substitution to Produce 200 Pounds of Gain on Good-to-Choice Beef Calves Weighing 400 Pounds at the Outset

Feed Combinations to Produce 200 Pounds of Gain			$\frac{\delta C}{\delta P}$*	$\frac{\delta P}{\delta C}$†
Lbs. of forage	Lbs. of corn	Lbs. of L.O.M.		
300	902	100	2.59	0.39
300	791	150	1.86	0.54
300	715	200	1.16	0.86
500	721	100	3.40	0.42
500	769	150	2.67	0.37
500	455	200	1.99	0.50
700	580	100	4.18	0.24
700	390	150	3.43	0.29
700	235	200	2.74	0.36

*Derived as the first derivative of C in respect to P from the basic production function equation 13.1.

†Reciprocal of $\frac{\delta C}{\delta P}$.

with hay fixed at given levels. This prediction conforms with existing nutrition logic and previous research, namely, additional pounds of protein substitute for less corn as the quantity of protein in beef rations increases.

It is of particular interest to compare the feed substitution relationships on increasingly higher gain isoquants when corn and protein are held in a constant ratio to each other. Figure 13.5 indicates iso-gain contours and protein to corn ratio lines of 15, 25, and 33 per cent, when hay is constant at 700 pounds. Some of the ration lines extend past the range of the data. In all cases, however, where consecutive points (with regard to gain levels) on the same ration line can be compared, the slope ($\frac{\delta C}{\delta P}$), i.e., the marginal rate of substitution of L.O.M. for corn, is greatest for the low gains and decreases for each successively higher gain contour. In other words, for beef calves fed a fixed corn to protein ration, with appropriate amounts of forage, each additional pound of protein substitutes for less corn as the animal matures and puts on additional gain. These empirical results are consistent with the physiological needs of the animal. A younger animal has a greater relative need for protein for building body tissue and growth; a more mature animal requires relatively more carbohydrates for adding fat and finish. The negatively sloped and curved lines are isoquants while the linear and positively sloped lines are ration lines. If the marginal rate of substitution between corn and L.O.M. did not change with animal weight, gain isoquants would have equal slopes where they are intersected by a given ration line. However, as the graph illustrates, the slope of

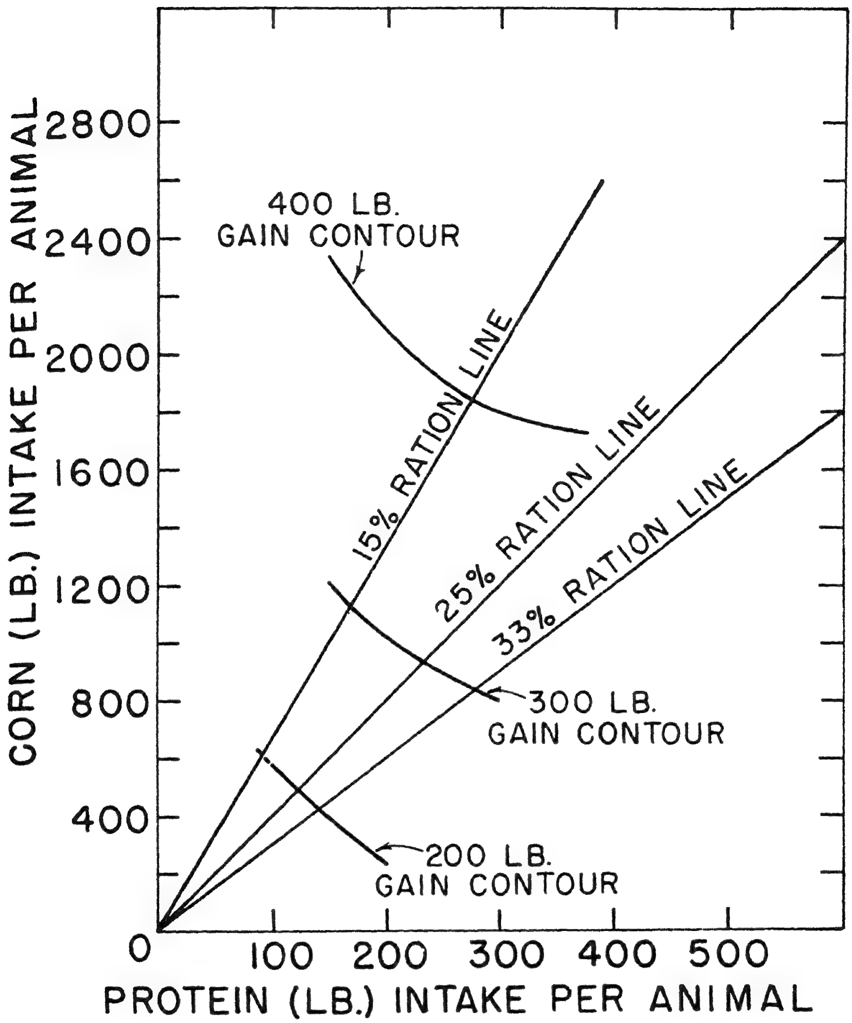

Figure 13.5. Beef gain contours and corn protein ration lines for good to choice beef calves weighing 400 lbs. at the outset (forage held constant at 700 lbs.)

isoquants representing greater gains have less slope than those representing smaller gains; indicating a declining marginal rate of substitution, $\frac{\delta C}{\delta P}$, as the animal progresses in weight.

TOTAL AND MARGINAL GAIN FOR SELECTED FIXED RATIONS

The livestock gain surface is explored further in this section by deriving total and marginal gain estimated for feeder calves fed selected fixed rations. It should be pointed out that no empirical observations were available for calves fed a single ration over the entire feeding program. Hence, the following estimates of feed-gain relationships with fixed rations are interpolations of the derived livestock-feed surface based on the available 272 observations. Until further empirical verification has been completed from the newly designed experiments, great practical importance is not attached to these estimates as the basis for feeding recommendation. However, prediction from the current data of feeder calf gain from different "fixed" rations provides some insights into the relative efficiency of various rations.

Gain Equations for Given Ration Lines

Using the over-all gain equation, equation 13.1, it is possible to derive total and marginal gain equations for various rations (feed components, corn, L.O.M., and forage held in fixed proportions). For each total gain equation, the procedure used was to define algebraically all three feed variables in terms of a new composite variable, α. Then by substituting the redefined variables into the over-all equation, equation 13.1, a sub-function in terms of Y (gain) and the composite variable α is derived.

Total and marginal gain equations, derived in the manner outlined above, are given in Table 13.6. Eight different rations were selected to represent a cross section of the calf gain surface from which observations were available. Equation 13.12 in Table 13.6, for example, expresses the relationship between feeder calf gain in pounds and pounds of ration A (α_A). One hundred pounds of ration A (α_A) consists of 57 pounds of corn, 14 pounds L.O.M., and 29 pounds of forage. Equation 13.20 represents the marginal gain equation corresponding to the fixed gain equation 13.12. Equation 13.20 is derived by taking the first derivative of Y_A with respect to α_A. In other words, it represents the change in feeder calf gain with an infinitely small change in the quantity of ration A (α_A) fed.

Tables 13.7 and 13.8 are the numerical counterparts of the equations shown in Table 13.6. Table 13.7 shows predicted total feeder calf gain in pounds when the quantity of each ration is varied from 100 pounds to 4,000 pounds. For example, 100 pounds of ration A (α_A) substituted into equation 13.12 gives 20.9 pounds of gain (Y_A) indicated in Table 13.7 under "ration A" (across from 100 "pounds of feed"). The remainder of Table 13.7 is constructed similarly.

Table 13.8 includes predicted marginal gains for the several rations. Substituting 100 pounds of ration A (α_A), for example, into equation 13.12 yields .172 pounds of gain (Y_A), shown in Table 13.8 under

Table 13.6. Predicted Total and Marginal Gain Equations for Selected Rations Fed Good-to-Choice Feeder Calves Weighing 400 Pounds at the Outset

Feed Composition per 100 Pounds of Ration	Prediction Equation for Total Gain	Prediction Equation for Marginal gain
Ration A Corn - 57 lbs. L.O.M. - 14 lbs. Forage - 29 lbs.	(13.12) $Y_A = .174768\alpha_A - .00001040\alpha_A^2 + 3.587530$	(13.20) $\frac{dY_A}{d\alpha_A} = .174768 - .00002080\,\alpha_A$
Ration B Corn - 54 lbs. L.O.M. - 14 lbs. Forage - 32 lbs.	(13.13) $Y_B = .171393\alpha_B - .00000746\alpha_B^2 + 3.587530$	(13.21) $\frac{dY_B}{d\alpha_B} = .171393 - .00001492\,\alpha_B$
Ration C Corn - 50 lbs. L.O.M. - 12 lbs. Forage - 38 lbs.	(13.14) $Y_C = .166964\alpha_C - .00000457\alpha_C^2 + 3.587530$	(13.22) $\frac{dY_C}{d\alpha_C} = .166964 - .00000914\,\alpha_C$
Ration D Corn - 68 lbs. L.O.M. - 10 lbs. Forage - 22 lbs.	(13.15) $Y_D = .168657\alpha_D - .00001085\alpha_D^2 + 3.587530$	(13.23) $\frac{dY_D}{d\alpha_D} = .168657 - .00002170\,\alpha_D$
Ration E Corn - 61 lbs. L.O.M. - 9 lbs. Forage - 30 lbs.	(13.16) $Y_E = .162974\alpha_E - .00000714\alpha_E^2 + 3.587530$	(13.24) $\frac{dY_E}{d\alpha_E} = .162974 - .00001428\,\alpha_E$
Ration F Corn - 57 lbs. L.O.M. - 8 lbs. Forage - 35 lbs.	(13.17) $Y_F = .160079\alpha_F - .00000594\alpha_F^2 + 3.587530$	(13.25) $\frac{dY_F}{d\alpha_F} = .160079 - .00001188\,\alpha_F$
Ration G Corn - 53 lbs. L.O.M. - 8 lbs. Forage - 39 lbs.	(13.18) $Y_G = .156301\alpha_G - .00000532\alpha_G^2 + 3.587530$	(13.26) $\frac{dY_G}{d\alpha_G} = .156301 - .00001064\,\alpha_G$
Ration H Corn - 46 lbs. L.O.M. - 8 lbs. Forage - 46 lbs.	(13.19) $Y_H = .151264\alpha_H - .00000574\alpha_H^2 + 3.587530$	(13.27) $\frac{dY_H}{d\alpha_H} = .151264 - .00001148\,\alpha_H$

"ration A" and across from 100 "pounds of feed." This means that the predicted additional beef gain on a feeder calf for the 100th pound of ration A (α_A) is .172 pounds (i.e., .172 pounds of gain for the 100th pound of the specified ration). The other columns in Table 13.8 are constructed and interpreted similarly.

Feed Efficiency

In livestock feeding, the term "feed efficiency" has many meanings. A common definition of feed efficiency is pounds of gain per pound of ration. Using this definition, Table 13.7 would indicate that ration C

Table 13.7. Predicted Total Gains for Different Feed Combinations (Rations) With Good-to-Choice Feeder Calves Weighing 400 Pounds at the Outset.

Pounds of Feed	Total Gain in Pounds for Corn-Protein-Forage Rations* (A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	
100	20.9	20.7	20.2	20.3	19.81	19.54	19.16	18.7	
200	38.1	37.6	36.8	36.9	35.90	35.37	34.63	33.6	
300	55.1	54.3	53.3	53.2	51.84	51.08	50.00	48.5	
400	71.8	71.0	69.6	69.3	67.63	66.67	65.26	63.2	
500	88.4	87.4	85.9	85.2	83.29	82.14	80.41	77.8	
600	104.7	103.7	102.1	100.9	98.80	97.50	95.45	92.3	
700	120.8	119.9	118.2	116.3	114.17	112.73	110.39	106.7	
800	136.7	135.9	134.2	131.6	120.40	127.85	125.22	120.9	
900	152.5	151.8	150.2	146.6	144.48	142.85	139.95	135.1	
1,000	168.0	167.5	166.0	161.4	159.42	157.73	154.57	149.1	90 days
1,100	183.2	183.1	181.7	176.0	174.22	172.49	169.08	163.0	
1,200	198.3	198.5	197.4	190.4	188.87	187.13	183.49	176.8	
1,300	213.2	213.8	212.9	204.5	203.39	201.65	197.79	190.5	
1,400	227.9	228.9	228.4	218.4	217.76	216.06	211.98	204.1	
1,500	242.3	243.9	243.8	232.2	231.98	230.34	226.07	217.6	
1,600	256.6	258.7	259.0	245.7	246.07	244.51	240.05	230.9	
1,700	270.6	273.4	274.2	258.9	260.01	258.55	253.92	244.1	
1,800	284.5	287.9	289.3	272.0	273.81	272.48	267.69	257.3	
1,900	298.1	302.3	304.3	284.9	287.46	286.29	281.35	270.3	180 days
2,000	311.5	316.5	319.2	297.5	300.98	299.98	294.91	283.2	
2,100	324.7	330.6	334.1	309.9	314.34	313.56	308.36	295.9	
2,200	337.7	344.5	348.8	322.1	327.57	327.01	321.70	308.6	
2,300	350.5	358.3	363.4	334.1	340.66	340.35	334.94	321.1	
2,400	363.1	372.0	378.0	345.9	353.60	353.56	348.07	333.6	
2,500	375.5	385.4	392.4	357.4	366.40	366.66	361.09	345.9	
2,600	387.7	398.8	406.8	368.7	379.05	379.64	374.01	358.1	
2,700	399.6	412.0	421.1	379.9	391.57	392.50	386.82	370.2	
2,800	411.4	425.0	435.3	390.8	403.94	405.24	399.52	382.1	270 days
2,900	423.0	437.9	449.3	401.4	416.16	417.86	412.12	349.0	
3,000	434.3	450.6	463.3	411.9	428.25	430.36	424.61	405.7	
3,100	445.4	463.2	477.3	422.2	440.19	442.75	437.00	417.3	
3,200	456.3	475.7	491.1	432.2	451.99	455.01	449.27	428.9	
3,300	467.1	487.9	504.8	442.0	463.65	467.16	461.45	440.3	
3,400	477.6	500.1	518.4	451.6	475.16	479.19	473.51	451.5	
3,500	487.9	512.1	532.0	461.0	486.53	491.10	485.47	462.7	
3,600	498.0	523.9	545.4	470.1	497.76	502.89	497.32	473.7	
3,700	507.9	535.6	558.8	479.1	508.84	514.56	509.07	484.7	360 days
3,800	517.5	547.2	572.1	487.8	519.79	526.11	520.71	495.5	
3,900	527.0	558.6	585.2	496.3	530.59	537.55	532.24	506.2	
4,000	536.3	569.8	598.3	504.6	541.24	548.86	543.67	516.8	

*See Table 13.6 for total gain predicting equations and composition of the rations.

was the most efficient ration and ration D the least efficient. That is, the predicted gain for feeder calves fed 4,000 pounds of ration C is 598 pounds compared to 505 pounds for calves fed ration D. Likewise, the marginal gain for the 4,000th pound of ration C is .130 pounds compared to .081 pounds for the 4,000th pound of ration D. This comparison of "efficiency" of two rations is not entirely adequate. First, the gain

Table 13.8. Predicted Marginal Gains for Different Feed Combinations (Rations) With Good-to-Choice Feeder Calves Weighing 400 Pounds at the Outset

Pounds of Feed	Marginal Gain in Pounds for Corn-Protein-Forage Rations* (A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	
100	0.172	0.169	0.166	0.166	0.162	0.159	0.155	0.150	
200	0.170	0.168	0.165	0.164	0.160	0.158	0.154	0.148	
300	0.168	0.166	0.164	0.162	0.159	0.157	0.153	0.147	
400	0.166	0.165	0.163	0.159	0.157	0.155	0.152	0.146	
500	0.164	0.163	0.162	0.157	0.156	0.154	0.151	0.145	
600	0.162	0.162	0.161	0.155	0.154	0.153	0.150	0.144	
700	0.160	0.160	0.160	0.153	0.153	0.152	0.149	0.143	
800	0.158	0.159	0.159	0.151	0.152	0.151	0.148	0.142	
900	0.156	0.157	0.158	0.149	0.150	0.149	0.147	0.140	
1,000	0.153	0.156	0.158	0.146	0.149	0.148	0.146	0.139	
1,100	0.151	0.154	0.156	0.144	0.147	0.147	0.145	0.138	
1,200	0.149	0.153	0.155	0.142	0.146	0.146	0.144	0.137	
1,300	0.147	0.152	0.155	0.140	0.144	0.145	0.142	0.136	
1,400	0.145	0.150	0.154	0.138	0.143	0.143	0.141	0.135	200 lbs.
1,500	0.143	0.149	0.153	0.136	0.142	0.142	0.140	0.134	
1,600	0.141	0.147	0.152	0.133	0.140	0.141	0.139	0.132	
1,700	0.139	0.146	0.151	0.131	0.139	0.140	0.138	0.131	
1,800	0.137	0.144	0.150	0.129	0.137	0.139	0.137	0.130	
1,900	0.135	0.143	0.149	0.127	0.136	0.138	0.136	0.129	
2,000	0.133	0.141	0.148	0.125	0.134	0.136	0.135	0.128	
2,100	0.131	0.140	0.147	0.123	0.133	0.135	0.134	0.127	300 lbs.
2,200	0.129	0.138	0.146	0.120	0.132	0.134	0.133	0.126	
2,300	0.126	0.137	0.145	0.118	0.130	0.133	0.132	0.124	
2,400	0.124	0.135	0.145	0.116	0.129	0.132	0.131	0.123	
2,500	0.122	0.134	0.144	0.114	0.127	0.130	0.130	0.122	
2,600	0.120	0.132	0.143	0.112	0.126	0.129	0.129	0.121	
2,700	0.118	0.131	0.142	0.110	0.124	0.128	0.128	0.120	
2,800	0.116	0.129	0.141	0.107	0.123	0.127	0.127	0.119	
2,900	0.114	0.128	0.140	0.105	0.122	0.126	0.125	0.117	
3,000	0.112	0.126	0.139	0.103	0.120	0.124	0.124	0.116	400 lbs.
3,100	0.110	0.125	0.138	0.101	0.119	0.123	0.123	0.115	
3,200	0.108	0.123	0.137	0.099	0.117	0.122	0.122	0.114	
3,300	0.106	0.122	0.137	0.097	0.116	0.121	0.121	0.113	
3,400	0.104	0.120	0.135	0.094	0.114	0.120	0.120	0.112	
3,500	0.101	0.119	0.134	0.092	0.113	0.118	0.119	0.111	
3,600	0.099	0.117	0.134	0.091	0.112	0.117	0.118	0.110	
3,700	0.097	0.116	0.133	0.088	0.110	0.116	0.117	0.109	
3,800	0.095	0.114	0.132	0.086	0.109	0.115	0.116	0.108	500 lbs.
3,900	0.093	0.113	0.131	0.084	0.107	0.114	0.115	0.106	
4,000	0.091	0.111	0.130	0.081	0.106	0.113	0.114	0.105	

*See Table 13.6 for marginal gain predicting equations.

produced by these two rations is, as any cattle feeder knows, not identical in quality. Second, the time variable, which is also correlated with quality, confounds the comparison (i.e., it takes less time for a feeder calf to consume 100 pounds of a high concentrate ration than a bulky forage ration). As discussed previously, estimates of beef grades were not provided in data upon which this study is based. However, estimates

of time required by the feeder calf to consume various quantities of feed were made. Accordingly, the "horizontal" lines in Table 13.7 indicate the estimated time required by the feeder calf to consume various quantities of the fixed rations.[8] For example, 90 days is the time predicted for consuming 1,400 pounds of ration D (68 per cent corn, 10 per cent L.O.M., and 22 per cent forage). The corresponding gain is 218 pounds. For an equal time period, 1,000 pounds of ration H (46 per cent corn, 8 per cent L.O.M., and 46 per cent forage) produces a gain of 149 pounds. Thus, in the same time period; the predicted gain is 69 pounds greater for the heavy concentrate ration compared to the bulky forage ration. However, the feeder animal fed the concentrate ration (ration D) is predicted to consume an additional 400 pounds of feed.

TIME FACTOR IN BEEF GAINS

Previous sections have indicated how given beef gains can be produced with varying combinations of corn, L.O.M., and hay. However, as mentioned above, different proportions of these three feeds in a ration affects the rate at which the beef calf gains weight. Hence, rations that minimize costs may not necessarily coincide with the ration that produces the most rapid gains. As cattle feeders are well aware, time of marketing may affect profits as much or more than feed costs.

Time Function

To allow analysis of these considerations, a quadratic function was used to express the relationship between corn (C), L.O.M. (P), forage (F), and the number of days (T) required to consume various quantities of these feeds. The estimated relationship is indicated by equation 13.28 where

$$\begin{aligned} (13.28) \quad T = {} & .039328C + .076652P + .134446F - .00000130C^2 \\ & + .00019013P^2 + .00001578F^2 - .00008414CP \\ & + .00000568CF + .00001115PF + 3.095509 \end{aligned}$$

T is time in days for the feeder calf to consume a given quantity of the three feeds, starting from the beginning of the feeding period. Consecutive observations were taken at 28-day intervals on a given ration line, with each observation being the accumulated sum of previous time intervals. For example, T = 28 days for the first observation, with the second observation being the sum of the first time interval plus the second time interval or 56 days, etc.

C is total intake of corn in pounds from the time grain feeding is started to the particular weighing date (each weighing period is separated by a 28-day time

[8] Equation 13.28 in the following section provides the basis for these time estimates.

interval). Hence, the first observation of corn is from the beginning of the experiment to the first weighing date (28 days), the second corn observation is from the beginning of the experiment to the second weighing date (56 days), etc.

P and F are total intake of L.O.M. and forage measured in pounds. Observations are measured as outlined for corn.

The coefficient of determination for equation 13.28 is .99, indicating that the three feed variables explain a major proportion of variation in the dependent variable T. Standard errors of the regression coefficients for equation 13.28 are given in Table 13.9.

Table 13.9. Standard Errors for Regression Coefficients in Equation 13.28

Regression Coefficient for	Standard Error
C	.00392100
P	.01790900
F	.00922700
C^2	.00000377
P^2	.00004390
F^2	.00000345
CP	.00001669
CF	.00000591
PF	.00002372

Time Equations for Specific Rations

Equation 13.28, the over-all time function, was reduced to several individual time equations; one for each of the eight different feed combinations or rations discussed in the previous section. These time equations are shown in Table 13.10. Estimates of the total time required to consume quantities of feed for the eight different rations are shown in Table 13.11.

Table 13.11 shows, as previous research would suggest, that the amount of time required to consume a "bulky" roughage ration is greater than that required to consume the same amount of a higher concentrate ration. More specifically, calves require 392 days to consume 4,000 pounds of ration H (1:1 corn-forage ratio). Calves fed ration D (3:1 corn-forage ratio) require only 221 days to consume the same quantity of feed. The quantity of L.O.M. in the ration appears to have little effect on the time required for the calf to consume the ration. Rations A and E both have approximately a 2:1 corn-forage ratio. However, ration A contains 14 pounds of L.O.M., as compared to 9 pounds for ration E, per hundred pounds of total feed (see Table 13.10). The predicted time for feeder calves to consume 4,000 pounds of ration A is 281 days, about the same number of days required for feeder calves to consume an equal quantity of ration E.

Table 13.10. Predicted Time Equations for Good-to-Choice Feeder Calves to Consume Selected Rations Indicated in Table 13.6

Feed Composition per 100 Pounds of Ration		Predicted Time Equations in Days
Ration A	(13.29)	$T_A = .072138\,\alpha_A - .00000069\,\alpha_A^2 + 3.095509$
Ration B	(13.30)	$T_B = .074991\,\alpha_B + .00000008\,\alpha_B^2 + 3.095509$
Ration C	(13.31)	$T_C = .079663\,\alpha_C + .00000119\,\alpha_C^2 + 3.095509$
Ration D	(13.32)	$T_D + .063986\,\alpha_D - .00000256\,\alpha_D^2 + 3.095509$
Ration E	(13.33)	$T_E = .071223\,\alpha_E - .00000080\,\alpha_E^2 + 3.095509$
Ration F	(13.34)	$T_F = .074261\,\alpha_F + .00000012\,\alpha_F^2 + 3.095509$
Ration G	(13.35)	$T_G = .079410\,\alpha_G + .00000121\,\alpha_G^2 + 3.095509$
Ration H	(13.36)	$T_H = .086068\,\alpha_H + .00000279\,\alpha_H^2 + 3.095509$

Least-Cost Rations

The methods used in previous chapters could be used to predict least-cost rations and optimum marketing weights. However, these steps are not taken here because of space limitations and because new 3-year experiments are being conducted to allow a better basis for such estimates.

APPENDIX: Reduced Gain Equation for Given Ration Line

The following notes explain the method of deriving the quantities in tables 13.6 to 13.8 from equation 13.1.

With a feed combination of X per cent corn, K per cent L.O.M., and Z per cent forage, a new variable (α) is defined with the following conditions:

$$\text{Corn (C)} = X_i\alpha_i$$

$$\text{Protein (L.O.M.)} = K_i\alpha_i$$

$$\text{Forage (F)} = Z_i\alpha_i\ .$$

Substituting the above conditions into equation 13.1 gives the general form of the reduced equation in terms of Y_i and α_i.

$$\begin{aligned}(13.37)\quad Y_i = {} & .157402(X_i)\alpha_i + .361070(K_i)\alpha_i + .112332(Z_i)\alpha_i \\ & - .00001612(X_i^2)\alpha_i^2 - .00090147(K_i^2)\alpha_i^2 - .00005958(Z_i^2)\alpha_i^2 \\ & - .00003612(X_iK_i)\alpha_i^2 - .00000637(X_iZ_i)\alpha_i^2 \\ & + .00054144(K_iZ_i)^2\alpha_i^2 + 3.58753\end{aligned}$$

Table 13.11. Predicted Total Time Required for Good-to-Choice Feeder Calves, Weighing 400 Pounds at the Outset, to Consume Various Fixed Rations

Pounds of Feed	(A)	(B)	(C)	(D)	(E)	(F)	(G)	(H)	
	Total Days Required To Feed Quantities of Rations*								
100	10.3	10.6	11.1	9.5	10.2	10.6	11.0	11.7	
200	17.5	18.1	19.1	15.9	17.3	18.1	19.0	20.4	
300	24.7	25.6	27.1	22.2	24.4	25.6	27.0	29.2	
400	31.8	33.1	35.2	28.4	31.5	33.1	35.1	38.0	
500	39.0	40.6	43.2	34.6	38.5	40.5	43.1	46.8	
600	46.1	48.1	51.3	40.8	45.5	48.0	51.2	55.7	
700	53.3	55.6	59.4	46.9	52.6	55.5	59.3	64.7	
800	60.4	63.1	67.6	53.0	59.6	63.0	67.4	73.7	
900	67.5	70.7	75.8	59.0	66.5	70.4	75.5	82.8	
1,000	74.5	78.2	84.0	65.0	73.5	77.9	83.7	92.0	
1,100	81.6	85.7	92.2	70.9	80.5	85.4	91.9	101.1	
1,200	88.7	93.2	100.4	76.7	87.4	92.8	100.1	110.4	
1,300	95.7	100.7	108.7	82.6	94.3	100.3	108.4	119.7	
1,400	102.7	108.2	117.0	88.4	101.2	107.7	116.6	126.0	200 lb.
1,500	109.8	115.8	125.3	94.1	108.1	115.2	124.9	139.5	gain
1,600	116.8	123.3	133.6	99.7	115.0	122.6	133.2	147.9	
1,700	123.7	131.7	142.0	105.4	121.9	130.0	141.6	157.5	
1,800	130.7	138.3	150.4	110.9	128.7	137.5	150.0	167.1	
1,900	137.7	145.9	158.8	116.4	135.5	144.9	158.3	176.7	
2,000	144.6	153.4	167.2	121.9	142.3	152.4	166.8	186.4	
2,100	151.5	160.9	175.7	127.4	149.1	159.8	175.2	196.1	300 lb.
2,200	158.5	168.5	184.1	132.7	155.9	167.2	183.7	205.9	gain
2,300	165.4	176.0	192.6	138.0	162.7	174.6	192.1	215.8	
2,400	172.3	183.5	201.2	143.3	169.4	182.1	200.6	225.7	
2,500	179.1	191.1	209.7	148.6	176.2	189.5	209.2	235.7	
2,600	186.0	198.6	218.3	153.7	182.9	196.9	217.7	245.7	
2,700	192.8	206.2	226.9	158.8	189.6	204.3	226.3	255.8	
2,800	199.7	213.7	235.5	163.9	196.2	211.7	234.9	266.0	
2,900	206.5	221.2	241.4	170.0	202.9	219.1	243.6	276.2	
3,000	213.3	228.8	252.8	173.9	209.6	226.5	252.2	286.4	400 lb.
3,100	220.1	236.3	261.5	178.9	216.2	233.9	260.9	296.7	gain
3,200	226.9	243.9	270.2	183.8	222.8	241.3	269.6	307.1	
3,300	233.6	251.4	279.0	188.6	229.4	248.7	278.3	317.5	
3,400	240.4	259.0	287.7	193.4	236.0	256.1	287.1	328.0	
3,500	247.1	266.5	296.5	198.1	242.6	263.5	295.9	338.5	
3,600	253.8	274.1	305.3	202.8	249.1	270.9	304.7	349.1	
3,700	260.6	281.7	314.2	207.4	255.7	278.2	313.5	359.7	
3,800	267.3	289.2	323.0	212.0	262.2	285.6	322.3	370.4	500 lb.
3,900	273.9	296.8	331.9	216.5	268.7	292.9	331.2	381.2	gain
4,000	280.6	304.3	340.8	221.0	275.2	300.4	340.1	392.0	

*See Table 13.10 for prediction equation corresponding to each ration.

Simplifying 13.37 yields:

$$(13.38)\quad Y_i = \alpha_i(.157402X_i + .361070K_i + .112332Z_i) + \alpha_i^2(-.00001612X_i^2 - .00090147K_i^2 - .00005958Z_i^2 - .00003612X_iK_i - .00000637X_iZ_i + .00054144K_iZ_i) + 3.58753\ .$$

For example, with 100 pounds of ration (i.e., feeds combined in fixed proportions) composed of 57 pounds of corn, 14 pounds of L.O.M., and 28 pounds of forage, X = .57, K = .14, and Z = .28.

Substituting the above values for X, K, and Z into equation 13.38 gives:

$$(13.39)\qquad Y_A = .174768\alpha_A - .00001040\alpha_A^2 + 3.587530\ .$$

In addition, marginal gain was predicted for each ration by taking the total derivative of Y_i (gain in pounds) with respect to α_i (feed variable with components held in fixed proportions). For predicted gain equation 13.39 the marginal gain equation becomes:

$$(13.40)\qquad \frac{dY_A}{d\alpha_A} = .174768 - .00002080\alpha_A\ .$$

Similarly, total and marginal gain equations were derived for seven other rations (see tables 13.6 and 13.7) with constant proportions of corn, protein, and forage.

Recent Data

Preliminary results of an experiment conducted with 300 steers over the period 1956-59 are similar to those presented above. Two major inputs, pasture forage (F) in dry weight and corn supplemented with protein (C), were fed along six ration lines. One resulting function, computed through the origin, is as follows:

$$(13.41)\qquad \begin{aligned} Y = \ & \underset{(.0127)}{.1246C} + \underset{(.0019)}{.0231F} - \underset{(.000005)}{.00001240^2} \\ & - \underset{(.0000002)}{.00000111F^2} - \underset{(.0000015)}{.00000246CF} - \underset{(.4033)}{1.3631T} \end{aligned}$$

where Y is gain in pounds and T is temperature above the summer mean. R^2 is .995. Isoquants again indicate that large animals readily substitute one feed for another. While details will be explained elsewhere, it is interesting to note that temperature is a significant explanatory variable, gains being less as temperature rises.

CHAPTER 14

Earl O. Heady
John T. Pesek
William G. Brown
John P. Doll

Crop Response Surfaces and Economic Optima in Fertilizer Use

THIS CHAPTER reports initial research on crop response surfaces initiated in Iowa in 1952. Since the original experiment in 1952, many additional studies employing several types of designs have been completed for different crops at various locations in the state. Results from a few of these experiments are reported in Chapters 14 and 15 to illustrate (a) the nature of estimated fertilizer response surfaces, (b) the nature of isoclines and other relationships derived for various soil and moisture conditions by different algebraic functions, and (c) the optimum quantities and use of fertilizer as specified by particular analysis of response. Experiments which have been completed include those with two of the nutrients N and K_2O and P_2O_5 variable, with all three variable, and with one or more nutrients variable while stand, moisture, or certain other inputs also are variable. Only a few examples from the many experiments will be discussed in these two chapters.

Fertilizer is an important area for production function research. Greater use of fertilizer has been one of the important developments in agriculture over the past decades. It is an input which is clearly divisible and for which profit-maximizing principles, for either limited or unlimited capital situations, can easily be applied. Because of space limitations, however, only a select set of the economic principles discussed in Chapter 2 have been applied to the several sets of quantitative results from production function studies.

In conformity with accepted economic terminology, nutrient combinations can be expressed in terms of their substitution or replacement rates. In the chemical processes of the plant one element may not substitute for another; however, it is true that moderate yield increases may be attained with several combinations of elements. A farmer may obtain a 5-bushel increase in corn from use of ammonium nitrate alone, from phosphate alone, or from a mixed fertilizer such as 20-20-0 or 8-8-8. If all of the mixtures give the 5-bushel increase, they can be looked upon as substitutes for each other in attaining the given yield even though physiological substitution does not actually take place. Elements Na and K may be real substitutes over wide ranges in the chemical processes of some plants. However, even though plant nutrients such as N, P, or K do not directly serve as substitutes in the chemical

functions of the plant, the fact that similar yield increases can be attained with different combinations of nutrients causes them to serve as substitutes in the decision-making framework of the farmer. Within limits, he can use more of one nutrient and less of another in attaining yield increases under many soil situations. While the terms "substitution" or "replacement rates" thus may not represent an entirely accurate physiological concept, they are employed in the remainder of this study in the absence of more appropriate terms. While substitution is discussed in forthcoming sections, the biological exceptions mentioned above should be kept in mind. From the standpoint of fertilizer ratios, the problem is perhaps as much one of finding "optimum combinations of nutrients" (least-cost combinations for a given yield) as in determining "substitution" rates.

INITIAL EXPERIMENTS

The experiments reported in this chapter were conducted in 1952 with corn on calcareous Ida silt loam in western Iowa and with alfalfa and red clover on Webster and Nicollet loam in north central Iowa. Major emphasis will be given to corn on Ida silt loam. Two variable nutrients were used on each experiment. Nitrogen in the form of ammonium nitrate and P_2O_5 in the form of concentrated superphosphate were applied to corn while K_2O in the form of potassium chloride and P_2O_5 in the form of concentrated superphosphate were applied to both alfalfa and red clover. Observations were obtained from an incomplete factorial experimental design of the nature indicated by Table 14.1.

The same design was used for alfalfa and red clover except that the second variable nutrient was K_2O. With replication, there were 114 observations for each of the three experiments.[1] Two cuttings were

Table 14.1. Design of Experiment for Corn; Each "X" Represents an Experimental Plot

Pounds P_2O_5 per Acre	Pounds Nitrogen per Acre								
	0	40	80	120	160	200	240	280	320
0	XX	XX	XX	XX	XX	XX	XX	XX	XX
40	XX	XX			XX	XX			XX
80	XX		XX		XX		XX		XX
120	XX			XX	XX			XX	XX
160	XX	XX	XX	XX	XX	XX	XX	XX	XX
200	XX	XX			XX	XX			XX
240	XX		XX		XX		XX		XX
280	XX			XX	XX			XX	XX
320	XX	XX	XX	XX	XX	XX	XX	XX	XX

[1] The treatments were assigned at random (completely randomized block design).

obtained from both the alfalfa and red clover. Yield measurements for hay were in terms of 12 per cent moisture. This design, with randomized plots, allows continuous observations at the extremes of application rates with combinations of the various nutrients. It also provides sufficient observations over other points of the production surface for estimation of the two-variable nutrient function. In the experiments, all resources or inputs except fertilizer were held constant except for the variable quantities of labor and machine services for application and harvesting; seeding rates were constant.

Weather in Experimental Year

The 1952 growing year was one favorable for the use of fertilizer. The spring was fairly cool and wet. Rainfall was ample to mid-August when a 2-month drought began. For these reasons, the experimental data do not necessarily serve as a basis for inference to average years. On corn the constant plant population of 18,000 plants per acre for all treatments may have limited the response obtained from heavy fertilization rates.

We again point out that fertility nutrients may not substitute in the biological processes involved in producing a given amount of a specified part of the plant. However, they do serve as substitute means of attaining specified yield responses. With corn, for example, an average yield of 24.8 bushels per acre was obtained on the plots receiving 120 pounds of nitrogen and no P_2O_5. The plots receiving 40 pounds of P_2O_5 averaged 28.6 bushels. With slightly fewer pounds of phosphoric acid, equal increments in yield might have been attained with entirely different nutrient combinations of nitrogen and P_2O_5. With 160 pounds of N and 40 pounds of P_2O_5, the plots averaged 101.5 bushels; with 240 pounds of P_2O_5 and 80 pounds of N, the average was 102.5 bushels. Similarly, for clover, the plots receiving 120 pounds of P_2O_5 and 160 pounds of K_2O averaged 3.66 tons while those receiving 160 pounds of P_2O_5 and 80 pounds of K_2O averaged 3.68 tons. Thus, while the nutrients may not serve as substitutes in the chemical process of the plants, they do serve as substitute means of attaining given yield increases. These are the kind of data needed in farmer decision-making; it is the cost of producing a given yield, rather than the chemical process itself, which directly concerns him.

While nutrients may serve as substitutes over a limited range in attaining given levels of crop response, the data also show how they eventually serve as technical complements as one is increased alone. By technical complementarity, we refer to the situation where an increase in one element without an increase in the other either (a) does not add anything to total yield or (b) actually decreases total yield. On corn, for example, any increase in N alone (a path followed horizontally from left to right in Table 14.1 with P_2O_5 held "fixed" at any level in the table) causes first an increase and then a decrease in total yield. The

same situation holds true for P_2O_5. That is, yield increases and then decreases down the column of the table with N fixed at specified levels, and this decrease indicates that N also is a limitational nutrient with P_2O_5. That the two nutrients serve as limitational resources or technical complements to each other also was illustrated by the fact that yields were taken to successively higher levels with diagonal movements from northwest to southeast in the table; under this "movement" over the cells and columns of the table, the two elements are, in effect, increased simultaneously and in fixed proportions.

DERIVATION OF PRODUCTION OR YIELD FUNCTIONS

After collection of yield observations, the next step was that of deriving production functions, input-output or response coefficients. Since, at the time of this experiment, practically no empirical analysis had been made of fertilizer production surfaces, numerous functional forms were fitted to the data. For surface estimates, forms employed included the Cobb-Douglas and general polynomial forms, all with linear terms but variously using exponents representing powers of .5, 1.5, and 2. The first will be termed a square root equation; the second, a "1.5" or "3/2" function, and the latter, a quadratic. Some estimates were made for individual nutrients where Spillman types of functions also were fitted to the data. In the equations which follow, Y refers to total bushels or tons per acre for corn or hay respectively, Y′ refers to yield above check plot, while N, P, and K refer, respectively, to pounds per acre of nitrogen, P_2O_5, and K_2O.

PRODUCTION FUNCTIONS FOR CORN

The variables for corn on Ida silt loam were nitrogen and P_2O_5. The experimental field, operated by a farmer in the usual manner except for fertilization, was very low in fertility. The four production surface equations estimated for corn are:

$$(14.1)\quad Y' = .442P^{.4090}N^{.2877}$$

$$(14.2)\quad Y = -7.51 + .584N + .664P - .0016N^2 - .0018P^2 + 00081NP$$

$$(14.3)\quad Y = -5.68 - .316N - .417P + 6.3512N^{.5} + 8.5155P^{.5} + .3410N^{.5}P^{.5}$$

$$(14.4)\quad Y = -13.62 + .984N + 1.129P - .0500N^{1.5} - .0576P^{1.5} + .0008NP\ .$$

The R and t values for the four functions are shown in Table 14.2. All of these are significant at a one per cent level of probability.

Table 14.2. Values of R for Two-Variable Nutrients and Values of t for Individual Regression Coefficients

Equation	Value of R	Value of t for Coefficients in Order Listed in Equations				
14.1	.9255*	18.62*	15.23*	--	--	--
14.2	.9122*	9.21*	10.46*	5.24*	8.96*	10.19*
14.3	.9582*	7.91*	10.44*	7.32*	9.81*	8.85*
14.4	.9434*	8.21*	10.35*	6.21*	7.31*	10.12*

*$0 < P < 0.01$.

Analyses of regression for three functions are shown in Table 14.3. Data are not included for the Cobb-Douglas function since it was not fitted to all yield observations. It was fitted by subtracting the average yield of check plots from the yield of plots with fertilizer above zero level. Observations for yields smaller than check plot levels were thus omitted and the R^2 for Cobb-Douglas, .86, based on the total sums of squares associated with the 105 observations to which the function was fitted cannot be compared with those of other equations. The proportions of variance in corn yield, R^2, explained by variables in the other three equations are .86 for equation 14.2, .91 for equation 14.3, and .89 for equation 14.4. The percentage of treatment sums of squares explained by regression equations decreases as the power of the equation increases. The mean squares for derivations from regressions were 625, 215, and 495, respectively, for these three equations. Using these and other statistics, it was decided that square root equation 14.3 served most efficiently in predicting the corn production surface. All three of the last mentioned equations have negative yield intercepts, with the largest negative value being for the equation with the 1.5 powers on N and P. Since the constants are statistically significant,

Table 14.3. Analyses of Variance of Regression for Equations Fitted to All Yield Observations, Ida Silt Loam, 1952

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Squares
Total	113	242,707	4,175
Treatments	56	233,811	
Due to regression of quadratic equation 14.2	(5	201,943	40,389
Deviations from regression	(51	31,868	625
Due to regression of square root equation 14.3	(5	222,828	44,566
Deviations from regression	(51	10,983	215
Due to regression of "3/2" equation 14.4	(5	208,582	41,716
Deviations from regression	(51	25,229	495
Among plots treated alike	57	8,896	156

they were retained in the equations, although the functions might have appeared more sensible if they had been forced to have zero intercepts. However, the magnitude of the constant does not affect marginal productivities and the recommendations which arise from curves which might have different constants but the same regression coefficients. The analysis relating to optima which follows is presented accordingly. However, in a later section, we analyze regression equations based only on non-zero fertilizer treatments.

Although the square root functions were selected as most appropriate for specifying economic optima in this section, we do turn to some comparisons of estimates made by the four equations. Specification of economic optima then follow in a later section.

COMPARISON OF PREDICTED YIELDS FROM COMPLETE PRODUCTION SURFACES FOR CORN

Corn yields predicted from equations 14.1 to 14.4 are included in Table 14.4. Production surfaces representing these predicted yields geometrically in three dimensions are presented in figures 14.1 through 14.4. Predictions of the equations differ considerably. The quadratic and "1.5" functions are most nearly alike. The square root and Cobb-Douglas functions differ most.

The surface for the square root function rises more rapidly than for

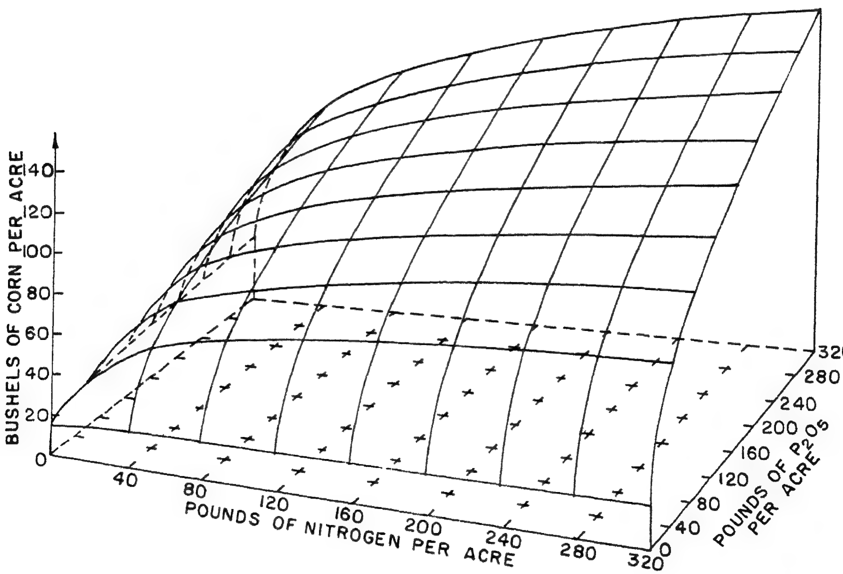

Figure 14.1. Production surface for corn predicted by Cobb-Douglas function 14.1, Ida silt loam, 1952, based on yield above check plots

Table 14.4. Bushels of Corn per Acre Predicted by Four Types of Production Functions for Specified Nitrogen and P_2O_5 Applications, Ida Silt Loam, 1952

Pounds of P_2O_5 per Acre	Pounds of Nitrogen per Acre								
	0	40	80	120	160	200	240	280	320
					Cobb-Douglas function 14.1				
0	15.4	18.5	19.2	19.7	20.0	20.4	20.6	20.9	21.1
40	21.8	51.5	59.5	65.0	69.2	72.8	75.9	78.6	81.1
80	24.0	63.4	74.0	81.2	86.9	91.6	95.7	99.4	102.6
120	25.5	72.0	84.5	93.1	99.9	105.4	110.2	114.5	118.4
160	26.8	79.1	93.1	102.8	110.3	116.6	122.0	126.9	131.2
200	27.8	85.2	100.6	111.1	119.4	126.3	132.2	137.5	142.3
240	28.8	90.6	107.2	118.5	127.5	134.9	141.3	147.0	152.1
280	29.7	95.5	113.1	125.2	134.7	142.6	149.5	155.5	161.0
320	30.5	100.0	118.6	131.4	141.4	149.8	157.0	163.4	169.2
					Quadratic function 14.2				
0	-7.5	13.3	29.2	39.8	45.5	46.1	41.7	32.2	17.6
40	16.2	38.3	55.4	67.4	74.4	76.3	73.1	64.9	51.6
80	34.1	57.5	75.9	89.2	97.5	100.7	98.8	91.9	79.9
120	46.3	71.0	90.7	105.3	114.9	119.4	118.8	113.2	102.5
160	53.0	78.7	99.8	115.6	126.6	132.3	133.0	128.7	119.3
200	53.4	80.7	103.0	120.2	132.3	139.4	141.5	138.4	130.4
240	48.3	76.9	100.5	119.0	132.5	140.8	144.1	142.4	135.7
280	37.5	67.4	92.3	112.0	126.8	136.5	141.1	140.7	135.2
320	20.9	52.1	78.3	99.4	115.4	126.4	132.3	133.2	129.0
					Square root function 14.3				
0	-5.7	21.8	25.8	25.9	24.0	20.9	16.8	12.1	6.8
40	31.5	72.6	82.3	88.7	88.5	88.6	87.4	85.3	82.5
80	37.1	83.9	95.9	102.1	105.4	106.8	106.9	105.9	104.1
120	37.5	88.7	102.4	110.1	114.5	116.9	117.9	117.8	116.8
160	35.3	90.1	105.4	114.2	119.6	122.9	124.6	125.2	124.9
200	31.6	89.3	105.9	115.7	122.0	126.1	128.5	129.7	130.0
240	26.1	87.0	104.8	115.6	122.6	127.4	130.4	132.2	133.0
280	19.9	83.6	102.5	114.1	121.9	127.2	130.8	133.2	134.5
320	13.1	79.2	99.2	111.5	120.0	126.0	130.1	132.9	134.7
					"1.5" power function 14.4				
0	-13.6	13.1	29.3	38.8	42.7	41.9	36.8	27.8	15.3
40	16.9	44.9	62.5	73.2	78.4	78.9	75.1	67.4	56.2
80	35.4	64.7	83.6	95.6	102.1	103.8	101.3	95.0	85.0
120	46.0	76.6	96.8	110.1	117.9	120.9	119.7	114.6	106.0
160	50.3	82.2	103.6	118.2	127.3	131.6	131.7	127.9	120.6
200	49.1	82.3	105.0	120.9	131.3	136.9	138.3	135.8	129.7
240	42.9	77.4	101.4	118.6	130.3	137.2	139.9	138.7	133.9
280	32.3	68.1	93.4	111.9	124.9	133.1	137.0	137.1	133.6
320	17.6	54.7	81.3	101.0	115.3	124.8	130.1	131.4	129.2

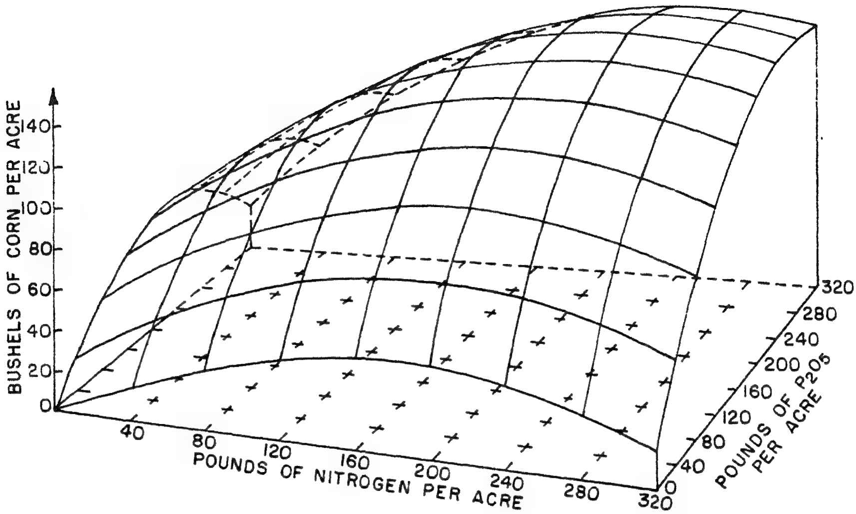

Figure 14.2. Production surface for corn predicted by quadratic function 14.2, Ida silt loam, 1952

the other functions when nutrient applications are small. But as input levels increase the square root surface has less slope than those of the other functions. When either of the nutrients is at a zero rate, so that interactions are not present, the square root function predicts yields

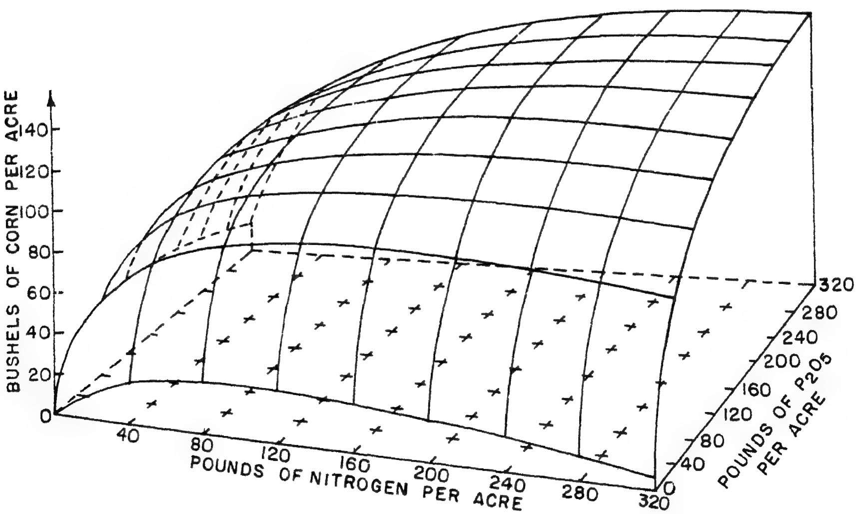

Figure 14.3. Production surface for corn predicted by square root function 14.3, Ida silt loam, 1952

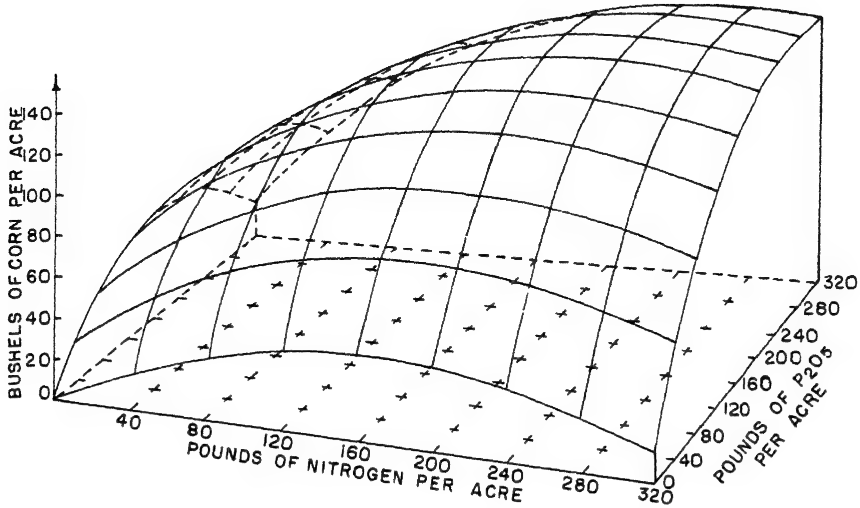

Figure 14.4. Production surface for corn predicted by "1.5" function 14.4, Ida silt loam, 1952

which are of a smaller magnitude. It has a surface which attains a maximum at lower input levels than other functions. When interactions are present, the square root function again predicts maximums, in reading across a row or down a column of the tables, at lower yield levels but at higher input levels.

The quadratic and "1.5" functions have lower yield intercept values than the square root function. While yields from the square root increase at initially high rates but decrease rapidly as the maximum yield is approached, these two functions increase to the maximum at fairly uniform rates. As explained in Chapter 3, the quadratic has a linear marginal product equation, indicating a constant deceleration. These two functions predict higher maximum yields at lower input levels than the square root function. The quadratic function predicts a maximum yield at input and yield levels slightly higher than the "1.5" function.

Predictions from the Cobb-Douglas function are subject to constant elasticity and thus do not attain a maximum. As mentioned previously, this function is not entirely comparable with the other equations. When both nutrients are present, predictions of the Cobb-Douglas function increase relative to the other functions until they surpass all other predictions at the highest input rates. For example, at 320-pound application rates for nitrogen and P_2O_5, the yield prediction for the Cobb-Douglas function is 169.2 bushels per acre, comparable figures for the square root, quadratic, and "1.5" functions are 134.7, 129.0, and 129.2 bushels, respectively.

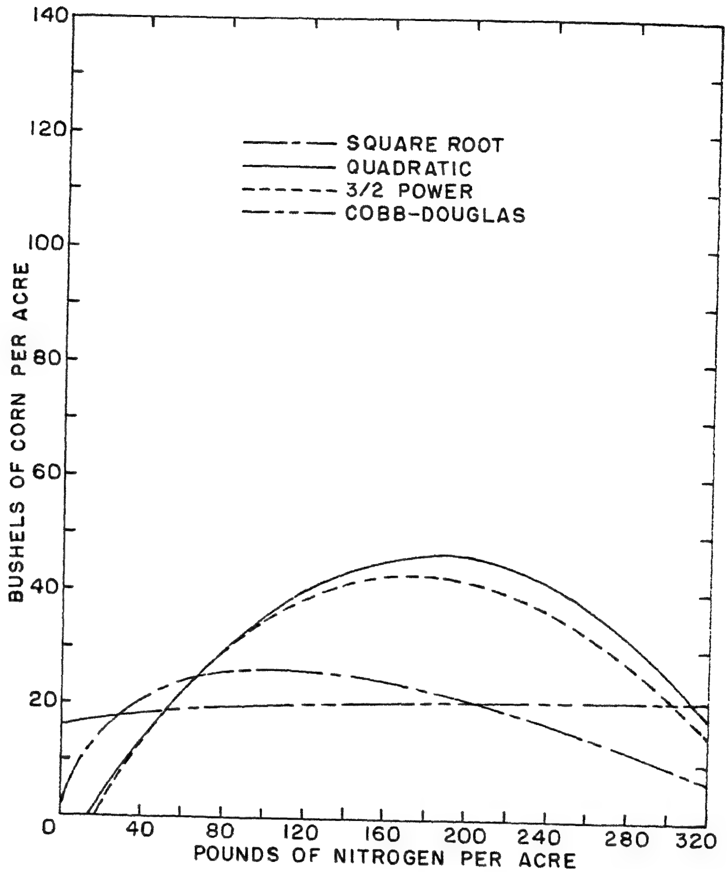

Figure 14.5. Corn yield response to nitrogen when P_2O_5 is constant at 0 pounds per acre predicted by four types of production functions, Ida silt loam, 1952

To show these differences more clearly, some curves of yield responses, with one input constant, are presented in figures 14.5, 14.6, and 14.7, remembering the Cobb-Douglas is based on Y.'

Marginal Physical Products and Yield Maxima

Functions 14.5 through 14.8 are marginal physical product equations for nitrogen, corresponding to equations 14.1 through 14.4. Marginal physical product equations, derivatives of production functions, predict the slope or rate of change of yield predictions of the functions for any rate of nutrient input. Interactions between the nutrients cause all of the marginal product equations to include a P_2O_5 term. Hence, the magnitude of the marginal product of one nutrient is dependent upon the

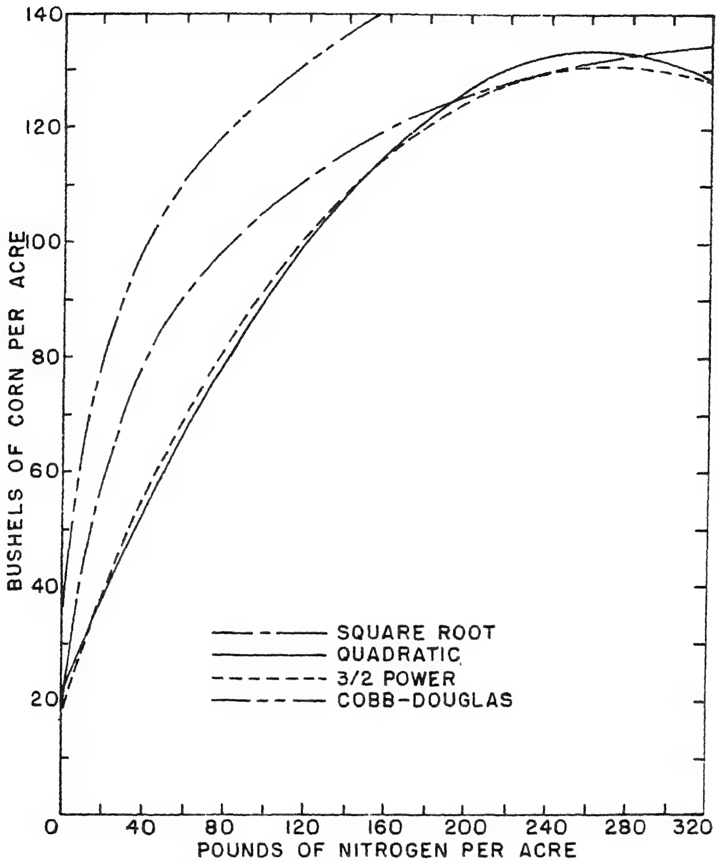

Figure 14.6. Corn yield response to nitrogen when P_2O_5 is constant at 320 pounds per acre predicted by four types of production functions, Ida silt loam, 1952

$$\frac{\delta Y'}{\delta N} = .2877N^{-1}Y' \tag{14.5}$$

$$\frac{\delta Y}{\delta N} = .5843 - .0032N + .0008P \tag{14.6}$$

$$\frac{\delta Y}{\delta N} = .3161 + 3.1756N^{-.5} + .1705P^{.5}N^{-.5} \tag{14.7}$$

$$\frac{\delta Y}{\delta N} = .9839 - .0749N^{.5} + .0008P \tag{14.8}$$

level at which the other is considered to be fixed. The marginal product equation for the square root function is discontinuous at zero input rates and forms a curve convex to the origin. When P_2O_5 is zero,

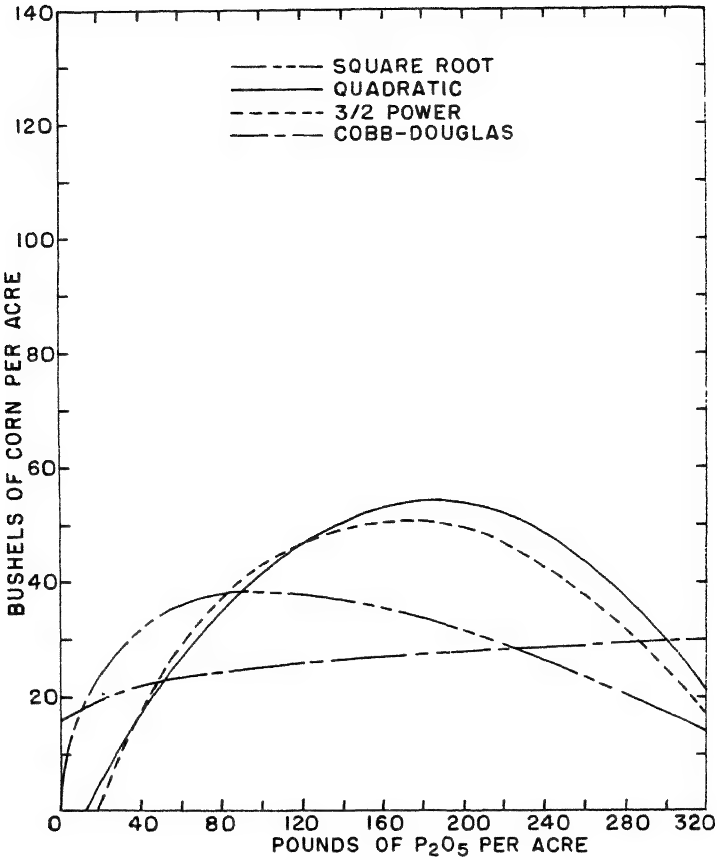

Figure 14.7. Corn yield response to P_2O_5 when nitrogen is constant at 0 pounds per acre predicted by four types of production functions, Ida silt loam, 1952

marginal products of nitrogen predicted by the square root function are at first larger (the marginal product of 1 pound of nitrogen is 2.86 bushels) but rapidly decrease and are smaller than marginal products of the other polynomials for input rates between 40 and 120 pounds. The square root function does not have marginal products which "diminish as markedly" as the other polynomials. Hence, its marginal product decreases less rapidly as nutrient inputs increase to high levels. Its marginal products again become larger at these high levels. As the rate of P_2O_5 applications increase, similar relationships are present but the size of the marginal products for the square root function increase relative to those of other functions.

Marginal products predicted by the quadratic and the "1.5" function are similar. Those for the "1.5" function are larger than those of the quadratic function for nitrogen inputs up to 40 pounds, smaller between

80 and 200 pounds, about equal at 240 pounds, and larger for higher nitrogen inputs. In relation to the polynomials, marginal products predicted by the Cobb-Douglas function are lower for low input rates but increase consistently until they are well above all others at high input rates.

Setting the marginal product equations for both nutrients equal to zero and solving simultaneously, the rates of nutrient application which maximize yield can be obtained for 14.2, 14.3, and 14.4. The square root function predicts a maximum yield of 135.7 bushels of corn per acre with nitrogen and P_2O_5 rates of 398 and 337 pounds per acre, respectively. This predicted yield is not as large as some yields actually obtained in the experiment but both input rates are outside the range of observations and should be interpreted accordingly. The quadratic function predicts a maximum of 144.2 bushels per acre with 246 and 240 pounds per acre, respectively, of nitrogen and P_2O_5. The predicted maximum for the "1.5" function is 139.7 bushels, with 244 pounds of nitrogen and 235 pounds of P_2O_5. The highest average yield obtained on the experimental plots, 141.3 bushels, was obtained with 200 pounds of each nutrient. Input combinations derived for maximum yields are well above this 200 pound figure but predicted maximum yields are all reasonably close to the realized yield figure. Evidently the production surfaces predicted by different equations display similar characteristics at similar yield levels, but at different nutrient input levels.

Yield Isoquants and Marginal Rates of Substitution

Isoquant equations 14.9 through 14.12 were derived from the production functions in 14.1 through 14.4, respectively. In these equations, P_2O_5 is expressed as a function of nitrogen and yield.

$$\log P = \frac{\log Y' - .2877 \log N - \log .4417}{.4090} \tag{14.9}$$

$$\begin{aligned} P = 184.7001 + .2257N \pm 278.2415(&.3867 \\ &+ .0053N - .00001N^2 - .0072Y)^{.5} \end{aligned} \tag{14.10}$$

$$\begin{aligned} P = [10.2007 + .4088N^{.5} \pm 1.1979(&16.4115N^{.5} \\ &- .4115N - 1.6696Y + 63.0275)^{.5}]^2 \end{aligned} \tag{14.11}$$

$$\begin{aligned} .0576P^{1.5} - (1.1285 + .0008N)P - [&-13.6238 \\ &+ .9839N - .0500N^{1.5} - Y] = 0 \end{aligned} \tag{14.12}$$

Isoquants predicted by these equations are presented in figures 14.8 through 14.11, respectively. Convex to the origin, all sets of isoquants indicate decreasing marginal rates of nutrient substitution. They are

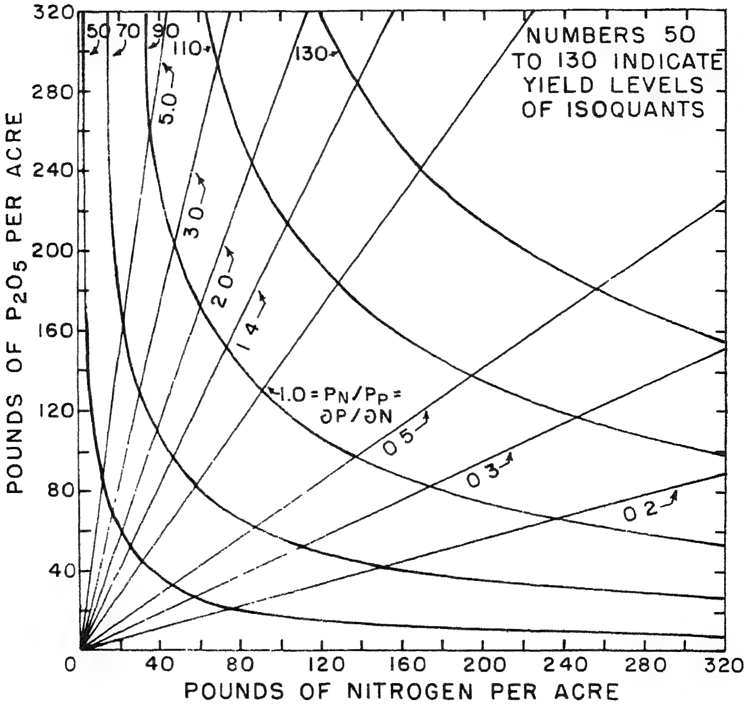

Figure 14.8. Isoquants and isoclines for corn predicted by Cobb-Douglas function 14.1, Ida silt loam, 1952

successively farther apart, on any straight line through the origin, indicating decreasing returns to fertilizer inputs as the yield level increases. Isoquants of the square root function appear almost horizontal for considerable distances as they approach the ridgelines but curve sharply at their centers. The larger distance between isoquants for the 110- and 130-bushel yields indicates extremely low marginal products at these high yield levels. Isoquants for the quadratic and "1.5" functions curve more gradually than those of the square root equation. Also, isoquants of the "1.5" function are located slightly lower in the input plane than like isoquants for the quadratic function. Isoquants for the two are very similar, however. Isoquants for the Cobb-Douglas function are asymptotic to the nutrient axes.

Input combinations which produce given yield levels and marginal rates of substitution for these combinations, tabular counterparts of the isoquants, are presented in Table 14.5 for three yield levels. The marginal rates of substitution, ratios of marginal products, were predicted by equations 14.13 through 14.17 for the respective production functions in equations 14.1 through 14.4.

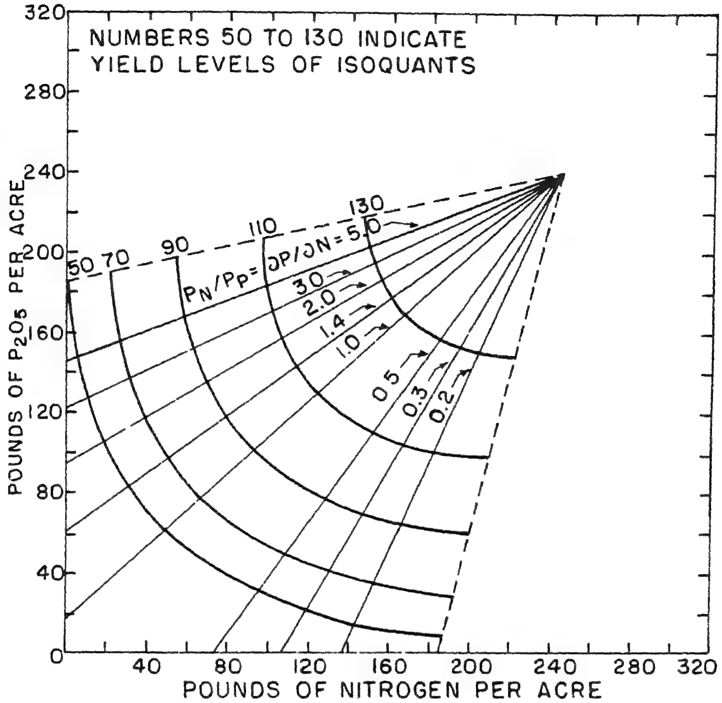

Figure 14.9. Isoquants and isoclines for corn predicted by quadratic function 14.2, Ida silt loam, 1952 (Dashed lines are ridge lines.)

Table 14.5. Combinations of Nitrogen and P_2O_5 Required To Produce Specified Corn Yields and Corresponding Marginal Rates of Substitution Predicted by Four Types of Production Functions, Ida Silt Loam, 1952

Pounds of Nitrogen per Acre	Square Root Function 14.3		Quadratic Function 14.2		"1.5" Function 14.4		Cobb-Douglas Function 14.1	
	Pounds of P_2O_5 per acre	MRS$(\frac{\delta P}{\delta N})$	Pounds of P_2O_5 per acre	MRS$(\frac{\delta P}{\delta N})$	Pounds of P_2O_5 per acre	MRS$(\frac{\delta P}{\delta N})$	Pounds of P_2O_5 per acre	MRS$(\frac{\delta P}{\delta N})$
				Y = 50				
20	16	-0.66	102	-1.92	75	-1.79	59	-2.06
40	9	-0.20	72	-1.18	49	-0.98	36	-0.63
60	6	-0.08	52	-0.83	33	-0.63	27	-0.32
80	5	-0.04	38	-0.65	22	-0.42	22	-0.19
				Y = 70				
40	34	-0.70	116	-1.98	95	-1.84	110	-1.93
60	25	-0.30	86	-1.16	68	-0.99	83	-0.97
80	21	-0.15	67	-0.79	52	-0.63	67	-0.59
100	19	-0.08	54	-0.57	42	-0.41	58	-0.41
				Y = 90				
60	76	-2.73	161	-3.94	131	-2.72	177	-2.08
80	59	-0.55	118	-1.40	97	-1.15	145	-1.27
100	51	-0.29	96	-0.86	79	-0.68	123	-1.07
120	46	-0.17	82	-0.58	68	-0.43	109	-0.64

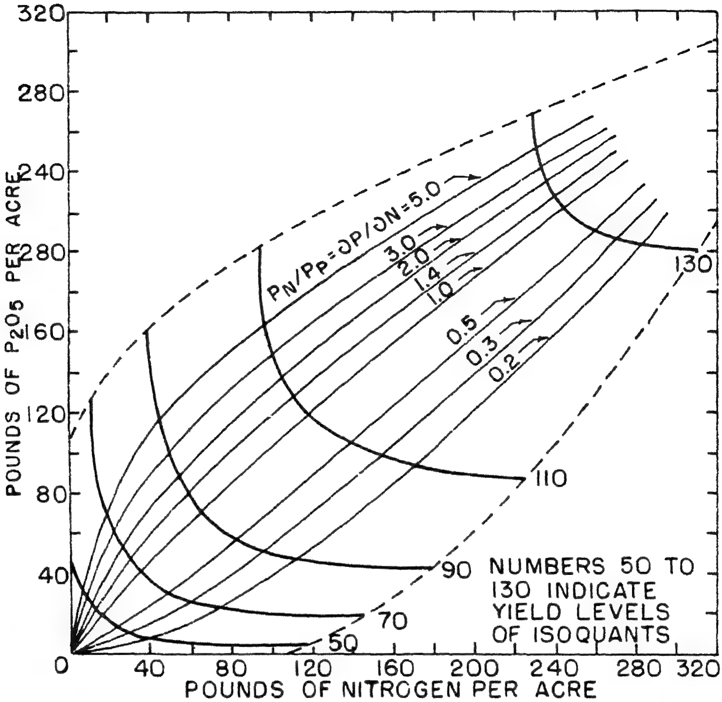

Figure 14.10. Isoquants and isoclines for corn predicted by square root function 14.3, Ida silt loam, 1952 (Dashed lines are ridge lines.)

$$\frac{\delta P}{\delta N} = .7034PN^{-1} \tag{14.13}$$

$$\frac{\delta P}{\delta N} = \frac{.5843 - .0032N + .0008P}{.6638 - .0036P + .0008N} \tag{14.14}$$

$$\frac{\delta P}{\delta N} = \frac{-.3161 + (3.1756 + .1705P^{1/2})N^{-1/2}}{.4174 + (4.2578 + .1705N^{1/2})P^{-1/2}} \tag{14.15}$$

$$\frac{\delta P}{\delta N} = \frac{.9839 - .0749N^{1/2} + .0008P}{1.1285 - .0864P^{1/2} + .0008N} \tag{14.16}$$

Yield Isoclines

Equations 14.17 through 14.20 are the isocline equations for the respective production functions where k equals the price of nitrogen divided by the price of P_2O_5.

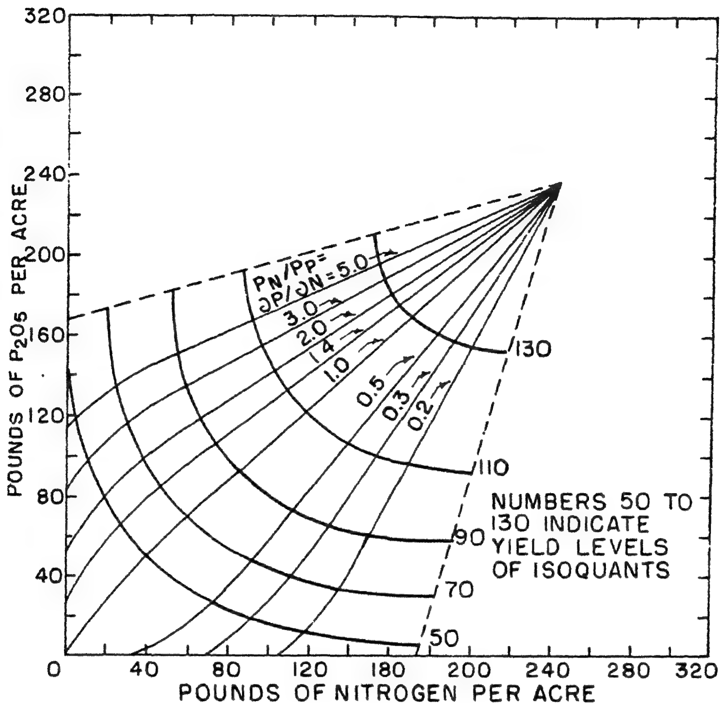

Figure 14.11. Isoquants and isoclines for corn predicted by "1.5" function 14.4, Ida silt loam, 1952 (Dashed lines are ridge lines.)

$$P = 1.4212kN \tag{14.17}$$

$$P = \frac{.6638k - .5843 + (.0008k + .0032)N}{(.0008 + .0036k)} \tag{14.18}$$

$$\begin{aligned} P = [&-9.3126 + (-1.2240k + .9271)N^{.5} \\ &\pm 2.9326(10.0845 + [-2.0078 + 5.5548k]N^{.5} \\ &+ [(.3161 + .4174k)^2 + .1163k]N)^{.5}]^2 \end{aligned} \tag{14.19}$$

$$\begin{aligned} P = [&-53.4932k \pm 6.88119(.0075k^2 - .0032 \\ &+ .0036k + .0002N^{.5} + .000003kN)^{.5}]^2 \end{aligned} \tag{14.20}$$

The shape of the isocline families, illustrated in figures 14.8 through 14.11, is determined jointly with the curvature or slope of isoquants (the magnitudes of marginal rates of substitution). The marginal rate of substitution is, in turn, a ratio of marginal products. Because the square root function predicts initially large marginal products but

which decrease rapidly, isoquant slopes for this function are almost vertical (or horizontal) along their extremities (near ridgelines) but curve sharply in the center. Hence, isoclines for square root functions tend to pass through the centers of isoquants and spread only for extreme price ratios and medium or high input rates. They necessarily begin at the origin, spread out as nutrient applications increase, and converge at the point of maximum yield. (The point of convergence, considerably outside the input observations at a yield of 141.3 bushels, is not shown.) The quadratic function has linear marginal equations and, therefore, linear isoclines which originate from either axis and converge at the point of maximum yield. The "1.5" function has isoclines which curve sharply as they intersect the nutrient axes but approach linearity for yields approaching the maximum. The Cobb-Douglas isoclines are linear, pass through the origin, and diverge with yield level.

DETAILED ANALYSIS WITH SQUARE ROOT EQUATION

Previous analysis has indicated similarities and differences in production functions estimated by the four equations. While each might be appropriate for certain purposes, the square root function appears to have a slight superiority in statistical efficiency in predicting the production surface for the particular environmental conditions under which the experiment was conducted. Hence, equation 14.2 is used in this section for examining certain additional details in physical relationships and in specifying some economic optima.

The predicted yield levels for this function, for various levels on N and P, are given in Table 14.4. These quantities are the counterpart of a production surface, except that they represent distinct points on it.

Since both nutrients were present in the soil in limited amounts, yields were not high for either nutrient used alone. With no P_2O_5, 120 pounds of nitrogen gives a maximum yield of 25.9 bushels in Table 14.4; with no nitrogen, 120 pounds of P_2O_5 gives a maximum of 37.5 bushels.[2] However, with the addition of 40 pounds of P_2O_5, a large yield increase takes place across the nitrogen columns; a similar change takes place for P_2O_5 down the first column. In other words, the productivity of one nutrient is highly limited by the amount of the other with which it is combined. With both nutrients variable, the predicted maximum yield

[2] These fertilizer quantities do not represent the exact maximum yield. The maximum yields for nitrogen variable with P_2O_5 fixed at zero or P_2O_5 variable with nitrogen fixed at zero are determined by setting the derivatives for each variable nutrient equal to zero and solving for N or P, respectively, as in (a) and (b) below. The maximum for P_2O_5 is with 104.3 pounds; the maximum for N is with 101.0 pounds of this nutrient. The corresponding yields are 37.7 and 26.4 bushels, respectively.

(a) $0 = +.316 - 3.1756N^{-.5}$ $\quad N = 101.0$ lbs.

(b) $0 = +.417 - 4.2578P^{-.5}$ $\quad P = 104.3$ lbs.

is 135.8 bushels with 397.6 pounds of nitrogen and 336.6 pounds of P_2O_5.[3]

Diminishing total yields for nitrogen, as the variable nutrient, are indicated up to 200 pounds of P_2O_5 as the fixed nutrient. Similarly, negative marginal products hold true for P_2O_5 as the variable nutrient, for up to 280 pounds of nitrogen as the fixed nutrient. Diminishing total yields are not predicted, within the range of the observation, when both nutrients are variable in a 1:1 ratio. Just as these two nutrients interact to affect the productivity of each other, another variable resource, such as stand, might well have caused different productivity coefficients for either nitrogen or P_2O_5.

Figure 14.3 is the response surface showing these productivity relationships more vividly. A vertical slice through this surface perpendicular with the P_2O_5 axis is the counterpart of a single-nutrient response curve with nitrogen as the variable nutrient; a slice perpendicular to the nitrogen axis represents P_2O_5 as the variable resource and nitrogen as the fixed nutrient. A vertical slice intersecting the origin is the counterpart of a response curve with both nutrients variable in fixed proportions. Horizontal slices through the surface provide yield isoquants showing all possible combinations of the two nutrients which will produce a given yield; these quantities are provided in later paragraphs.

Table 14.6 indicates the marginal products or yields corresponding to the total yields of Table 14.4; they are the counterparts of the slopes of vertical slices through Figure 14.3, at the yield levels of Table 14.4. These figures again illustrate that the quantity of one nutrient affects the productivity of the other. For example, movement down any column of Table 14.4 represents an increase in the ratio of P to N; movement across a row represents a decrease in the P to N ratio. Down any column, the marginal product of P_2O_5 decreases while the marginal product of nitrogen increases; across rows the opposite holds true. Marginal yields per pound of nutrient are equal for the two nutrients when the quantity of each is 120 pounds. The negative marginal products represent diminishing total yields; the small positive marginal products in much of the table correspond to the fact that the production surface is quite flat over a large section.

Single Variable Input-Output Curves

Figures 14.12 and 14.13 provide total response as yield curves when one nutrient is fixed at specified levels and the other is variable. With

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[3] The predicted maximum yield, an extrapolation beyond the observations of the experiment, was obtained as follows. The partial derivatives (the marginal products) for each nutrient were set at zero; the quantity of each nutrient, to give a partial derivative of zero, was then computed. These are the quantities of nutrients which give a maximum yield. They were substituted back into the original function and the maximum yield was predicted accordingly.

Table 14.6. Marginal Product or Yield (Bushel per Pound of Fertilizer Nutrient) for Combinations Indicated in Rows and Columns; Upper Figure for Nitrogen; Lower Figure for P_2O_5*

Pounds of P_2O_5	Pounds of Nitrogen								
	0	40	80	120	160	200	240	280	320
0	--	.19	.04	.02	-.07	-.09	-.11	-.12	-.14
	--	--	--	--	--	--	--	--	--
40	--	.36	.16	.07	.02	-.02	-.04	-.06	-.08
	.26	.43	.49	.55	.60	.64	.67	.71	.74
80	--	.43	.21	.11	.06	.01	-.01	-.04	-.05
	.06	.17	.23	.27	.30	.33	.35	.38	.40
120	--	.48	.24	.14	.08	.04	.01	-.01	-.03
	-.03	.07	.11	.14	.17	.19	.21	.23	.25
160	--	.52	.28	.17	.11	.06	.03	.01	-.02
	-.08	.01	.04	.07	.09	.11	.13	.14	.16
200	--	.57	.31	.19	.13	.08	.04	.02	-.01
	-.11	-.04	-.01	.02	.03	.05	.07	.09	.10
240	--	.60	.33	.21	.14	.10	.06	.03	-.01
	-.14	-.07	-.04	-.02	-.01	.02	.03	.04	.05
280	--	.63	.36	.23	.16	.11	.07	.04	.02
	-.16	-.10	-.07	-.05	-.03	-.02	-.01	.01	.02
320	--	.67	.37	.25	.18	.12	.09	.06	.03
	-.18	-.12	-.09	-.07	-.06	-.04	-.03	-.03	.01

*These figures are the derivatives of yield in respect to the single-nutrient variable while the other is fixed. They are derived from equation 14.3, with the nitrogen and P_2O_5 quantities shown at the top of the columns and to the left of the rows. The .36 in the cell where both nutrients are 40 pounds is the derivative or marginal product for nitrogen as the variable nutrient while P_2O_5 is fixed at 40 pounds. The .43 is the marginal product for P_2O_5 as the variable nutrient while nitrogen is fixed at 40 pounds.

a zero nitrogen input for Figure 14.12, the P_2O_5 curve falls low in the plane with diminishing total yield indicated for small inputs of P_2O_5. With nitrogen input at 160 and 320 pounds, the response curves for P_2O_5 cross each other. This is due to the fact that, with small quantities of P_2O_5, 320 pounds of nitrogen gives an excessive quantity of nitrogen; with larger quantities of P_2O_5, the two nutrients interact to give slightly higher yields for 320 than for 160 pounds of nitrogen. A similar situation exists for nitrogen as the variable nutrient. With P_2O_5 fixed at 160 and 320 pounds, the nitrogen response curves in Figure 14.13 again cross each other. An increase in P_2O_5 from 160 to 320 pounds adds nothing to yield if nitrogen inputs are small. The fact that the maximum yield from nitrogen, with no P_2O_5, is lower than the maximum of P_2O_5, with no nitrogen, suggests that the soil, while deficient in both nutrients, was lacking especially in P_2O_5.

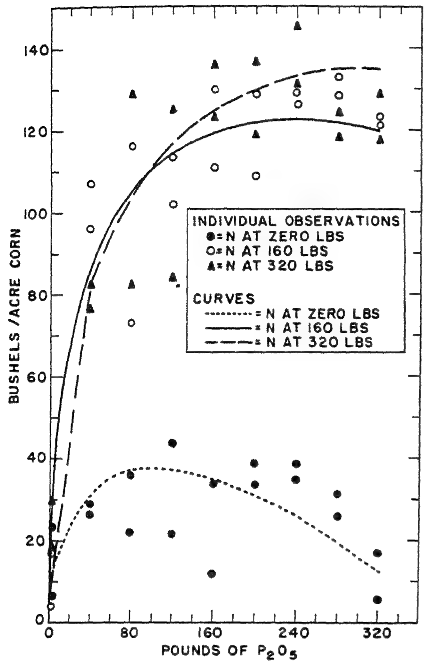

Figure 14.12. Total yield with P_2O_5 variable and nitrogen fixed at three levels

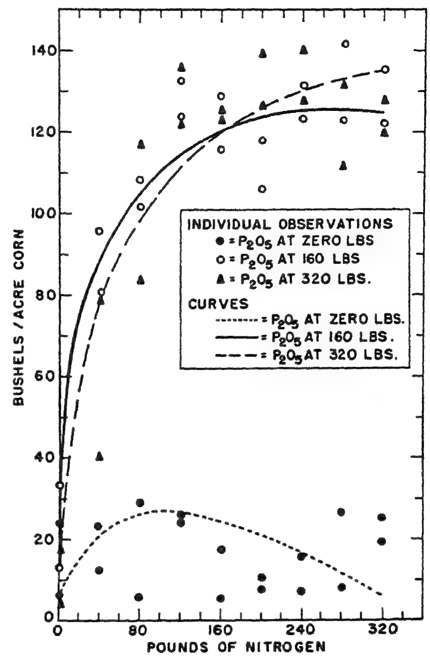

Figure 14.13. Total yield with nitrogen variable and P_2O_5 fixed at three levels

"Scale Line" Response Curve With Both Nutrients Variable

Figures 14.14 and 14.15 show predicted input-output or response curves when the two nutrients are increased in fixed ratios. The amount of one element is always in a fixed ratio to the amount of the other, as indicated on the bottom of the graphs. In Figure 14.14, for example, the ratio line of 1P = 2.0N means that 2 pounds of P_2O_5 is used for each pound of nitrogen; with a nitrogen input of 160 pounds, input of P_2O_5 is 320 pounds; and with nitrogen at 320 pounds, input of P_2O_5 is 640 pounds. These two figures indicate that greatest yields can be obtained from use of the two nutrients in a 1:1 ratio. For light applications of fertilizer, greater response per pound may be obtained with nutrient ratios differing from 1:1.

Yield Isoquants

Yield isoquants derived from the same basic yield surface equation were shown in Figure 14.10. The isoquants show that as higher and

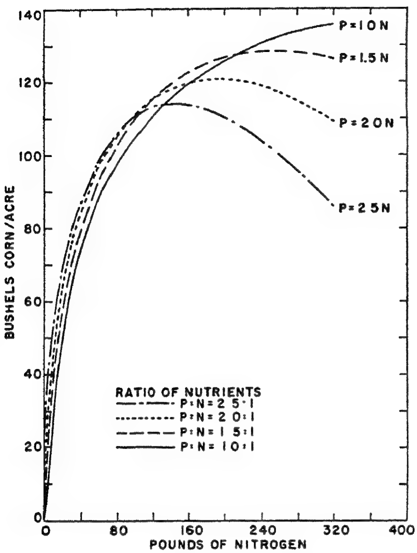

Figure 14.14. Yield of corn with nutrients increased in fixed proportions

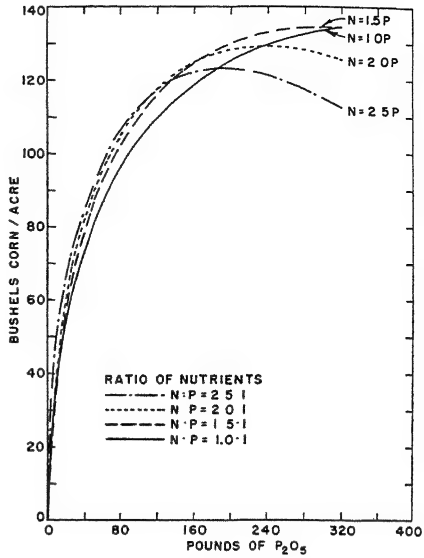

Figure 14.15. Yield of corn with nutrients increased in fixed proportions

higher yields are attained, the marginal rates of substitution between P_2O_5 and nitrogen change along a scale line (a fixed nutrient ratio line). In other words, the slopes of successively higher isoquants are different at the points where they are intersected by an imaginary straight line through the origin. This change in the slopes of the yield isoquants indicates that the combinations of nutrients (the fertilizer ratio) which gives lowest cost for one yield level is not the same mixture which gives lowest cost for another yield level. In other words, the least-cost combination is not the same for yields of 60 and 120 bushels. This same point is illustrated in Table 14.5 which shows several predicted combinations of the two nutrients which will produce the same yield and the marginal rates of nutrient substitution for the indicated combinations. Figures for isoquants indicate, on the one hand, the minimum quantity of nitrogen and the maximum quantities of P_2O_5 which will produce the stated yield and, on the other hand, the maximum quantities of P_2O_5 and the minimum quantities of nitrogen. More P_2O_5 must be used with a stated amount of nitrogen for a higher yield as compared to a lower yield. With 160 pounds of nitrogen, 165 pounds of P_2O_5 allows a yield of 120 bushels; only 64 pounds of P_2O_5 is required with 160 pounds of nitrogen to produce 100 bushels. The yield isoquants also indicate that the range of N/P ratios, over which the two nutrients can be substituted in obtaining a given yield, narrows as higher yield levels are attained. For higher yields, the nutrients become limitational in nature as the "upper ends" of the isoquants take on an infinite slope and as the

"lower ends" take on zero slopes. Low yields can be attained by addition of one nutrient alone, but high yields can be attained only with some minimum quantity of either nutrient. The maximum yield per acre, as predicted from the equation, can be produced by only one combination of P_2O_5 and nitrogen (i.e., the isoquant for a yield of 135.8 bushels reduces to a single point corresponding to 397.6 pounds of nitrogen and 336.6 pounds of P_2O_5).

Economic Optima

The various quantities and relationships derived above provide a basis for specifying selected economic optima, applying to the particular environmental conditions of the experiment. We first examine the optimum level of applying one nutrient when the level of the other is considered to be fixed. We suppose N to be variable but P_2O_5 to be at zero. The optimum level of nutrient application depends on the magnitude of marginal products and the nutrient to corn price ratio. With corn at \$1.40 per bushel and nitrogen at \$.18 per pound (including nitrogen and the cost of application) the price ratio is $\frac{.18}{1.40}$ or .129. Hence the derivative of equation 14.3 for corn yield with respect to nitrogen is set at this quantity in equation 14.21 below, with P_2O_5 at zero level. Solving 14.21 for N, 53.3 pounds of nitrogen equates the marginal product, and therefore is the most profitable quantity of this nutrient when no P_2O_5 is used. The corresponding yield, from equation 14.3, is 24.8 bushels.

$$\frac{\delta Y}{\delta N} = .316 + 3.1756N^{-.5} = .129 \tag{14.21}$$

With corn at \$.80 and nitrogen at \$.18, the price ratio is $\frac{.18}{.80}$ or .225 and 34.8 pounds of N is the level of fertilization to maximum profits. With the price ratio at $\frac{.10}{1.40}$ or .071, 67.4 pounds of N is most profitable. However, when 80 pounds of P_2O_5 are used and the price ratio is $\frac{.18}{1.40}$, 136.9 pounds of nitrogen represents the optimum.

Using the same price ratios for P_2O_5, we obtain the values below. For any one of the nutrient to crop price ratios shown, the optimum quantity of P_2O_5 with a stated amount of nitrogen, is slightly greater than the optimum quantity of nitrogen with the same amount of P_2O_5.

Zero input of nitrogen

Price ratio of .129; P_2O_5 optimum is 60.8 pounds

Price ratio of .225; P_2O_5 optimum is 44.0 pounds

Price ratio of .071; P_2O_5 optimum is 76.1 pounds

160 pounds of nitrogen

Price ratio of .129; P_2O_5 optimum is 140.5 pounds

Price ratio of .225; P_2O_5 optimum is 101.6 pounds

Price ratio of .071; P_2O_5 optimum is 175.9 pounds

Minimum Costs for a Specified Yield

Selection of the optimum quantity of a single nutrient is only a partial solution of the economic problem of fertilizer use. Still to be solved is (a) the optimum quantity of each nutrient or rate of application when both nutrients are variable and (b) the best combination of nutrients for any one yield level. The change in the slopes of the yield isoquants along a scale line in Figure 14.10 suggests that the combination of the two nutrients which will give the lowest cost, for a stated yield, changes with the level of yield. The nutrient combination which is best for a 100-bushel yield is not also best for a 50-bushel yield. This point also is illustrated with the isoquant and substitution data of Table 14.7. The least-cost resource combination for a given yield is attained when the marginal rate of substitution of the resources (i.e., the derivative of one nutrient in respect to the other, with yield at stated levels) is equal to the inverse price ratio. Hence, we can illustrate that the proportion of the two nutrients, to give the least cost, differs with yield level. First, we derive the equation of marginal rates of substitution (the first derivatives of change in one nutrient with respect to the other) as in equation 14.15. Second, we set it equal to the particular price ratio for the nutrients and solve for the nutrient combination which minimizes cost for the particular yield. Or, we can set the derivative of P with respect to N equal to the price of nitrogen divided by the price of P_2O_5. Using the equation in 14.22, and equating it to different price ratios, and solving for N and P, we obtain the results in Table 14.7.

$$(14.22) \quad \frac{\delta P}{\delta N} = -\frac{3.1756N^{-.5} + .1705N^{-.5}P^{.5} - .316}{4.2578P^{-.5} + .1705N^{.5}P^{-.5} + .417} = -\frac{N}{P_2O_5} \text{ price ratio}$$

These are the least-cost combinations for the specified price ratios and yield isoquants. With a N/P price ratio of 1.5, the combination for a 50-bushel yield should total 36.1 pounds; 32.7 per cent of this should be nitrogen and 67.3 per cent should be P_2O_5. For the same price ratio, a total of 180.9 pounds, composed of 43.8 per cent nitrogen and 56.2 per cent P_2O_5, should be used to minimize fertilizer cost for a 100-bushel yield. The mixture, to minimize cost for a given yield, should contain relatively more nitrogen for higher yield levels. Traditionally, this distinction has not been made in fertilizer recommendations; the same fertilizer mix has, for a given soil and productivity situation, usually been recommended for numerous yield levels.

Table 14.7. Combinations of Nitrogen and P_2O_5 Needed to Minimize Fertilizer Costs per Specified Yield Level for Different Price Ratios

Yield Level	Optimum Pounds of N	Optimum Pounds of P
	Price of \$.18 per lb. for N and \$.12 per lb. for P ($\frac{N}{P}$ ratio of 1.5)	
50 bu.	11.8	24.3
100 bu.	79.3	101.6
	Price of \$.12 per lb. for N and \$.18 per lb. for P ($\frac{N}{P}$ ratio of .67)	
50 bu.	19.8	16.3
100 bu.	99.1	82.7

Similarly, with a change in the price ratio from 1.5 to .67, the percentage of nitrogen, for a 50-bushel yield, should change from 32.7 per cent to 54.8 per cent. For a 100-bushel yield, similar changes in the price ratio should cause the nutrient combination to change from 43.8 per cent to 54.5 per cent nitrogen.

Solution for Two-Variable Nutrients

In the analysis above, principles of profit maximization were used to independently specify (a) the optimum quantity of one-variable nutrient, with yield as a variable and the second nutrient fixed and (b) the optimum combination for two-variable nutrients for a given or fixed yield. However, these conditions need to be imposed simultaneously if the economic optimum usage of fertilizer is to be determined. In other words, we must simultaneously determine the optimum (a) combination of nutrients and (b) level of application. It was explained in an earlier section that the combination of nutrients which gives lowest cost for one yield level does not similarly give the least cost for other yield levels. This is true since the slopes of the yield isoquants, and hence the marginal rates of substitution between nutrients, change with higher yield levels.

One approach to determining the dual solution outlined above is that of successive approximation. One can use the principle that application of more fertilizer is profitable (for a farmer with unlimited capital) as long as the marginal product of a fertilizer nutrient is greater than the nutrient to crop price ratio. Hence, with a price of \$1.40 per bushel for corn, \$.18 per pound for nitrogen, and \$.12 for P_2O_5, we can obtain solutions by successive approximations using Table 14.6. The $\frac{P_2O_5}{\text{corn}}$ price

ratio is .085; we can move down the first column until we find a marginal product for P_2O_5 which is greater than .085. The marginal product of the 40th pound of P_2O_5 is .26 — hence, it is profitable. The 80th pound of P_2O_5 is not profitable since its marginal product of .06 is less than the price ratio of .85. Starting from zero nitrogen, we can then move across the second row to determine the amount of nitrogen which is profitable, with 40 pounds of P_2O_5 already applied. Since the nitrogen to corn price ratio is .125, the 80th pound of nitrogen is profitable; the 120th pound is not since the marginal product of .07 is less than the price ratio of $\frac{.18}{1.40}$ or .125.

Now, with 40 pounds of P_2O_5 and 80 pounds of nitrogen, we move down the third column. With 80 pounds of nitrogen, the 120th pound of P_2O_5 becomes profitable since its marginal product of .11 is greater than the price ratio of .085. With 120 pounds of P_2O_5, the 120th pound of nitrogen also becomes profitable. From the data in Table 14.4 and with the prices quoted, the method of "successive approximation" indicates that 120 pounds of each nutrient is profitable. However, the successive approximation may require added steps in arithmetic before the final solution is attained.

The successive approximation indicates only which of the combinations in the table are most profitable. It does not indicate the exact combinations which might be more profitable. The exact fertilizer combination can be solved by setting the marginal products or partial derivatives for both nutrients equal to the price ratios and simultaneously solving for the quantity of the nutrients to apply for maximum profits. These optima are attained when the partial derivatives (the marginal products) for both nutrients are equal to the nutrient to corn price ratio. Hence, with a price of \$1.40 for corn, \$.18 for nitrogen, and \$.12 for P_2O_5, the equations become 14.23 and 14.24 below.

$$\frac{\delta C}{\delta N} = -.316 + \frac{3.1756}{\sqrt{N}} + \frac{.1705\sqrt{P}}{\sqrt{N}} = \frac{.18}{1.40} \tag{14.23}$$

$$\frac{\delta C}{\delta P} = -.417 + \frac{4.2578}{\sqrt{P}} + \frac{.1705\sqrt{N}}{\sqrt{P}} = \frac{.12}{1.40} \tag{14.24}$$

From simultaneous solution of these equations, we obtain the figures for situation A in Table 14.8: 298.93 pounds of fertilizer should be used, including 156.45 pounds of P_2O_5 and 142.48 pounds of nitrogen.

The same procedure has been used for the other price situations in Table 14.8. With a decline in corn price by 36 per cent (from situation A to situation B), total usage of fertilizer should decline by 30 per cent. Input of nitrogen should decline 34 per cent and input of P_2O_5 should decline 26 per cent if profit is to be at a maximum; inputs should not be reduced by the same proportions. With a 43 per cent increase in corn price (from situation A to situation C), total input of fertilizer should increase by 25 per cent; input of nitrogen should increase by 30 per cent

Table 14.8. Optimum Quantity of Fertilizer and Optimum Combination of Nutrients for Specific Price Relationships

Price Situation	Optimum Yield in Bushels	Optimum Fertilizer Use		
		Total pounds	Pounds N	Pounds P_2O_5
A: corn at $1.40; N at .18; P at .12	117.21	298.93	142.48	156.45
B: corn at $ .90; N at .18; P at .12	104.99	209.27	94.06	115.21
C: corn at $2.00; N at .18; P at .12	124.22	374.84	185.04	189.80
D: corn at $1.40; N at .12; P at .12	122.30	349.50	180.19	169.31
E: corn at $2.00; N at .12; P at .18	124.91	384.18	208.72	175.46

and input of P_2O_5 should increase by only 21 per cent. With corn at $1.40 and a 1:1 price ratio for nutrients (situation D), input of nitrogen should be greater than input of P_2O_5. With a N/P price ratio of 1.5 (situation C), input of phosphate should be about 5 pounds greater than input of nitrogen. However, with a N/P price ratio of .667 (situation E), input of nitrogen should exceed input of phosphate by 33 pounds.

We have illustrated that simultaneous solution of (a) the optimum rate of fertilization and (b) the optimum combination of nutrients is possible from appropriate experimental data. We also have illustrated some points ordinarily overlooked in both economic and agronomic recommendations: a reduction in product price not only may call for a reduction in the total quantity of fertilizer used on corn, it also may specify a change in the fertilizer grade. These and many other basic principles can be applied when fertilizer experiments are designed to provide relevant marginal quantities and the corresponding economic analysis.

For high level yields and recent prices, the cost of the optimum nutrient combination (computed by both partial derivatives with their respective price ratios) for corn is only slightly less than numerous other nutrient combinations which will give yields in the neighborhood of 125 bushels. The reasons for this outcome are explained below.

Since an isocline connects all points of the same slope (i.e., equal substitution rates) on successive isoquants, the isocline conforming to a particular price ratio also is an expansion path. It traces all combinations of nutrients which give least-cost yields. If an isocline conforming to a particular price ratio is nearly straight, an increase in nutrients by a fixed proportion is nearly consistent with the least-cost

use of nutrients. If the isocline bends sharply, a fixed-ratio fertilizer increase will not give the most economic nutrient combinations. While little is yet known about them, isocline maps can take on many distinct forms. They can be established only by basic experiments. In a family of isoclines, one denoting a substitution ratio of 1.0 may be bent; one for a .5 substitution ratio may be linear. Hence, with a nutrient price ratio of 1, least-cost fertilizer mixture proportions will vary across the surface; with a .5 price ratio, the fertilizer mix should follow a fixed ratio line, although no particular ratio can be specified without knowledge of the function. One of a family of isoclines may be straight (although it need not be one along a 1:1 ratio); none may be straight. All may be straight but may not pass through the origin. The extent to which profit is depressed, in increasing or decreasing fertilizer inputs in fixed mixes as price levels change, depends on the slope and curvature of the relevant isocline. If it is curved and passes through the origin but has a slope which is nearly constant, the depression will not be great. If it intersects the nutrient axes near the origin and is a straight line, the same will hold true. Of course, the relative price of nutrients also is important in determining profit sacrifices from using a fixed nutrient mix for all yield levels.

PRODUCTION FUNCTIONS WITH OMISSION OF ZERO NUTRIENT APPLICATIONS

While economic optima are illustrated only for the quadratic function, the same procedure can be applied to the other equations from 14.1 through 14.4. However, because of algebraic properties, they suggest different nutrient mixes and amounts for any one price level. For example, Table 14.9 shows some of the differences in predicted inputs (N and P) and output (Y) for several price situations under each of the functions.

Because of the algebraic nature and forced condition of constant elasticity, the predictions from the Cobb-Douglas function are unreasonable. The square root function, equation 14.3, gives the most

Table 14.9. Predicted Optimum Inputs and Output for Several Price Levels Under the Cobb-Douglas (14.1), Quadratic (14.2), Square Root (14.3), and "1.5" (14.4) Equations

Price ($) of			Inputs and Outputs Specified by											
			Equation 14.1			Equation 14.2			Equation 14.3			Equation 14.4		
Y	N	P	N	P	Y	N	P	Y	N	P	Y	N	P	Y
1.60	.08	.09	12,539	15,832	2,195	225	220	143	240	212	129	216	207	138
1.00	.08	.09	2,666	3,369	757	213	203	141	185	168	123	196	192	136
1.60	.17	.13	1,757	3,266	665	205	208	141	162	166	120	190	192	135
1.00	.17	.13	374	644	236	180	189	135	108	121	108	161	169	129

conservative estimates of input applications. Changes in input levels are not great, for selected changes in price ratios, because of the nature of the production functions under the environmental conditions studied.

It has been suggested that with marginal productivities for initial nutrient inputs of the magnitude realized in this study, less difference in the optima specified by the various equations might occur if zero rates of application were omitted from the analysis. This procedure is used in this section.

Regression Equations

The types of equations shown in equations 14.1 through 14.3 are repeated in equations 14.25 through 14.28, respectively, with observations for zero rates omitted.

$$Y' = 19.94N^{.2119}P^{.1246} \tag{14.25}$$

$$Y = 38.3122 + .5093N + .2788P - .0013N^2 - .0007P^2 + .0004NP \tag{14.26}$$

$$Y = -37.5002 + 15.5508N^{.5} + 5.7321P^{.5} - .7077N - .3536P + .3941N^{.5}P^{.5} \tag{14.27}$$

$$Y = 25.2709 + .9279N + .5088P - .0449N^{1.5} - .0255P^{1.5} + .005NP \tag{14.28}$$

The coefficients of determination are .45, .77 .84, and .78, respectively, for the four equations. The R^2's for the polynomials are about 10 per cent below those resulting when all observations were included. The difference in R^2's of the quadratic and "1.5" equations are negligible. The square root equation again has the highest R^2. (Yield intercept values have no meaning because there were no observations for zero nutrient levels.) Analyses of variance of regression for the four equations are presented in Table 14.10. The square root equation again has the smallest figure for deviations from regression.

The number of observations was reduced from 114 to 80 by the deletion. Because these observations represented yields which were relatively large distances from the mean yield level, their omission greatly reduced total and treatment sums of squares. Also, the mean corn yield was increased from 86.07 to 113.39 bushels per acre. Because treatment sums of squares were reduced relatively more than the sums of squares among plots treated alike, R^2's of the equations were lowered by dropping the observations.

Table 14.10. Analyses of Variance of Regression for Equations Fitted to Selected Corn Yield Observations, Ida Silt Loam, 1952

Source of Variation	Degrees of Freedom	Sums of Squares	Mean Squares
Total	79	38,439	
Treatments	39	31,001	795
Due to regression of Cobb-Douglas equation 14.25	2	17,217	8,609
Deviations from regression	37	13,784	373
Due to regression of quadratic equation 14.26	5	23,859	4,772
Deviations from regression	34	7,142	210
Due to regression of square root equation 14.27	5	25,939	5,188
Deviations from regression	34	5,062	149
Due to regression of "1.5" equation 14.28	5	24,274	4,855
Deviations from regression	34	6,727	198
Among plots treated alike	40	7,438	186

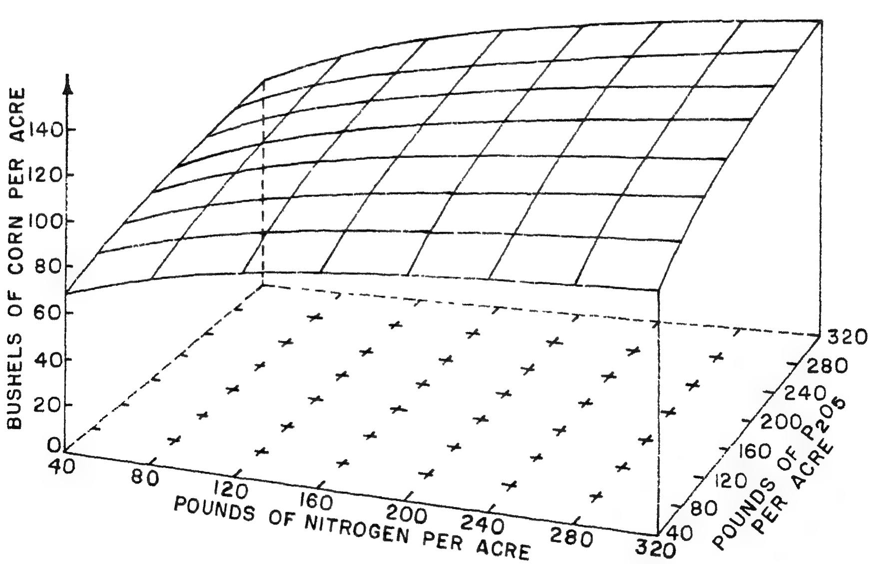

Figure 14.16. Production surface for corn predicted by Cobb-Douglas function 14.25, Ida silt loam, 1952

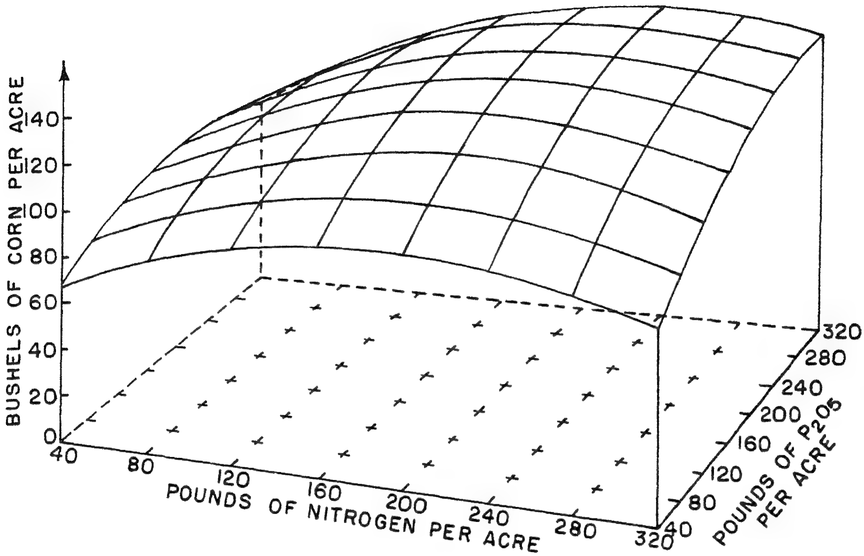

Figure 14.17. Production surface for corn predicted by quadratic function 14.26, Ida silt loam, 1952

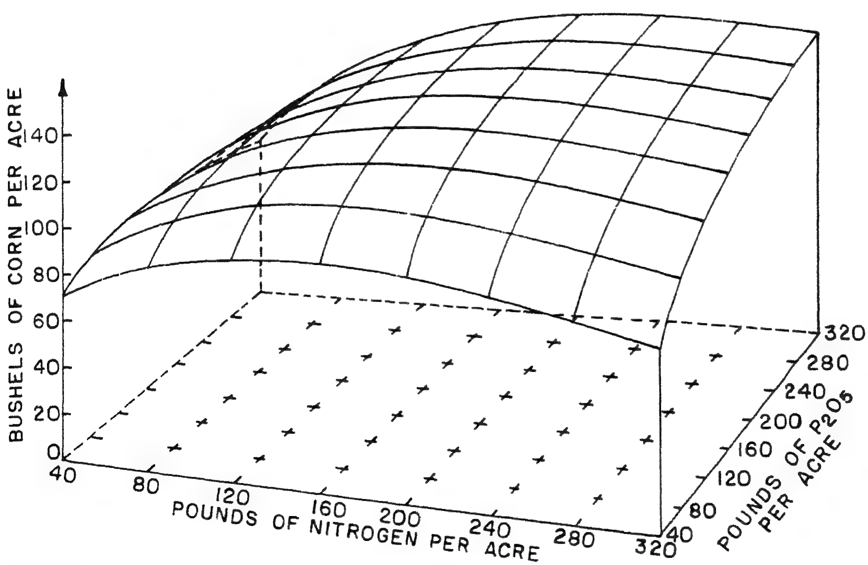

Figure 14.18. Production surface for corn predicted by square root function 14.27, Ida silt loam, 1952

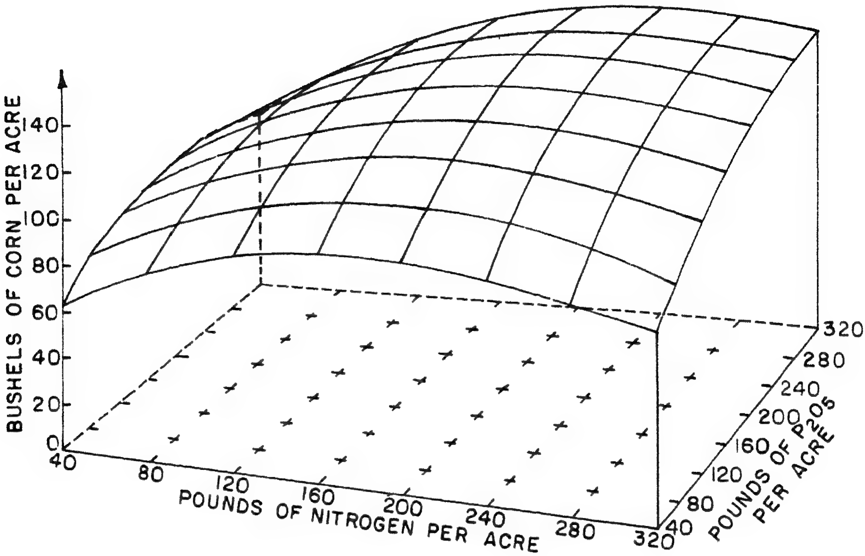

Figure 14.19. Production surface for corn predicted by "3/2" function 14.28, Ida silt loam, 1952

Predicted Yields and Production Surfaces

Production surfaces for the four equations are presented in figures 14.16 through 14.19. They are, of course, flatter than the earlier surfaces. Yields predicted by the four functions are now more comparable than before the zero observations were dropped. Again, predictions of the quadratic and "1.5" functions are most similar and, in this case, are almost identical. The square root function, which characteristically predicts higher, lower, and, finally, higher yields than the other polynomials as nutrient inputs increase, still follows this pattern, but the differences are now quite small. For example, for 80 pounds of both nutrients, the square root function predicts 8.3 bushels more than the quadratic function; before the zero rates were dropped this difference was 20 bushels. Predictions of the Cobb-Douglas function are more similar to those of other functions than before but, due to the constant elasticity assumption, still vary widely for some nutrient levels. The predicted yields of the functions for 40-pound nutrient rates are all very close to the mean yield for that treatment level, 71.4 bushels.

As Figure 14.20 indicates, the single nutrient input-output curves are made much more similar through deletion of zero rates. However, the Cobb-Douglas curve still is highly different from the others.

In summary, dropping the observations changed the response patterns in this way: predicted responses to nitrogen have been increased relative to P_2O_5 responses. Marginal products of all functions were, in

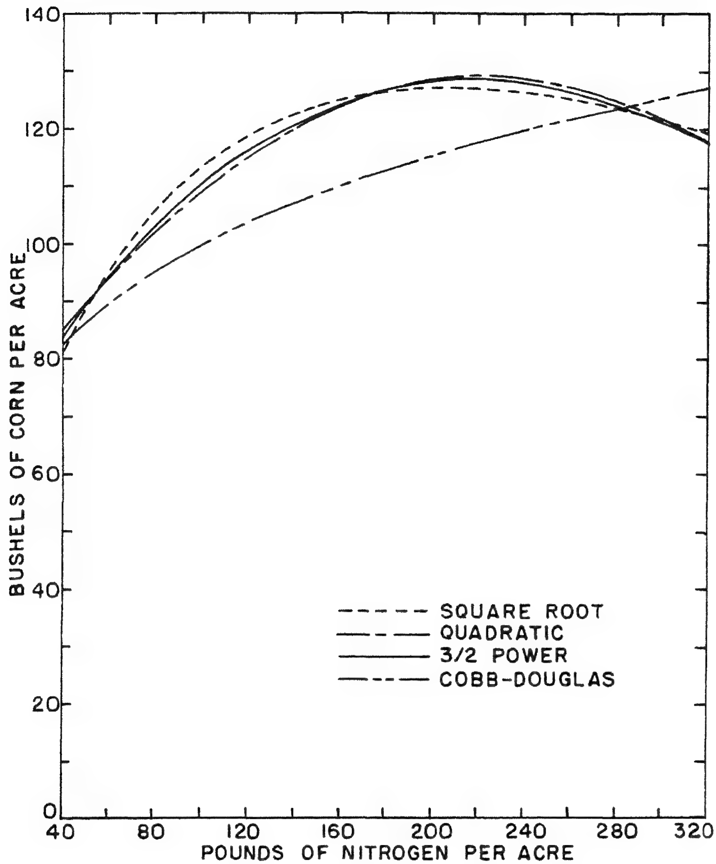

Figure 14.20. Corn yield response to nitrogen when P_2O_5 is constant at 160 pounds per acre predicted by four types of production functions, Ida silt loam, 1952

general, lowered. The deletion affected the square root function more than the other polynomials. Most significant was the change in the input rates at which marginal products of this function were forced to zero. Ridgelines for it became less curved and converged at nutrient input rates similar to the other functions. Therefore, diminishing returns of yields predicted by this function became more similar to those of other functions and input levels at which maximum yields occur became almost identical for the three functions. Hence, while the pattern of yield response differs between the two sets of production functions, omitting the observations has made predictions of the polynomial equations as a group more similar. The isoquant and isocline maps for the quadratic equation 14.26, square root equation 14.27, and "1.5" equation 14.28 functions are shown in figures 14.21 through 14.23. The Cobb-Douglas

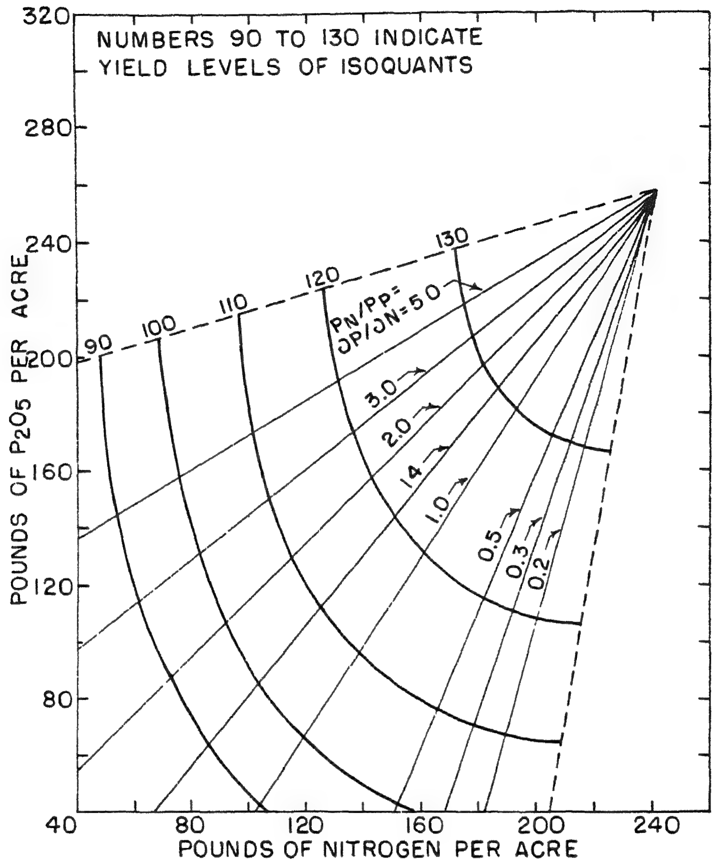

Figure 14.21. Isoquants and isoclines for corn predicted by quadratic function 14.26, Ida silt loam, 1952 (Dashed lines are ridge lines.)

isoquants and isoclines for equation 14.25 are not presented since they fan out, rather than converge to a yield maximum. The isoclines for the square root function now converge to a yield maximum within the observed range of inputs.

Isoclines predicted by equations 14.25 to 14.28 are more similar than those obtained before the zero observations were omitted. Previously isoclines and ridgelines of the square root function tended to diverge for the inputs of the experiment. They now converge at a maximum yield similar to those of other functions and ridgelines are narrowed accordingly and limited the divergence of the isoclines. Isoclines for the "1.5" function have also been changed, and now are nearly linear. Isoclines of all functions have shifted due to the change in the response pattern.

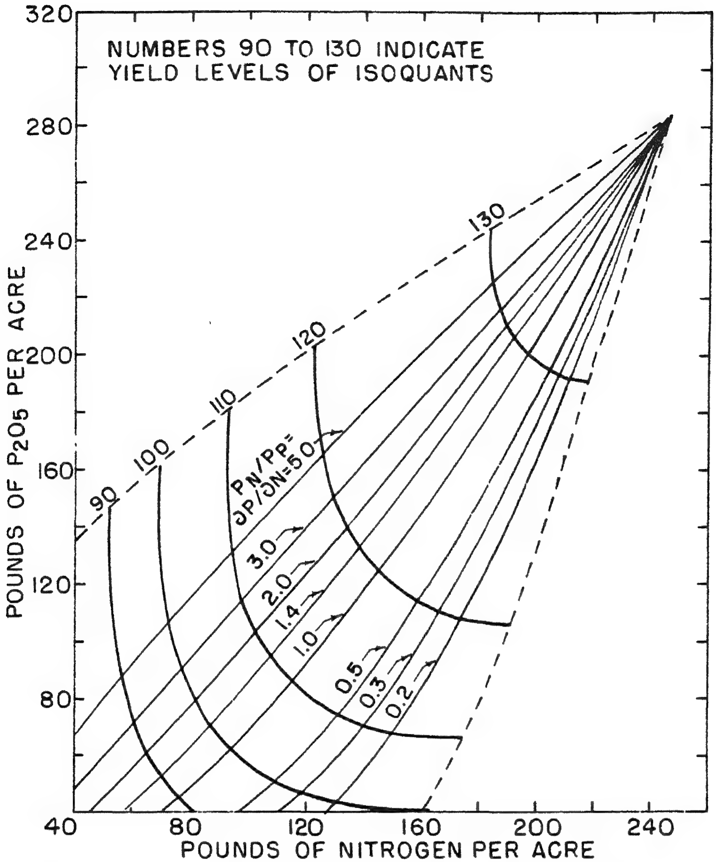

Figure 14.22. Isoquants and isoclines for corn predicted by square root function 14.27, Ida silt loam, 1952. (Dashed lines are ridge lines.)

Economic Optima

Combinations of nitrogen and P_2O_5 which minimize the cost of producing 90 and 130 bushels of corn for price ratios (P_n/P_p) of 5/1, 2/1, 13/9, 1/2, and 1/5 are presented in Table 14.11.

Amounts of nitrogen predicted by the polynomial functions are quite similar for all yield and price situations. For instance, for a yield of 130 bushels and a price ratio of 1/2, 207, 207, and 206 pounds of nitrogen are recommended by the square root, quadratic, and "1.5" function, respectively. Recommendations for P_2O_5 are somewhat more variable; the amounts being 195, 173, and 178 pounds per acre, respectively. Combinations predicted by the three polynomial functions are most

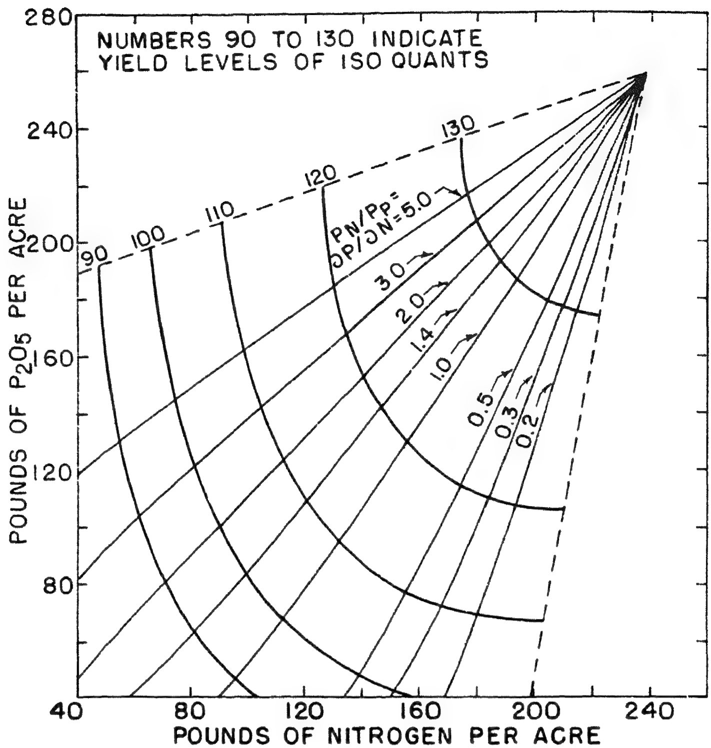

Figure 14.23. Isoquants and isoclines for corn predicted by "1.5" function 14.28, Ida silt loam, 1952 (Dashed lines are ridge lines.)

similar for the 130-bushel yield level, indicating that predictions of these functions become more comparable as yields increase. The Cobb-Douglas function again differs considerably from the polynomials.

Combinations of nitrogen and P_2O_5 which maximize profits per acre are presented in Table 14.12 for various corn and nutrient prices. These combinations were derived from functions 14.25 to 14.28 in the manner explained previously.

Predictions of the quadratic function and the "1.5" function are closely comparable for all price situations; those of the quadratic being slightly higher in all cases. Nutrient combinations and yields predicted by the square root function are, in all situations, more conservative than those of the other polynomials. Predicted profits are almost identical for all three polynomial functions except in the first price situation where the quadratic function predicts $5.69 more profit per acre than the square root function. Differences among predictions of these functions increase as nutrient prices rise relative to corn prices but this increase is small. Hence, the functions are about equally sensitive

Table 14.11. Cost Minimizing Combinations (Pounds) of Nitrogen and P_2O_5 per Acre for Corn Yields of 90 and 130 Bushels per Acre Predicted by Four Types of Production Functions (Fitted to Selected Yield Observations), Ida Silt Loam, 1952

Price Ratio	Cobb-Douglas		Quadratic		Square Root		"1.5" Function	
$(\frac{P_n}{P_p})$	N	P_2O_5	N	P_2O_5	N	P_2O_5	N	P_2O_5
				Y = 90				
5	19	173	54	146	57	88	55	130
2	83	98	74	87	65	60	71	80
13/9	94	79	87	64	70	53	81	63
1/2	138	40	--*	--	--	--	--	--
1/5	--	--	--	--	--	--	--	--
				Y = 130				
5	--	--	177	218	186	228	178	207
2	248	291	184	196	190	211	184	200
13/9	279	237	188	188	195	207	190	193
1/2	--	--	207	173	207	195	206	178
1/5	--	--	218	168	215	193	217	175

*Input combinations are beyond limits of isoquant and isocline map.

to price changes. Predictions of the Cobb-Douglas function have been reduced relative to those of the other functions by dropping the zero observations. However, nutrient predictions for many price situations still fall beyond the limits of observation for the experiment.

In summary, differences in recommended optimum nutrient inputs of the functions have not been eliminated by omitting the zero observations. These differences have been made more consistent, however. Yield and profit predictions of the polynomials are closely comparable after the deletions.

Table 14.12. Predicted Optimum Inputs and Outputs for Several Price Levels Under Equations 14.25 Through 14.28

Price ($) of:			Inputs and Outputs Specified by											
			Equation 14.25			Equation 14.26			Equation 14.27			Equation 14.28		
Y	N	P	N	P	Y	N	P	Y	N	P	Y	N	P	Y
1.60	.08	.09	709	370	166	215	212	134	191	186	128	208	208	132
1.00	.08	.09	349	183	132	198	184	131	167	149	125	189	175	129
1.60	.17	.13	245	188	123	189	187	130	157	150	124	179	177	128
1.00	.17	.13	-	-	-	159	145	121	126	110	78	148	135	119

Profit Maximizing Combinations — Limited Capital

Quantities derived for limited capital situations, by the methods outlined in Chapter 3, are presented in Table 14.13 when zero application

Table 14.13. Corn Yields, and Nitrogen and P_2O_5 Combinations Predicted for Limited Capital Situations When Nitrogen and P_2O_5 Cost \$.13 and \$.09 per Pound, Respectively

Function	Capital	Pounds per Acre		Corn Yield
		N	P_2O_5	
Quadratic equation 14.26	20	98	80	97
	40	181	182	129
Square root equation 14.27	20	97	81	105
	40	179	186	128
"1.5" equation 14.28	20	97	81	99
	40	181	183	129

rates are not included in the analysis. The figures show the optimum amounts of N and P_2O_5 which should be used if capital for fertilizer is limited to \$20 and \$40 per acre for three functions.

For capital levels of \$20 and \$40 per acre, nutrient recommendations are almost identical. The above comparison has been made only for one set of nutrient prices. However, this set of prices typified fertilizer prices current at the time of the study. With greatly different nutrient prices and amounts of capital, the predictions by the three equations would differ more.

RED CLOVER

The general empirical procedures for red clover were the same as for corn except that only the three equations below were derived. The methodological steps in fitting functions only to observations of nonzero fertilizer inputs were not repeated for red clover and alfalfa. (The experiments were designed for both alfalfa and red clover as outlined in Table 14.1.) Here Y refers to yield in tons, Y′ refers to yield above check plot, P refers to P_2O_5, and K refers to K_2O in pounds.

$$(14.29) \qquad Y' = .36551K^{.0384}P^{.1864}$$

$$(14.30) \qquad Y = 2.657 + .0019K + .0079P - .0000018K^2 - .0000167P^2 - .0000031KP$$

$$(14.31) \qquad Y = 2.46 - .000073K - .003952P + .028141K^{.5} + .128004P^{.5} - .000980K^{.5}P^{.5}$$

Basic statistics of the equations are given in Table 14.14. The clover data were relatively more variable than were the corn data. For the preceding three algebraic functions, the largest portion of variance explained by fertilizer nutrients was the 64 per cent for the square root

Table 14.14. Values of R and t for Individual Regression Coefficients

Equations	Value of R	Value of t for Coefficients in Order Listed				
14.29	.7510*	15.16*	3.12*			
10.30	.7622*	2.17*	9.20*	.76‡	6.99*	1.48†
14.31	.8016*	.10‡	5.52*	1.81†	8.23*	1.82†

*$0 < P < .01$
†$.10 < P < .20$
‡$P > .40$

functions. The t value for one regression coefficient in both the cross-product and square root function was not significant at the 40 per cent level of probability. The K^2 term was dropped from the quadratic equation and the K term from the square root equation (see Table 14.15) since these terms were not significant. The new regression coefficients are shown in equations 14.32 and 14.33.

$$(14.32)\quad Y = 2.68 + .0013K + .0079P - .00000017P^2 - .0000031KP$$

$$(14.33)\quad Y = 2.468 - .003947P + .026834K^{.5} + .127892P^{.5} - .000979K^{.5}P^{.5}$$

As Table 14.15 indicates, dropping one term from each of the equations did not result in a significant decrease in the R^2 values.[4] After comparing response curves and isoquants from the new two-variable functions with (a) individual observations from the experiment and (b) similar estimates from the single-variable function, it was decided to use the latter four-term square root function for the estimates which follow, i.e., to use equation 14.33.

Table 14.15. Sum of Squares and Value of F in Analysis of Variance for Added Regression Term

Item*	Quadratic	Square Root
Deviation from regression, four terms	15,026.07	12,751.18
Deviation from regression, five terms	14,946.95	12,749.88
Reduction due to added term	79.12	1.30
Value of F	.572	.011

*Degrees of freedom are 109 for four terms, 108 for five terms, and 1 for regression term analyzed.

[4] The R values are .76 and .80, respectively, for the new quadratic and square root functions. In order of the coefficients in the regression equations, the t values are: quadratic = 3.21, 9.20, 6.98, and 1.50: square root = 5.55, 3.08, 8.29, and 1.82.

Production Surface Estimates

The surface of Figure 14.24 shows the surface to be relatively flat for large inputs of either or both nutrients. Diminishing total yields are attained with extremely large quantities of P_2O_5. While the marginal products of K_2O decline for large inputs, negative marginal products do not exist, on the predicted surface, for this nutrient. The marginal products for small nutrient inputs are largest for P_2O_5. Hence, it is the most limiting of the two nutrients. However, with more than 160 pounds of both nutrients, K_2O has higher marginal products than P_2O_5. The first 40 pounds of P_2O_5 have a greater effect in increasing total yields than for K_2O although increases from P_2O_5 are smaller as K_2O is increased. This is because P_2O_5 and K_2O substitute more for each other in red clover production than did P_2O_5 and N for corn.

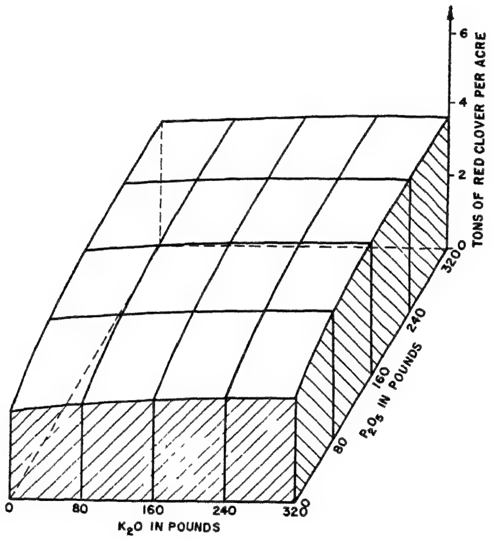

Figure 14.24. Predicted yield response surface for red clover

Yield Isoquants and Substitution Ratio

Figure 14.25 includes the product isoquants predicted for yields of 2.8, 3.1, and 3.4 tons per acre.

Only slight quantities of P_2O_5 alone or P_2O_5 in combination with K_2O are predicted to produce a yield of 2.8 tons. This yield can be produced with about 8 pounds of P_2O_5 alone; for any quantity of P_2O_5, 1 pound of the K_2O replaces less than 1 pound of P_2O_5. Although the isoquant extends out as far as 80 pounds on the K_2O axis, this quantity of K_2O

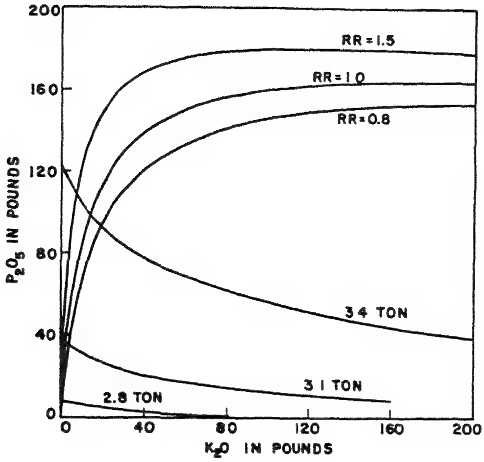

Figure 14.25. Yield isoclines showing points of equal slope and replacement rate for red clover isoquants

would never be profitable in producing a 2.8-ton yield. On this portion of the isoquant, 1 pound of K_2O substitutes for only a very small fraction of a pound of P_2O_5. Actually, economic combinations of nutrients for a 2.8-ton yield do not exist away from the P_2O_5 axis. Only this nutrient should ever be used for a 2.8-ton yield; for all practical purposes, the 2.8-ton isoquant does not exist. Somewhat the same situation holds true for a yield of 3.1 tons. A 3.1-ton yield is predicted to be attained with 37 pounds of P_2O_5 and none of K_2O, or any of the other combinations indicated on the middle isoquant. The replacement rate of K_2O for P_2O_5 becomes less than 1:1 with 36 pounds of P_2O_5 and 2 pounds of K_2O. While the isoquant is predicted to extend far out toward the K_2O axis, the substitution rates are extremely low at these extremes.

The isoquant for a 3.4-ton yield has greater slope than the 3.1-ton isoquant. Accordingly, K_2O replaces P_2O_5 at a higher rate over a wider range of K_2O inputs. One pound of K_2O replaces more than 1 pound of P_2O_5 up to 14 pounds of K_2O; the substitution rate becomes 1:1 with 14 pounds of K_2O and 98 pounds of P_2O_5. But only a very slight amount of K_2O will, according to the predictions of the production relationships, be profitable at the usual ratio of prices for the two nutrients. However, larger amounts of K_2O are predicted to be necessary for higher yields.

This point is illustrated by the lines labeled RR. They are isoclines indicating the path of a given replacement rate over the map of yield isoquants or contours. They indicate the point on any yield isoquant where the replacement rate, for the stated yield, is that indicated by the isocline. For yields of 3.4 tons or less, the three isoclines (which represent a range of price ratios which might be attained for the two

nutrients) are close to the phosphate axis. For higher yields, they are predicted to veer in a direction specifying a larger proportion of K_2O. Yields as high or greater than 3.4 tons would never be profitable; the nutrient to hay price ratio has never been low enough and the marginal response for fertilizer is not high enough at this yield level. However, the isoclines do predict the least-cost nutrient combination for each possible yield level. With P_2O_5 and K_2O costing the same amount per pound, the price ratio is 1.0. Since the least-cost nutrient combination is attained when the price ratio is equal to the replacement or substitution ratio, each isocline traces out the path of least-cost nutrient combinations as higher yields of red clover are attained.

ALFALFA

Two-variable functions derived from the alfalfa yield data are listed below. Predictions from these were compared with (a) predictions from 35 single-variable functions and (b) a scatter diagram of observations. These comparisons, along with the statistics of Table 14.16, again suggested that the square root function provided the best estimates of the production or yield surface and related quantities.

$$Y' = .87935K^{.0542}P^{.1310} \tag{14.34}$$

$$Y = 2.2514 + .0033K + .0097P - .000007K^2 - .000020P^2 - .000001KP \tag{14.35}$$

$$Y = 1.8737 - .0014K - .0050P + .06731K^{.5} + .173513P^{.5} - .001440K^{.5}P^{.5} \tag{14.36}$$

Table 14.16. Values of R and t for Individual Regression Coefficients

Equation	Value of R	Value of t for Regression Coefficients in Order Listed				
14.34	.7329*	9.01*	4.29*	--	--	--
14.35	.8128*	3.31*	9.74*	2.50†	7.31*	.42‡
14.36	.8793*	1.99†	6.81*	3.85*	10.83*	2.03†

*$0 < P < .01$
†$.01 < P < .05$
‡$P > .05$

Production Surface Estimates

Optimizing quantities are shown in Table 14.17 for specified price combinations. The square root function was used in deriving these quantities and in providing the production surface of Figure 14.26.

Table 14.17. Optimum Rates and Combinations of Fertilizer for Specified Crop and Nutrient Prices

Price Situation	Price per Unit			Optimum Quantity			
	Lbs. K_2O	Lbs. P_2O_5	Tons hay	Total lbs. nutrient	Lbs. K_2O	Lbs. P_2O_5	Yield (tons)
A	.12	.09	$16	71.4	8.0	63.4	3.07
B	.12	.09	10	41.0	3.9	37.1	2.84
C	.12	.09	22	98.0	12.5	85.5	3.20
D	.12	.09	28	120.8	17.2	103.6	3.29
E	.09	.12	16	58.8	13.7	45.1	2.99
F	.09	.12	10	31.8	6.9	24.9	2.75
G	.09	.12	22	84.5	21.0	63.5	3.14
H	.09	.12	28	107.5	28.1	79.4	3.24
I	.08	.08	10	50.2	7.8	42.4	2.93

Diminishing marginal yields are indicated for either nutrient increased alone (with the other one fixed at the specified levels of the rows or columns) or for both nutrients increased in fixed proportion. The predicted maximum yield is forthcoming (i.e., the marginal products or first derivatives are zero) at a 3.64-ton yield with 232.2 pounds of P_2O_5 and 203.6 pounds of K_2O.

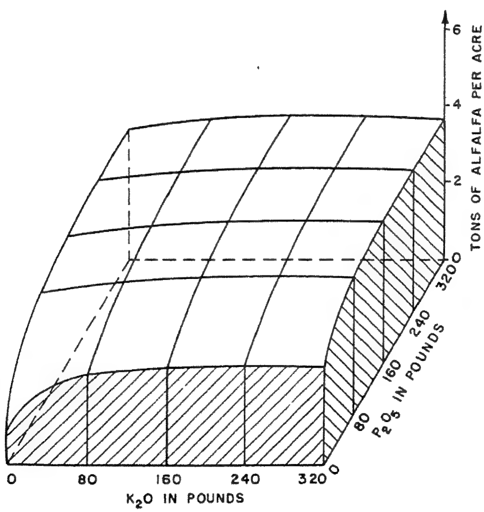

Figure 14.26. Predicted yield response surface for alfalfa

Yield Isoquants and Nutrient Combinations

Yield isoquants for alfalfa are shown in Figure 14.27. A 2.5- or 3.0-ton yield can be attained with P_2O_5 alone. However, the maximum P_2O_5 for a 3.5-ton yield is 225 pounds or 90 per cent of the total nutrient application. Hence, the range of possible substitution again declines with increased yield. For 3.6 tons, the substitution ratio drops below 1.0 with less than 202.9 pounds of P_2O_5 and more than 100 pounds of K_2O. The range of replacement possibilities for the 2.5-ton yield ranges only from zero to 120 pounds of K_2O.

The slopes of the contour or isoquant lines change at higher yields. The least-cost fertilizer mixture thus differs somewhat for 2.5-, 3.0-, or 3.5-ton yields. The isocline labeled RR of 1.0 indicates the path over which 1 pound of K_2O replaces 1 pound of P_2O_5. For a yield of 3.2 tons, the intersection point indicates that a replacement ratio of 1:1 is attained with about 78 pounds of P_2O_5 and 17 pounds of K_2O. For points on the 3.2-ton yield isoquant above this point of intersection, K_2O substitutes for P_2O_5 at a rate greater than 1:1; for points below the intersection point, 1 pound of K_2O replaces less than 1 pound of P_2O_5. Similarly, for a 3.6-ton yield, the replacement rate between nutrients is 1:1 with about 198 pounds of P_2O_5 and 107 pounds of K_2O (i.e., at the point where the yield isoquant and the isocline of 1.0 intersect). Proportionately more K_2O is required at higher yield levels if a 1:1 substitution ratio is maintained.

The isocline RR = 1.5 indicates the nutrient combination for each successive yield level where 1 pound of K_2O replaces 1.5 pounds of P_2O_5. Isocline RR = .8 indicates nutrient combinations where 1 pound of K_2O substitutes for .8 pound of P_2O_5. Again, these isoclines indicate the most economical combination (i.e., the least-cost combination) of nutrients for any specified yield level. With K_2O at 12 cents and P_2O_5

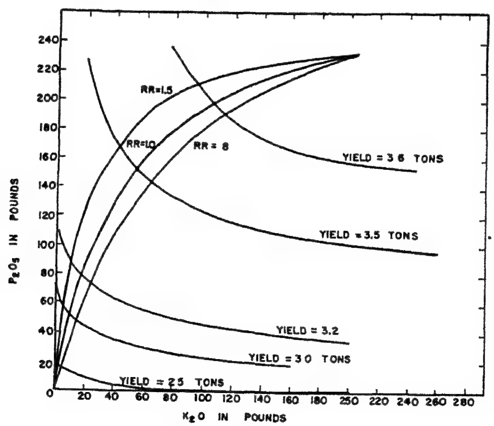

Figure 14.27. Yield isoquants and isoclines for alfalfa

at 8 cents per pound, the price ratio is $\frac{12}{8}$ or 1.5. The least-cost nutrient combination includes 88 pounds of P_2O_5 and 10 pounds of K_2O for a 3.2-ton yield; it includes 213 pounds of P_2O_5 and 94 pounds of K_2O for a 3.6-ton yield (although this yield level may not itself be profitable, the nutrient combination is the one allowing the lowest fertilizer cost for the particular yield). Since the isoclines "bend" towards the K_2O axis, proportionately more K_2O must be used for higher yields if the least-cost nutrient combination is to be attained. (A single nutrient combination would provide the least-cost combination for all yield levels only if the isoclines were straight lines passing through the origin.) The isoclines converge to a single point denoting the maximum possible yield; replacement of one nutrient by the other is not possible for 3.64 tons of alfalfa.

Optimum Rates for Alfalfa

Optimum rates of fertilizer application for alfalfa are predicted in Table 14.17 from equation 14.36. These estimates were obtained by setting the partial derivatives of hay yield in respect to K and P to equal the appropriate price ratios and solving for the magnitudes of the variables. Only this type of optimum is considered for alfalfa; although the alternatives considered for corn are equally appropriate.

Under price situation A, the nutrient input includes 87.4 per cent P_2O_5 and 12.6 per cent K_2O. With a fall in hay price from \$16 to \$10 (B) and nutrient costs remaining the same, only 41 pounds of total nutrients should be used. However, the total nutrient input now should be composed of 90.5 per cent P_2O_5 and only 9.5 per cent K_2O. An increase in hay price to \$28 (D) requires an increase to 120.8 total pounds of nutrients for economic optimum. The 120.8 pounds is composed of 85.8 per cent P_2O_5 and 14.2 pounds of K_2O.

RESIDUAL RESPONSE FUNCTIONS FOR CORN

Residual responses also are important in determining the economic optimum use of fertilizer. Preceding discussions related to responses in the year following fertilizer application. For the corn experiment on Ida silt loam it was possible to obtain second-year yields; the land was planted back to corn and no fertilizer was applied in the second year. Hence, the second-year response functions for 1953 reported below are due alone to the "carry-over" effect of fertilizer applied in 1952.

Response Functions and Related Data

Two methods were used in analyzing the 1953 corn response data: (a) total response surfaces for the 2 years were computed by adding the

1953 yields to the 1952 yields and fitting functions to these combined data, including zero application rates and (b) functions were fitted to the 1953 "carry-over" yields alone. The predicted production functions are given below where all observations, including zero rates, are used:

Two-year total (1952 and 1953 data pooled and functions fitted to total yield of 2 years):

$$Y = -.0965 + .6464N + .8140P - .00176N^2 - .00231P^2 + .00149NP \tag{14.37}$$

$$Y = 12.636 - .2213N - .4614P + 4.2464N^{.5} + 8.7506P^{.5} + .5603N^{.5}P^{.5} \tag{14.38}$$

Second-year residual (1953 yields only):

$$Y = 7.4177 + .0621N + .1502P - .000180N^2 - .000511P^2 + .000683PN \tag{14.39}$$

$$Y = 18.317 + .0948N - .0440P - 2.1047N^{.5} + .2352P^{.5} + .2193N^{.5}P^{.5} \tag{14.40}$$

The R^2 values are: quadratic 2-year total (14.37), 88 per cent; square root 2-year total (14.38), 92 per cent; quadratic residual (14.39), 81 per cent; and square root residual (14.40), 77 per cent.

Table 14.18. Values of R and t for Individual Regression Coefficients

Equation	Value of R	Value of t for Coefficients in Order Listed in Equations				
14.37	.940	8.82*	11.11*	8.65*	11.33*	8.37*
14.38	.961	4.11*	8.58*	3.63*	7.49*	10.80*
14.39	.900	2.00†	4.84*	2.08†	5.93*	9.03*
14.40	.878	3.09*	1.43‡	3.16*	.35§	7.41*

*$0 < P < .01$
†$.01 < P < .05$
‡$.10 < P < .20$
§$.50 < P$

Although the square root function did not fit the residual data quite so well as the quadratic, it did fit the 2-year total yields better than the crossproduct. Since the 2-year total yields combine both the first- and second-year response, the square root function was chosen for use in the following economic analysis.

If the coefficients of the second-year residual are added to the corresponding coefficients of the first-year function, the result is equal to the 2-year total functions given above. For example, the coefficients of

Table 14.19. Analysis of Variance for Total or 2-Year and for Second Year Residual Response Functions

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square
Quadratic 2-year total			
Total	113	466,207.68	
Treatments	56	450,794.90	8,049.91
Due to regression	5	411,938.30	82,387.66
Deviations from regression	51	38,856.60	761.89
Among plots treated alike	57	15,412.78	270.40
	F = 305		
Square root 2-year total			
Total	113	466,207.68	
Treatments	56	450,794.90	8,049.91
Due to regression	5	430,186.01	86,037.20
Deviations from regression	51	20,608.89	404.10
Among plots treated alike	57	15,412.78	270.40
	F = 318		
Quadratic second year residual			
Total	113	51,216.90	
Treatments	56	45,742.37	816.83
Due to regression	5	41,488.27	8,297.65
Deviations from regression	51	4,254.10	83.41
Among plots treated alike	57	5,474.53	96.04
	F = 86.4		
Square root second year residual			
Total	113	51,216.90	
Treatments	56	45,742.37	816.83
Due to regression	5	39,485.33	7,897.07
Deviations from regression	51	6,257.04	122.69
Among plots treated alike	57	5,474.53	96.04
	F = 82.2		

equation 14.40 plus those of equation 14.3 equal those of equation 14.38. This is to be expected; the production surface for 2 years is the sum of the surfaces for the first and second years.

Yield Isoquants

Yield isoquants were derived from the basic production function in the manner outlined in previous sections. The isoquants for the 2-year production surface denote diminishing productivity over the entire

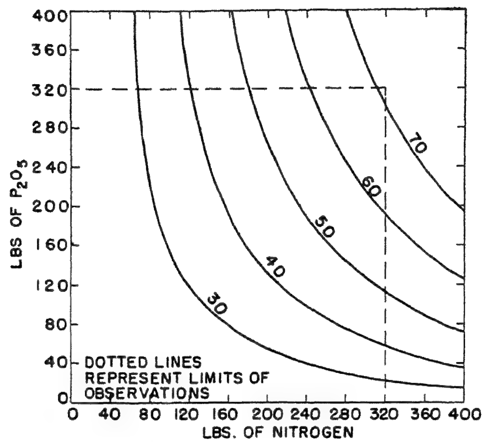

Figure 14.28. Corn yield isoquants for square root residual function 14.40, for second year alone

surface. The segments on scale or fixed ratio lines which are intersected by yield isoquants (representing equal increments in yield) become greater for higher yield levels. In the case of the residual or second-year response surface, however, the scale lines show slightly increasing returns for small applications of fertilizer (see Figure 14.28 and Table 14.20). However, decreasing returns in the second year might have occurred if heavier fertilization had been used in the first year. Also, the second-year response may have, first, a stage of increasing returns and, second, a stage of decreasing returns.

For economic decisions, residual response must be considered in conjunction with the first-year response. Therefore, isoquants for the

Table 14.20. Isoquant Combinations for Producing Specified Yields; Residual Function 14.40

40-Bushel Yield		50-Bushel Yield		60-Bushel Yield		70-Bushel Yield	
Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N
20	489.0	50	465.9	50	586.3	50	706.0
50	342.5	80	378.5	80	487.9	80	596.9
80	268.4	100	338.6	100	441.6	100	544.9
100	236.0	150	272.2	150	362.2	150	453.7
150	184.6	200	231.8	200	311.8	200	394.1
200	155.4	250	205.1	250	277.2	250	352.2
250	137.2	300	186.5	300	252.2	300	321.2
300	125.2	350	173.0	350	233.6	350	297.6
400	111.1	400	163.0	400	219.4	400	279.2

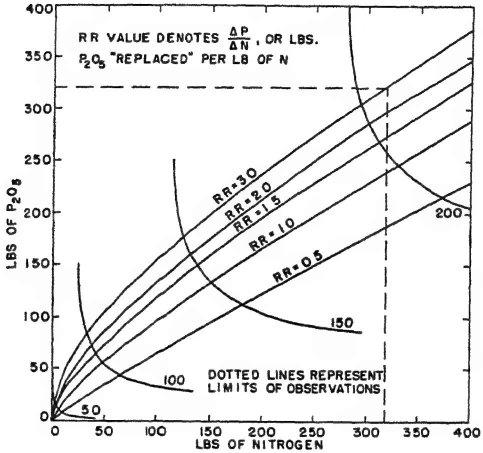

Figure 14.29. Corn yield isoquants and isoclines for 2-year total square root function 14.38

2-year total response surface are given in Figure 14.29 and Table 14.21. Five isoclines are presented along with four isoquants in Figure 14.29. The center isocline, RR = 1.5, is appropriate for an N to P_2O_5 price ratio of 1.5, approximately the present price relationship. If the price of N were twice that of P_2O_5, the isocline labeled RR = 2.0 would be the appropriate one. If some new process should make nitrogen one-half the price of available P_2O_5 then the isocline with RR = .5 would be the one to be followed to maximize profits (considering both the first- and second-year response). Of course, these isoclines will depend upon or differ with the soil type and fertility level. For other soil types and fertility conditions, a different production surface would be expected. It is obvious for the isoclines presented that any price ratio which would require expansion of fertilizer use along the isocline RR = .5 would give

Table 14.21. Isoquant Combinations for Producing Specified Yields; 2-Year Total Function 14.38

50-Bushel Yield		100-Bushel Yield		150-Bushel Yield		200-Bushel Yield	
Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N	Lbs. P_2O_5	Lbs. N
3	37.2	30	117.4	80	361.0	180	515.7
5	19.8	50	54.6	100	207.8	200	412.5
10	6.9	80	33.2	120	165.0	250	325.1
15	3.0	120	25.6	150	135.8	300	294.4
20	1.3	150	24.3	200	117.4	350	283.0
30	.2	180	24.7	250	113.1	400	281.3

about the same economic results as increasing nutrients in a P/N ratio of 1.7. This isocline deviates only slightly from a straight line through the origin. Also, some of the other isoclines have only slight curvature, denoting only slight profit depression if a fixed nutrient combination is used for increasing yield. Rates of fertilization were not great enough to define the point of maximum yield and convergence of isoclines for the 2-year total yield within the range of the experiment.

Economic Optima

Using the 2-year total function (equation 14.38), the optimum inputs of nitrogen and phosphorous can be found in the way shown in the preceding sections. Optimum inputs are determined by equating marginal physical products with their corresponding factor-product price ratios. This optimum solution is valid in the 2-year case only when the expected price of corn is the same for both years and the farmer does not discount the expected value of the second crop more than the first crop.

However, farmers generally discount the value of distant crops more than current crops. In this case, the problem can still be solved by adding the discounted residual production surface to the first-year surface. As an example, assume the farmer expects the price of corn to be $1.25 per bushel for both the first and second year. He discounts the value of the second crop by 20 per cent due to uncertainty or other reasons. This makes the present value of the second-year corn worth $1 per bushel. The response function is now the first-year response coefficients plus .80 times the second-year coefficients.

Optimum inputs of N and P for various prices of corn are in Table 14.22. The corn prices in this table are assumed to be the present discounted values of the crops at the time fertilizer is applied.

A more important problem is to determine the optimum amount of

Table 14.22. Optimum First-Year Applications of Fertilizer, Considering the Response From Both the First and Second Years for Specific Price Relationships*

Price Situation				Optimum Fertilizer Application	
First-Year Corn per Bushel	Second-Year Corn per Bushel	P_2O_5 per Pound	N per Pound	Pounds N	Pounds P_2O_5
$1.50	.75	.10	.15	290.1	259.6
1.25	.625	.10	.15	241.5	226.6
.625	.3125	.10	.15	101.5	120.8
.50	.25	.10	.15	72.6	95.1

*Prices of corn are assumed to be the present discounted value to the farmer at the time fertilizer is applied.

fertilizer to be applied in the second year. This can not be answered from the present data. One hypothesis is this: the optimum application for the second year will drop back to considerably less than that for a single year. More research regarding the relation of soil fertility to fertilizer response appears necessary before some of these problems can be adequately answered.[5]

In this section and in previous ones, the appropriate economic principle has been applied in specifying optima. It is recognized, of course, that uncertainty and other factors do not allow farmers to be so precise in their decision making. The purpose of this study, however, has been to apply appropriate methods. Mechanical or "rule of thumb" procedures can be developed for applying these basic principles with only slight depression of profit, once additional research provides added information on basic response functions.

LIMITATIONS AND EXPERIMENTAL NEEDS

The concepts and analytical procedures employed in this study are basic for determining economic optima in the use of fertilizer. Also, they provide the basic physical or structural relationships of crop responses in relation to fertilizer application. The predictions apply to particular soils in a particular year; production surfaces obtained under other rainfall and soil conditions can be expected to differ from those obtained in the experiments reported. These limitations are not, however, unique to the type of experiment and empirical procedure reported here. Traditional experimental procedures (wherein a few rates of one or more nutrients are applied) also refer to the rainfall, climatic, insect, and crop conditions of the particular year.

[5]See Jensen, D. and Pesek, J. Generalization of yield equations in two or more variables. Agronomy Jour., 51: 255-63. 1959.

CHAPTER 15

John T. Pesek
Earl O. Heady

Surfaces, Isoquants, and Iscolines From Fertilization

THIS CHAPTER summarizes several fertilizer production function studies in Iowa. The several sets of predictions are provided to illustrate the types of physical relationships which arise from various functional forms and magnitudes of regression coefficients which appear most efficient in response predictions under various environmental conditions.

The analysis also illustrates the types of empirical studies which often can be made from limited resources for experimentation or from data which already exist. Both situations held true for some of the experiments reported in this chapter. However, experiments of recent years emphasize the composite and rotatable designs discussed in earlier chapters.

Not all experiments reported provide such "neat" functions as the corn experiment reported in Chapter 14. However, this is the situation which faces the farmer: response may be large under favorable rainfall but small or nonexistent in dry years. Hence, it is as important to predict response under conditions where an experiment may provide few or no significant regression coefficients as it is to conduct experiments only for years where response is great. A variety of conditions in respect to nutrient combinations, stand levels, and harvesting are reported in this chapter.

CARRINGTON EXPERIMENT

This section reports a corn experiment on Carrington silt loam in 1953. The factorial experiment providing the data consisted of two randomized blocks, each block having five levels of N, four levels of P_2O_5 and three levels of K_2O. Yields were high in this experiment; plots without fertilizer averaged almost 98 bushels per acre. Large yield responses for fertilizer were not expected since the soil was at a relatively high fertility level (i.e., high yields were obtained on the check plots). However, an average increase of 9.8 bushels per acre

*This chapter summarizes a series of studies made by the authors in co-operation with William G. Brown, John P. Doll, Owen W. McCarthy, R. P. Nicholson, and Joseph A. Stritzel.

Table 15.1. Analysis of Variance of Corn Yields on Carrington Soil, Randomized Block Design

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Total	119	18,570.37		
Blocks	1	6,336.53	6,336.53	106.39*
Treatments	59	8,719.703	147.80	2.48*
N	4	514.042	128.51	2.16
P	3	79.285	26.43	.44
K	2	4,198.086	2,099.04	35.24*
N x P	12	523.262	43.61	.73
N x K	8	630.862	78.86	1.32
P x K	6	870.996	145.17	2.44*
N x P x K	24	1,903.170	79.30	1.33
Error	59	3,514.137	59.56	

*P < .01

was obtained from 40 pounds of K_2O. Application of 80 pounds of K_2O resulted in an average increase of 14.2 bushels over the plots with no potash. The significant potassium effect (Table 15.1) might have been anticipated since the experimental plot was, according to soil tests, low in K_2O.

Values of F in Table 15.1 provide information for variables to be used in the estimating equation or production functions which follow. Potash should be included since K_2O gives a consistent and statistically significant increase in yield. Phosphorus can be dropped from consideration because, even if all of the mean square due to P were explained by one regression term, its F value would not be significant. Nitrogen is an intermediate case; there is some logical justification for including it even though it is not significant at the .05 level of probability. Phosphorus x potash interaction is significant at the .05 level. However, it was not included in the regression because no term was found which would significantly account for the variance in yield due to this term. An analysis of covariance indicated that stand had a highly significant effect on yield. Similar results were obtained when stand was included as a variable in the multiple regression.

Regression Analysis for Carrington Soil

The basic purpose of this study is to estimate crop yield production functions for fertilizer. Accordingly, information on the derivation of the regression equations is included in this section. In the preliminary analysis for each experiment, two general types of equations were used: a quadratic equation and a square root equation.

The highly significant difference between the yields of the two randomized blocks (Table 15.1) raised a question as to whether the response surface differed significantly between the two blocks. To test

Table 15.2. Values for t for Coefficients of Individual Block Regressions and Test of Difference Between Corresponding Coefficients of the Two Blocks

Coefficient	Values of t for Equation 15.1	Significance Level*	Values of t for Equation 15.2	Significance Level*	Values of t for Difference Between Equations 15.1 and 15.2†	Significance Level*	Values of t for Pooled Regression Equation 15.3	Significance Level*
K	3.515	.001	2.006	.06	.739	.47	4.118	.0001
K^2	2.059	.05	.810	.43	.834	.41	1.965	.06
N	.737	.48	.996	.33	.211	.84	.316	.20
N^2	.719	.48	.620	.55	.045	.92	1.030	.31
S	3.757	.001	2.554	.02	.731	.46	4.556	.00002
B	--	--	--	--	--	--	9.570	.00001

*Probability of obtaining as large or larger value of t by chance, given the null hypothesis.
†These t's have been computed by subtracting each particular regression coefficient in equation 15.2 from the corresponding regression coefficient in equation 15.1 and dividing by the weighted standard error.

whether the response differed between blocks, regressions were calculated for each block separately as indicated in equations 15.1 and 15.2.

$$(15.1)\qquad \text{Block I: } Y = 57.97 + .3800\,K - .002711\,K^2 + .4365\,N^{.5} - .02638\,N + .002552\,S$$

$$(15.2)\qquad \text{Block II: } Y = 51.64 + .2702\,K - .001162\,K^2 + .6414\,N^{.5} - .02490\,N + .002081\,S$$

In the above equations, Y refers to predicted total yield in bushels per acre, K refers to pounds of K_2O per acre, N to pounds of elemental nitrogen per acre, and S refers to stalks per acre. The t values of the regression coefficients are given in the left half of Table 15.2. To help determine whether the two blocks should be pooled, t tests of the differences between corresponding regression coefficients were made (Table 15.2).[1] The t values for the difference between corresponding regression coefficients of the two blocks are small. A value of t as large or larger than the t value of difference for K^2, t = .834, could occur by chance 40 per cent of the time even though the population of K^2 coefficients was the same. The other coefficients had even smaller t values of difference. Since there was no evidence that the blocks had different response surfaces (different regression coefficients), a regression for the pooled data of the two blocks was computed as indicated in equation 15.3.

$$(15.3)\qquad Y = 77.866 + .3162\,K - .001813\,K^2 + .9190\,N^{.5} - .04453\,N + .002241\,S - 13.497\,B$$

In the pooled regression above, B represents the particular block; B is 1.0 for Block I and 2.0 for Block II. Stand and block are used here

[1] An analysis of variance was also computed to test for homogeneity of regression; the results were similar to those obtained from the t tests.

as a method of adjustment similar to covariance. The experiment was not designed to include stand as a variable, but variation in stand did occur. This experiment cannot be used to determine optimum stand. However, precision of estimates is considerably improved by including stand in the regression (as shown by its t value of 4.56 in Table 15.2).

Blocks were included in the regression to allow an estimating equation for either block and to increase precision of estimation. Including blocks in the regression is justified since it takes out the variability due only to the difference in stand and yield level of the two blocks. Predicting the actual yield is secondary to predicting the response of corn yield to fertilizer inputs. That is, more interest is in the slopes of the production surface rather than the absolute level of yield. The values of the N and K coefficients are important in determining the most profitable amount of nitrogen and potash to apply. Stand and blocks were introduced only to increase the precision of estimate of the N and K coefficients.

For an average stand and for Block I, the intercept (N = 0; K = 0) of equation 15.4 becomes 105.971. For an average stand and Block II, the intercept is 92.474. Equation 15.4 is the average of the two blocks with an average stand of around 18,000 stalks per acre and will be used in the later economic analysis.

$$Y = 99.223 + .3162\,K - .001813\,K^2 + .9190\,N^{.5} - .04453\,N \qquad (15.4)$$

The value of t (4.118) for the linear response of yield to potash in Table 15.2 is highly significant. Accordingly, greater reliability can be placed in the potash response than in the nitrogen response.

The analysis of variance for the regression estimates is presented in Table 15.3. The F value of 33 indicates that the proportion of variance explained by the regression equation 15.3 is highly significant. However, only 65 per cent of the total sum of squares is accounted for by equation 15.3.

Table 15.3. Analysis of Variance for Regression of Corn Yield, Carrington Soil

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Total	119	18,570.37		
Due to regression equation	6	12,013.09	2,002.18	33.04*
Deviation from regression	113	6,557.29	58.03	
Other treatment effects	55	3,043.15	55.33	
Error	58	3,514.14	60.59	

*$P \leq .01$

Production Surface for Carrington Soil

Equation 15.4 is used for the economic analysis of the experiment on Carrington soil. Since the soil was fertile, yields are predicted to start at 99 bushels per acre with no fertilizer. A yield of almost 118 bushels per acre is predicted at 80 pounds each of N and K_2O. The figures indicate ranges of both increasing and decreasing total yields (i.e., positive and negative marginal products).

A geometric view of the predicted production surface is provided in Figure 15.1. The slope of the surface indicates the response to both N and K_2O. The slope is greater along the K_2O axis than along the N axis. However, the surface in general is quite flat.

A slice through the surface parallel to the potash axis in Figure 15.1 would represent response of corn to K_2O at a fixed level of N. Individual yield response curves to potash for given levels of nitrogen remain the same distance from each other. This lack of "interaction" between N and K_2O was probably a characteristic of the experimental site. Previous experiments have provided production surfaces with important interactions between fertilizer nutrients. However, N and K may interact less with each other than N does with P or P does with K.

Figure 15.2 shows the 95 per cent confidence limits of the yield estimates for K_2O. The spread at the ends of the curve is due to the increased distance from the mean, as the response is extrapolated beyond the 80 pound limit of the K_2O application in the experiment. Confidence intervals for the N response are also relatively narrow, indicating some degree of precision in estimation.

Marginal physical products of N remain the same at all levels of K_2O because there is no interaction between N and K_2O in equation 15.4. Conversely, the marginal physical product of K_2O is not affected by the level of N. This is illustrated in the marginal product equations for N and K_2O:

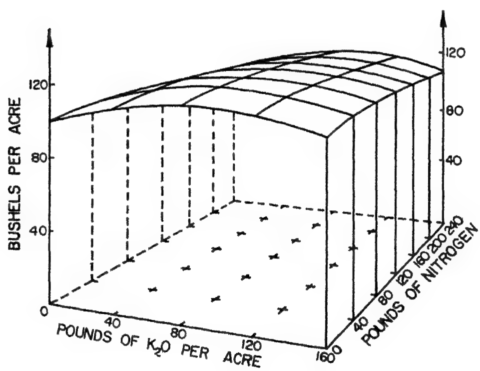

Figure 15.1. Perspective view of predicted yield surface for corn on Carrington soil predicted from equation 15.4

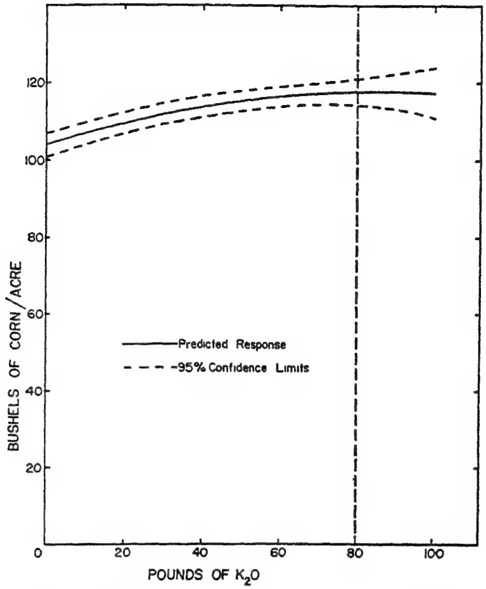

Figure 15.2. Confidence limits for corn response to K_2O at 104 pounds of N, Carrington soil (Dashed vertical line is limit of K_2O in experiment.)

$$\frac{\delta Y}{\delta K} = .3162 - .003626\ K \tag{15.5}$$

$$\frac{\delta Y}{\delta N} = -.04453 + .4595\ N^{-.5}. \tag{15.6}$$

The marginal yields from K correspond to the slope or incline of the "roof" in Figure 15.1 parallel to the N axis. At 0, 20, 40, 60, 80, and 100 pounds of K_2O, the marginal yields are .32, .24, .17, .10, .03, and -.05 bushel, respectively. Similarly, marginal yields for N, computed from equation 15.6 for 1, 20, 40, 60, 80, and 120 pounds of N, are .41, .06, .03, .01, .007, .001, and -.003 bushel, respectively. It can be seen from the N marginal yields that while N returns a fairly large increase in yield at small inputs the response soon levels out. Negative marginal products for either N or K indicate that further inputs at the particular levels cause a decline in total per-acre yield.

Yield Isoquants and Isoclines for Carrington Soil

Yield isoquants in Figure 15.3 are derived from equation 15.7.

$$K = 87.23 \pm \frac{\sqrt{.00666N^{.5} - .000323\ N + .8194 - .00725\ Y}}{.003626} \tag{15.7}$$

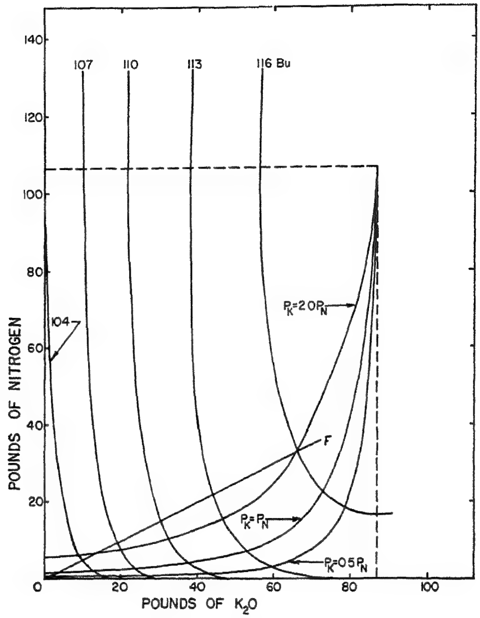

Figure 15.3. Isoquants and isoclines with dashed ridgelines, Carrington soil

The corresponding isocline equation k is the K_2O/N price ratio

(15.8) $$K = 87.20 + 12.28\,k - 126.72\,N^{-.5}k.$$

Isoclines are presented as the positively sloped curves in Figure 15.3. These are bent close to the K_2O axis, denoting a relatively greater deficiency of this nutrient in obtaining yield response. However, for the highest yields, both nutrients become relatively deficient.

The dashed lines in Figure 15.3 represent ridgelines which denote the economic limits of the isoclines. The ridgelines define the portion of the production surface included between the extremes of zero (or infinite) substitution rates for nutrients. (In other words, they are isoclines with zero substitution ratios, indicating the extreme limits of nutrient substitution in obtaining specified yields.) The ridgelines (isoclines of zero substitution rates) indicate the boundaries of the surface with positive slopes along both input axes; beyond the ridges, one or both slopes are negative. If nitrogen were free in price but K_2O were not, it would pay to expand production along the top ridgeline, always applying 106 pounds of N and purchasing K_2O according to its cost and return. On the other hand, if potash were free and nitrogen were not, production should be expanded along the right hand vertical ridgeline. Since N and K_2O were independent in basic surface equation 15.4, the ridgelines are straight and meet at a right angle.

All the isoclines (including ridgelines) converge and intersect at the point of maximum physical product. If both N and K_2O were free and cost nothing to apply, inputs should be extended to 87.2 pounds of K_2O and 106.5 pounds of N, the point of isocline convergence. A maximum physical yield of 117.76 bushels is predicted from these inputs of N and K_2O.

Carrington Soil Presentation for Practical Use

While the main purpose of this study is that of dealing with certain basic or methodological aspects of fertilizer response and economics, it is useful to indicate how the results can be presented for farmers or extension personnel. Since N and K_2O effects were independent in the production function, equation 15.4, the optimum rate for N can be selected without regard to the level of K_2O and vice versa. To find the optimum input of either nutrient, divide the price (per pound) of the nutrient by the price of corn. Selection of the corresponding ratio from one of the charts in Figure 15.4 then provides the optimum input.

Assuming N to be \$ .15 and K_2O to be \$ .08 per pound with corn at \$1.00 per bushel, the appropriate N/C price ratio is .15 and the appropriate K/C price ratio is .08. Use of these ratios in Figure 15.4 indicates an optimum input of about 65 pounds of K_2O and 6 pounds of N. The gain in yield from these inputs also can be estimated. A gain of about 13 bushels per acre from use of K_2O and about 1.5 bushels from the N application is predicted. Of course, such a chart should be used for a Carrington soil with fertility similar to the experimental field. Rainfall and biological conditions also would need to be as favorable as for the experimental field in 1953.

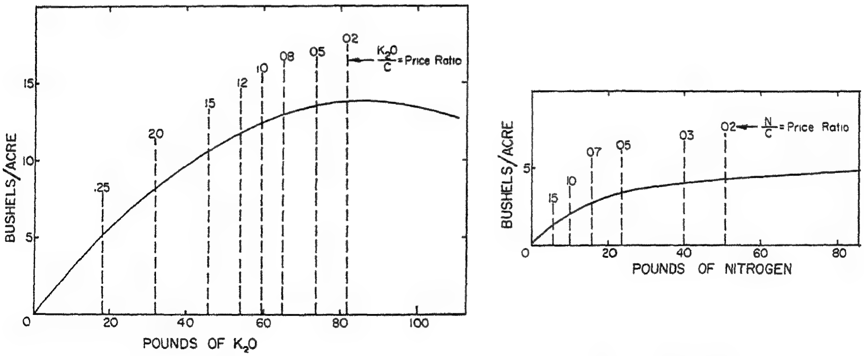

Figure 15.4. Added bushels from K_2O and N and optimum rates for specified price ratios of fertilizer nutrients and corn, Carrington soil

MOODY EXPERIMENT

The experiment reported in this section was made on Moody soil in 1953. Cropping history and soil tests indicated low availability of nitrogen and phosphorus, and high availability of potassium. Large responses in corn yields were obtained by adding nitrogen; in fact, yield was more than doubled by applying 40 pounds of N. Further increases in yield were given by 80 and 160 pounds of N. However, with 240 pounds of N, a slight decline resulted. Potassium had little effect on yield. Phosphorus also seemed to have only a small effect since yield was increased by less than 8 bushels in rates ranging from 0 to 120 pounds. However, examination of the average response to P_2O_5 over all levels of N and K_2O hides part of the actual effect. Actually, as careful examination of the yields indicates, there was a strong interaction between phosphorus and nitrogen. At zero level of N, P_2O_5 had a depressing effect on yield, but at 160 and 240 pounds of N it increased yield.

Table 15.4. Analysis of Variance of Corn Yields on Moody Soil, Randomized Block Design

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Total	117	53,164.96		
Blocks	1	432.82	432.82	4.57*
Treatments	59	47,330.08	802.20	8.46†
N	4	41,000.49	10,250.12	108.15†
P	3	1,051.22	350.41	3.70*
K	2	123.05	61.53	.65
N x P	12	3,333.88	277.82	2.93†
N x K	8	455.83	56.98	.60
P x K	6	99.55	16.59	.18
N x P x K	24	1,266.06	52.75	.56
Error	57	5,402.06	94.77	

*$P \leq .05$
†$P \leq .01$

The analysis of variance in Table 15.4 confirms the highly significant effect of nitrogen. The effect of P_2O_5 was significant at the .05 probability level. Interaction between N and P_2O_5 was highly significant. There also was a significant difference between the yield levels of the two randomized blocks. Potash and the remaining interactions accounted for no significant portion of yield variance. The lack of potash response was expected since the experimental plots tested high in K. Since the soil test for P was low, negative phosphorus response at low levels of N (see above) was unexpected. However, other evidence suggests that this can happen as a result of aggravating the nitrogen deficiency.

Regression Analysis for Moody Soil

Several algebraic forms of the yield predicting equation were tried before equation 15.9 was selected. Equation 15.9 had an R^2 of .827 and was logically more acceptable than certain other forms.

$$(15.9) \quad Y = 13.543 + .5340N - .001743\,N^2 - .0003549\,P^2 + .001069\,NP + .000873\,S.$$

In this equation, Y refers to total yield in bushels per acre, N refers to pounds of nitrogen per acre, P refers to pounds of P_2O_5 per acre, and S refers to stalks per acre. Values of t for the coefficients in the order that they appear in the equation are 12.56, 14.47, 1.68, 5.44, and 1.50. The preceding t values for N, N^2, and NP are significant at the .00001 level of probability. The terms for P^2 and S are significant at .10 and .14 probability levels, respectively, and are retained for logical reasons. The negative P^2 term is important because it forces diminishing returns to phosphorus inputs. Some of the functions fitted to the data, or the particular function without this term, did not have this characteristic. For example, the full five-term square root or regular quadratic functions gave increasing returns for part of the production surface; increasing returns make it difficult to secure determinate economic solutions. The five-term quadratic equation was as follows:

$$(15.10) \quad Y = 30.277 + .533\,N - .00175\,N^2 - .623\,P + .000066\,P^2 + .00116\,NP.$$

Equation 15.10 was rejected in favor of equation 15.9 since the latter gave diminishing returns and a determinate predicted maximum yield. The stand variable was included in equation 15.9 to increase the precision of fit of the nutrient response curves; equation 15.11 has been adjusted to an average stand and was used for the subsequent economic analysis. Equation 15.11 is the same as equation 15.9 except that average plot stand is fixed at 18,000 stalks per acre. With the coefficient for S significant at the .14 probability level, the writers adopted this procedure as being more efficient than the conventional procedure of adjusting individual plot yields for stand.

$$(15.11) \quad Y = 29.248 + .5340\,N - .0001743\,N^2 - .0003549\,P^2 + .001069\,NP.$$

The analysis of variance of the basic regression, equation 15.9, is given in Table 15.5. The F value of 91.20 for the over-all regression is highly significant. The mean square for deviations from regression is smaller than the within-plot estimate of experimental error.

Production Surface for Moody Soil

Estimated yields were predicted from equation 15.9 to provide the corn production surface for the Moody soil.

Table 15.5. Analysis of Variance for Regression of Corn Yield on Moody Soil

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Total	117	53,164.96		
Blocks	1	432.82	432.82	4.49*
Due to regression, equation 15.9	5	43,991.51	8,798.30	91.20†
Deviations from regression	111	9,173.45	82.64	
Other treatment effects	55	3,771.39	68.57	
Error	56	5,402.06	96.47	

*$P \leq .05$
†$P \leq .01$

The interaction of nitrogen and phosphorus can best be seen from the surface in Figure 15.5. Yields increase sharply as nitrogen is applied at the zero level of phosphorus. However, even higher yields are obtained from N as P_2O_5 is increased. Yield at zero level of P_2O_5 for different rates of N is represented by the edge of the surface directly above the nitrogen axis. A second line over the surface parallels the first and shows yield response to N at 40 pounds of P_2O_5. Thus, the strong positive interaction or complementarity of N and P can be seen from the high center ridge of the surface at large inputs of N and P.

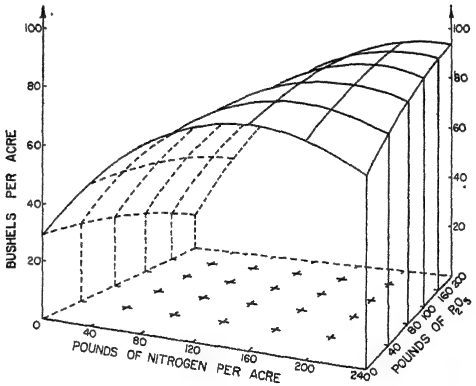

Figure 15.5. Perspective view of predicted yield surface for corn on Moody soil

Yield Isoquants and Isoclines for Moody Soil

In the yield isoquant equation 15.12, derived from the basic regression equation 15.11, P_2O_5 is expressed as a function of N. The yield isoquants in Figure 15.6 are based on equation 15.12.

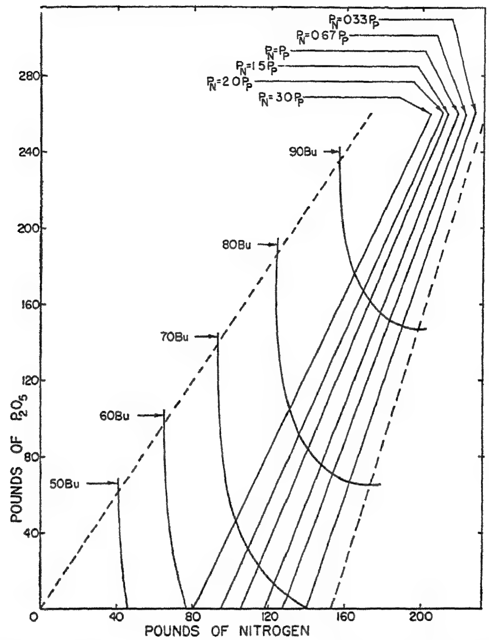

Figure 15.6. Yield isoquants and isoclines with dashed ridgelines, Moody soil

$$(15.12)\quad P = 1.506\,N \pm \frac{\sqrt{.0007581\,N - .000001332\,N^2 + .0415 - .00142\,Y}}{.0007098}$$

For yields as low as 50 or 60 bushels in Figure 15.6, isoquants are nearly vertical. These steep slopes for lower yields mean that many pounds of P_2O_5 are necessary to replace 1 pound of nitrogen in attaining the constant yield (or, practically, that added P_2O_5 does not substitute for N in attaining these yields when N input is low). As yield is increased to 70 bushels per acre and more N is used, the isoquant becomes more curved as it approaches the N axis. The 70-bushel isoquant intersects the N axis; 70 bushels per acre are predicted from the equation with all nitrogen and zero of P_2O_5. However, the 80-bushel isoquant requires P_2O_5 in addition to N; a yield this high requires the complementary effect of P_2O_5 with N.

Since the slopes of the isoquants show the change in amount of P_2O_5 required to maintain a given yield when another unit of N is added, the curvatures of the isoquants indicate the change in the rate of substitution of N for P_2O_5. Substitution or replacement rates predicted from equation 15.12 for a 70-bushel yield show that at 100 pounds of N, an additional unit of N replaces about 3.5 units of P_2O_5. However, at 140 pounds of N, an additional pound of N replaces only about one-third of a pound of P_2O_5, if an 80-bushel yield is to be retained. Since the slopes

of the isoquants change along a scale line (fixed nutrient combination) the combination of nutrients or fertilizer ratio which gives lowest cost for one yield level is not the same fertilizer ratio which gives lowest cost for another yield level.

Each isocline in Figure 15.6 intersects every isoquant at a point of specified slope on the isoquant. For example, the isocline labeled $P_n = 3.0\ P_p$, meaning a price of nitrogen equal to three times the price of phosphate, goes through each of the 70-, 80-, and 90-bushel isoquants at points where the slope (i.e., the marginal rate of substitution) is 3:1. On the isocline labeled $P_n = .33\ P_p$, each isoquant is intersected where the slope is 1:3. On this isocline, each pound of P_2O_5 would replace 3 pounds of N. Therefore, if the price of N were one-third the price of P_2O_5 per pound, production should be expanded along the isocline labeled $P_n = .33\ P_p$, if the path of fertilizer ratios for least-cost yields is to be traced out. Relevant isoclines in Figure 15.6 all intersect the nitrogen axis. This suggests that for economical yield increases, nitrogen is the most limiting factor at the outset. However, for higher yields which might be profitable, P_2O_5 is relatively more limiting, since the isoclines have slopes greater than 45 degrees.

HAYNIE EXPERIMENT

An experiment on Haynie soil in 1953 is reported in this section. Rates of application were 0, 40, and 80 pounds of each nutrient. Improved estimates of the N and P_2O_5 response would have resulted if N and P_2O_5 levels had extended higher. Average yields of the nitrogen responses are 8 bushels for the last 40 pounds of N applied. Similarly, 80 pounds of P_2O_5 gave almost 5 bushels more corn per acre than did 40 pounds. To estimate the N x P interaction, nitrogen and phosphorus inputs should go far enough to cause a decline, or at least a leveling out, of total yield. Additional increases in yield might have been obtained at heavier N and P_2O_5 combinations.

Of several algebraic forms examined, equation 15.13 was selected as most efficient for prediction.

$$(15.13)\quad Y = -.9751 + .7126\ N - .004352\ N^2 + .5255\ P - .003103\ P^2 + .2546\ K - .001624\ K^2 - .002255\ PK + .003863\ S$$

Rather than adjusting yield for stand, plant population was included as a variable because analysis of covariance indicated a highly significant stand effect. Ideally, stand should have been introduced as an experimental variable. However, this was not the case and stand levels in different plots varied only through weather and chance. Experiments are reported later where stand was included as a variable. If inputs of N and P_2O_5 had been extended to higher levels, the square root function might have given better results. Higher levels of N and P_2O_5 might have also revealed a significant N x P interaction.

Table 15.6. Values of t for Individual Regression Coefficients of Equation 15.13

Variable	N	N^2	P	P^2	K	K^2	PK	S
t value	5.67	2.89	3.72	1.96	1.90	1.06	2.10	2.99
P level	.00001	.005	.0004	.05	.07	.30	.04	.004

The term S again refers to stalks per acre while N, P, and K refer to pounds of N, P_2O_5, and K_2O per acre. The values of t for the regression coefficients are given in Table 15.6. They show that the coefficients for the N variables are significant in explaining yield variance. The value of t for K^2 is only 1.06. A value this large could occur by chance in about one-third of the time where K^2 had no real effect. Nevertheless, the K^2 term is retained for logical reasons. Without the negative K^2 term, an unlimited linear response to K would be implied.

The analysis of variance of equation 15.13 regression in Table 15.7 shows the over-all regression to be highly significant. The deviations from regression mean square are about the same as the estimate of experimental error from within plots.

Table 15.7. Analysis of Variance for Regression of Corn Yield on Haynie Soil

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	F
Total	53	13,876.57		
Due to 8-term regression equation 15.13	8	10,794.66	1,349.33	18.92*
Deviations from regression	45	3,081.91	68.49	
Other treatment effects	20	1,299.43	64.97	
Error	25	1,782.48	71.30	

*$P \leq .01$

In equation 15.14 stand is fixed at 9,220, the average stalk count for all experimental plots. If stand were included as a controlled variable in the original experiment, the optimum level of stand could be determined by economic analysis. However, none of the experiments analyzed in this study was so designed, and stand is used only to improve the precision of estimate of the fertilizer response.

$$(15.14) \quad Y = 35.0587 + .7126\,N - .004352\,N^2 + .5255\,P - .003103\,P^2 + .2546\,K - .001624\,K^2 - .002255\,PK$$

Production Surfaces

Equation 15.14 allows production of an infinite number of three dimensional production surfaces when one nutrient is held fixed and the other two are variable. Since the effects of N were predicted to be independent of P and K, the phosphorus-potassium surface retains the same shape or curvature at different levels of N. In Figure 15.7, the surface shows a greater rise in yield from inputs of P_2O_5 than from K_2O. Also, the surface is relatively flat over the top, indicating that yields do not change greatly for many combinations of P_2O_5 and K_2O. For a line stretched diagonally over the surface from the zero corner to the opposite corner, a sharp increase in yield is followed by a decrease. The dropping off at the opposite corner for high levels of both P_2O_5 and K_2O is due to the negative P x K interaction. The decline at high levels of P_2O_5 and K_2O is in contrast to the high ridge at high levels of N and P_2O_5 in Figure 15.5, caused by positive N x P interaction.

Yields are predicted to increase by about 21 bushels as nitrogen inputs are increased to 40 pounds. For 80 pounds of N, predicted yields are increased 29 bushels over corresponding P-K treatments receiving no nitrogen. These relationships between nitrogen and P-K responses for a particular angle of the P-K yield surface are shown in A, B, and C of Figure 15.7. In A, the P-K yield surface is shown with a zero level of N; B gives P-K yields with 40 pounds of N; C shows P-K yields with 80 pounds of N. The surface of B is 21 bushels higher than for A because of the response of the 40-pound application of N. The shapes of the three surfaces are exactly the same, but one may see more of the underside of the declining surface in the higher structures. If the P-K yield surface for 120 pounds of N were shown it would be of the same height as in B, since predicted yields start to decline around 82 pounds of N.

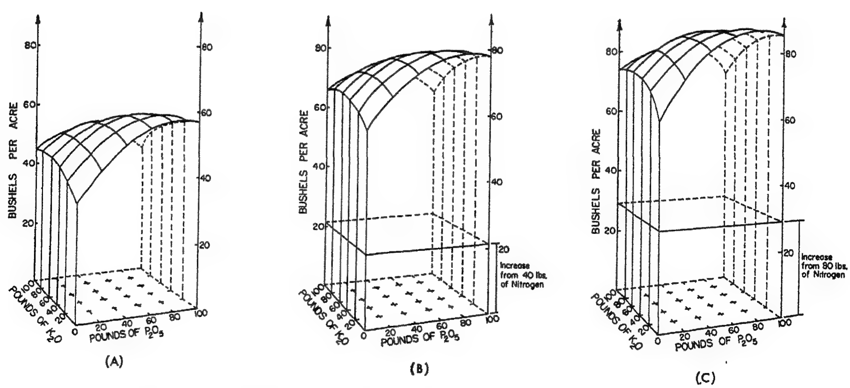

Figure 15.7. (A) Predicted yield surface for corn on Haynie soil with no N application, (B) Predicted yield surface at 40 pounds of N, (C) Predicted yield surface at 80 pounds of N

Isoquants and Isoclines

Three sets of isoquant and three sets of isocline equations can be derived from equation 15.14 and have been used in constructing figures 15.8, 15.9, and 15.10. The symbols P_N, P_P, and P_K indicate prices of the three nutrients and the lines so labeled designate the isoclines which serve as expansion paths for various price ratios for the nutrients. Both figures 15.8 and 15.9 have ridgelines which form 90 degree angles because interaction between nitrogen and the other two nutrients is lacking, where negative interaction between nutrients exists, as for P_2O_5 and K_2O in Figure 15.10, ridgelines meet at an angle greater than 90 degrees: Negative P x K interaction gives the production surface a comparatively flat top; economic limits of nutrient combination are wide. For close complementarity and positive interaction (i.e., between N and P_2O_5 for corn in Figure 15.6), the ridgelines are close together and meet at an angle of less than 90 degrees. With positive N x P interaction, as in Figure 15.6 where ridgelines are close together, a non-optimum nutrient combination could be very costly. However, to deviate slightly from the optimum fertilizer ratio line (isocline) in Figure 15.10, if the price ratio differs only slightly from the substitution ratio indicated by the isocline, may depress profits only slightly since rates of substitution change slowly along the isoquants.

Under prices where the price of K_2O is 80 per cent of the P_2O_5

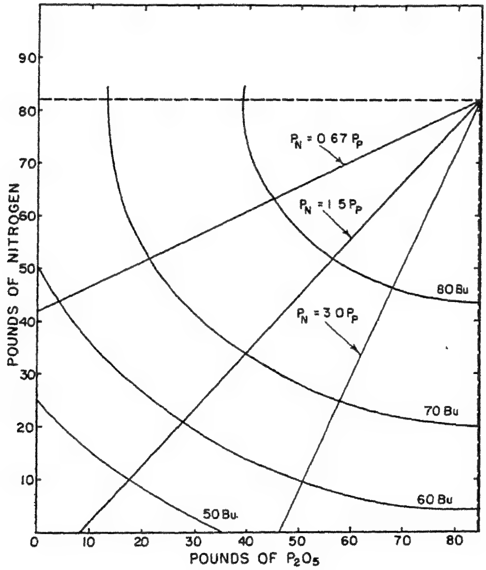

Figure 15.8. Yield isoquants and isoclines with dashed ridgelines at zero level of K_2O, Haynie soil

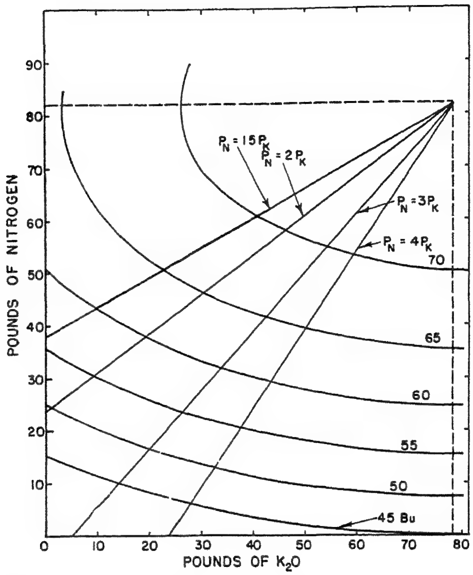

Figure 15.9. Yield isoquants and isoclines with dashed ridgelines at zero level of P_2O_5, Haynie soil

price, over 60 pounds of P_2O_5 should be applied before any K_2O is used (Figure 15.10). However, with K_2O at one-third the price of P_2O_5, it would pay to apply over 30 pounds of K_2O before any P_2O_5 is used.

STAND AND NITROGEN

Several studies with stand or plant population have been completed in Iowa. This section reports an experiment on Marshall silt loam where stand (S) in hundreds of plants per acre and nitrogen (N) in pounds per acre were variables. The crop is corn measured as yield (Y) per acre. Three nitrogen and five stand levels were used on each of the two varieties, A.E.S. 801 hybrid and Iowa 4397 hybrid. Plants were thinned to five stand levels. A split-split-plot design was employed with nitrogen rate, stand, and variety as the whole plots, subplots, and sub-subplots, respectively. With the temperature being equally favorable (or unfavorable) the potential production of A.E.S. 801 would be expected to be relatively higher in this experiment.

Yield observations suggested decreasing total returns for higher levels of stand, particularly when level of nitrogen is low. Response to nitrogen was quite large, and a strong interaction exists between nitrogen and stand. These data also suggested that yields for variety 801 were somewhat higher than those obtained from variety 4397.

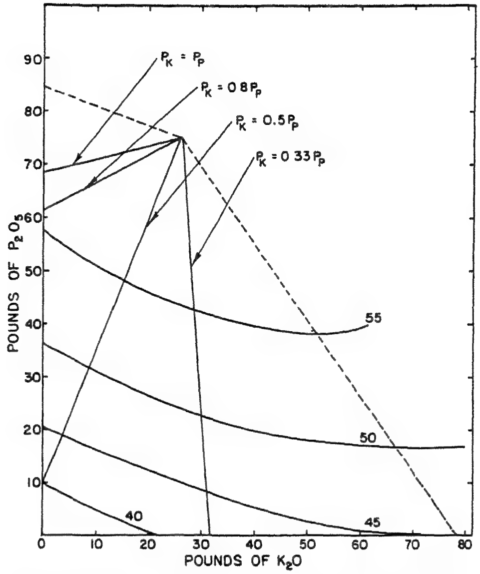

Figure 15.10. Yield isoquants and isoclines with dashed ridgelines at zero level of N, Haynie soil

Regression Analysis

Several regression equations were considered as a basis for estimating the production surface. Those appearing most appropriate are presented below. The first two functions are quadratic forms and are for varieties 4397 and 801, respectively.

$$(15.15) \quad Y = 18.9996 + .2253\ N + .4200\ S - .000720\ N^2 - .001415\ S^2 + .000361\ NS$$

$$(15.16) \quad Y = 29.3249 + .1990\ N + .389216\ S - .000565\ N^2 - .001503\ S^2 + .000800\ NS$$

Equation 15.15 has an R^2 of .6259 while equation 15.16 has an R^2 of .7222. The values of t and probability levels for the regression coefficients for each equation are included in Table 15.8. The level of significance of all coefficients appears high enough to warrant retaining them in the equation.

Parallel coefficients of the equations for the two different hybrids do not appear, by inspection, to be greatly different. Differences in yielding ability of the two varieties are important to research workers interested per se in yields from different hybrids. However, this study is concerned with the shape of the function, in terms of optimum stand and fertilizer levels. While the surfaces may have different vertical

Table 15.8. Values of t for the Coefficients of the Individual Quadratic Regression Equations Relating Corn Yields to Inputs of Stand and Nitrogen on Marshall Silt Loam

Regression Coefficient for	Variety Iowa 4397		Variety A.E.S. 801	
	t values for equation 15.15	Probability level*	t values for equation 15.16	Probability level*
N	3.01	.005	2.42	.025
S	2.21	.050	2.12	.050
N^2	2.63	.025	1.81	.100
S^2	2.21	.050	2.45	.025
NS	1.12	.300	2.26	.050

*Probability of drawing t as large or larger by chance, given the null hypothesis.

locations above the resource plane, the functions will still lead to similar conclusions on optimum resource inputs, if the shape or curvature of the surfaces are the same. Hence, a t test has been used to test the significance of the difference between regression coefficients in equations 15.15 and 15.16. The results of these tests (shown in Table 15.9) suggest that differences between parallel regression coefficients, as large as those obtained, might well have occurred by chance. Hence, the observations were pooled.

Table 15.9. Tests of Significance of Differences Between Parallel Coefficients in Equations 15.15 and 15.16

Regression Coefficient for	Value of t for Difference Between the Two Coefficients	Probability Level*
N	.34	.80
S	.17	.90
N^2	.53	.60
S^2	.14	.90
NS	1.29	.20

*Probability of drawing a t value as large or larger by chance, given the null hypothesis.

A square root function was fitted to the pooled data. However, it was logically unsatisfactory because of the signs of the coefficients. This quadratic equation was then fitted to the pooled data:

$$(15.17)\quad Y = 27.5963 + .2051\,N + .3553\,S - .000641\,N^2 - .001298\,S^2 + .000625\,NS.$$

The R^2 for equation 15.17 is .6125, and t values of the coefficients are given in Table 15.10. All of the terms are significant at a probability level at least as high as .05.

The significance levels of the coefficients of the pooled regression

Table 15.10. Values of t and Significance Levels of Coefficients of Quadratic Regression Equation 15.17

Regression Coefficient for	t Value	Probability Level*
N	3.14	.005
S	2.32	.025
N^2	2.63	.010
S^2	2.52	.025
NS	2.22	.050

*Probability of drawing t as large or larger by chance, given the null hypothesis.

in equation 15.17 are, for all terms, at least as high as for the individual regressions. Hence, through equation 15.17, the pooled regression on varieties 801 and 4397, is used in the analysis which follows.

Predicted Production Surface

Decreasing total returns to stand is predicted by equation 15.17 throughout the entire range of nitrogen applications. While there are decreasing total returns for nitrogen at the low levels of stand, this is not true for higher stand levels. Nitrogen and stand are predicted to have sufficient interaction to cause yields to increase continuously along the main diagonal of the input plane. The resulting surface is presented in Figure 15.11. The decreasing total returns to stand and nitrogen, for the latter when stand is held at low levels, are indicated by the negative slopes over some portion of the surface. The surface also slopes upward from the left-hand corner above the nutrient plane because of the stand by nitrogen interaction.

Marginal Physical Products

Equations for the marginal physical products of stand and nitrogen are, respectively:

$$\frac{\delta Y}{\delta S} = .3553 - .0026\,S + .0006\,N. \tag{15.18}$$

$$\frac{\delta Y}{\delta N} = .2051 - .0013\,N + .0006\,S\ . \tag{15.19}$$

The interaction term in the original production function causes the marginal equation for one input to contain a term due to the other input: when one input is varied, its marginal product is affected by the level at which the other input is fixed.

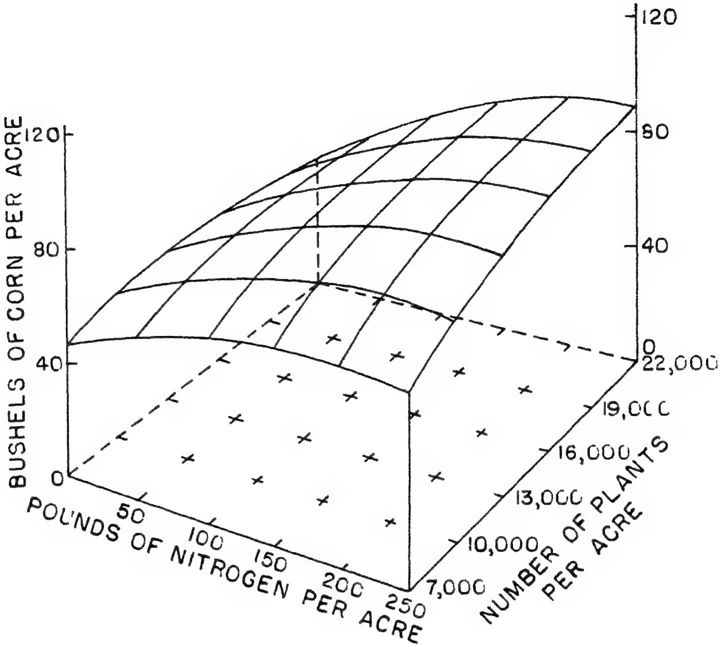

Figure 15.11. Production surface for stand and nitrogen on Marshall silt loam, equation 15.17

By setting the marginal equations equal to zero, and solving for the values of the inputs, the quantity of input at which the marginal product becomes zero is obtained. When no nitrogen is applied and stand is utilized to the point where its marginal product is zero (13,688 plants per acre), the predicted corn yield is 51.92 bushels per acre. On the other hand, when stand is utilized at a minimum rate for the experiment (i.e., 7,000 plants per acre since a zero rate of stand would result in no yield) and enough nitrogen is applied to drive the marginal product to zero, yield is predicted to be 70.26 bushels per acre. The input level at which yield is at a maximum (i.e., both partial derivatives are zero) is determined by equating both marginal product equations to zero and solving simultaneously for values of stand and nitrogen. Using equations 15.18 and 15.19 for this purpose, yield is predicted to be a maximum with 256.88 pounds of nitrogen and 19,873 plants per acre. The predicted yield is 89.25 bushels, an extrapolation along the nitrogen axis since the nitrogen input is above the limit of experimental observations.

Yield Isoquants and Isoclines

The curves of negative slope in Figure 15.12 are isoquants, depicting the horizontal sections through the surface in Figure 15.11. They

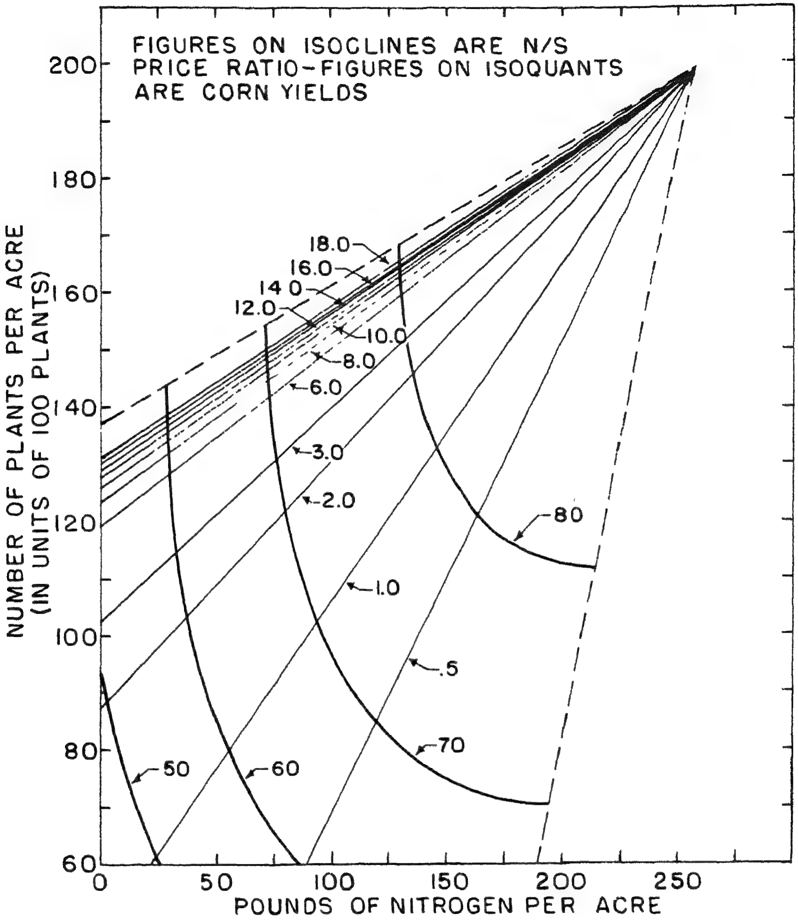

Figure 15.12. Isoquants and isoclines for stand and nitrogen on Marshall soil based on equations 15.20 and 15.21

indicate the various combinations of stand and nitrogen which will produce the specified or given yields and are based on the isoquant equation:

(15.20) $S = 136.8798 + .2408\,N \pm 385.2080\,(.2695 + .001509\,N - .000003\,N^2 - .005192\,Y)^{\frac{1}{2}}$

Since the isoquants are convex to the origin, stand and nitrogen are predicted to substitute at decreasing rates. Also, since isoquants representing equal increments in yield are successively farther apart (on any straight line through the origin or perpendicular to either axis), diminishing returns to the inputs are indicated.

The isocline equation derived from equation 15.20 is:

$$(15.21) \quad S = \frac{k\,(.355340) + k\,(.000625)N - .205114 + .001282\,N}{.000625 + k\,(.002596)}$$

where k is the P_S/P_N price ratio and the other terms are as defined previously. Isoclines (positively sloped lines) derived from this equation are presented in Figure 15.12. Again, these isoclines can be looked upon as representing either (a) marginal rates of substitution of given magnitudes or (b) price ratios representing particular expansion paths. Isoclines labeled 6.0 to 10.0 approximate the range of price ratios realized in the past. Thus, if a given yield is produced at minimum cost, stand should be increased until its marginal product is almost zero before applying more nitrogen. In other words, the least-cost combination of stand and nitrogen approaches the upper ridgeline (the dashed line where $\delta N/\delta S$ is zero).

The dashed lines in Figure 15.12 are the ridgelines (i.e., isoclines representing zero substitution rates) beyond which the inputs will not substitute for each other except at greater total physical inputs of both variables. Hence the ridgelines represent working limits beyond which only less favorable economic solutions can ever be found. Levels of input where the marginal physical products are zero fall exactly on one or the other of the ridgelines. This is true because the marginal rate of substitution between the two inputs is the ratio of their marginal physical products and the ridgelines represent all points at which the marginal product of either stand or nitrogen is zero. Convergence of isoclines is at a point in the input plane where the partial derivatives of both inputs are zero. In Figure 15.12, convergence of isoclines falls at a yield of 89.25 bushels and inputs of 256.88 pounds of nitrogen and 19,873 plants per acre. At this particular point, stand and nitrogen are technical complements or limitational resources; neither can be substituted for the other in producing the specified yield.

Economic Optima

Optimal conditions under certainty and unlimited capital concern selection of (a) the combination of stand and nitrogen levels which minimize the cost of a given output and (b) the levels of stand and nitrogen which will maximize the per-acre profits. These conditions are attained simultaneously when the partial derivatives of the two input categories are equated to the ratio formed by dividing their price (cost) by the price of corn per bushel.

Isoclines are expansion paths, indicating the combinations over which nitrogen and stand should be increased in attaining successively higher yield levels, if each yield is to be attained at a minimum cost. The isoclines representing least-cost expansion paths for reasonable prices of nitrogen and stand fall towards the upper ridgeline of Figure 15.12. For a price ratio of 9, the cost per pound of fertilizer divided by the cost of the amount of seed to provide 100 plants, the cost of stand

is so low that economic optima seem almost consistent with the use of planting rates which drive the marginal physical product of stand to zero.

Optimum nitrogen levels vary widely as both corn and nitrogen prices vary. When nitrogen costs \$ .08 per pound, nitrogen use ranges from 181 to 211 pounds per acre depending on corn and stand prices. When nitrogen costs \$ .15 per pound, the predicted optimum quantities of nitrogen are lower, ranging from 120 to 172 pounds per acre. Variations in nitrogen prices, within their relevant range, cause larger fluctuations in resulting profit than variations of stand prices within their relevant range.

Profit maximization, even with ability to predict yields and prices, as outlined above, is possible only for the farmer with unlimited capital. Farmers with limited funds may wish to apply some fertilizer, but divert the remainder of their capital to other uses. Also, given weather and price uncertainty, farmers not limited on capital also may use resources short of equating marginal value products with the cost of resources. Hence, we suppose a farmer has capital to use only one-half the profit maximizing nitrogen quantities for a given set of prices. We then ask: "What stand level would be optimum?" Computations show that stand costs are so small that most farmers could use levels which drive stand marginal productivity to zero regardless of availability of capital or price of seed and yield.

MULTIPLE HAY SURFACES

This section reports a fertilization and production function study for hay. This crop, with several products forthcoming during the year, presents analytical and decision problems which are not encountered with commodities harvested at a single time during the year. However, emphasis in this section is on basic production relationships rather than on the decision criteria which might be used for hay production when the crop has different values or prices throughout the season, the quality differs with the time of harvest, and other such problems.

The experiment from which the data were obtained was one of a series of pasture fertilization trials. The experiment analyzed was conducted on Weller silt loam in southeast Iowa in 1952. The design was a 3 x 3 factorial, replicated twice, giving a total of 18 observations. Two of these were checks. P_2O_5 and K_2O fertilizer were both applied at three levels. The alfalfa was cut three times during the course of the growing season. Three cuttings are considered normal in this area. While improved designs are now in use, the data which follows does provide some interesting insights into the nature of hay production surfaces.

Predicted Surfaces

Experimental yield observations were used to predict five production surface equations. First, an equation was fitted to each separate

cutting. Next, yields for the first and second cuttings were added together and an additional function was fitted to this total. Likewise a regression equation was computed for the sum of the yields of the three cuttings. Hay yield (Y) is measured in tons per acre while K_2O (K) and P_2O_5 (P) inputs are measured in pounds per acre.[2] The five equations, selected from among several algebraic forms tried, presented in the order mentioned above are

$$Y = .82235 + .007042\,P + .006181\,K - .000028\,P^2 - .000027\,K^2 - .000010\,PK \tag{15.22}$$

$$Y = .811947 + .001278\,P + .001403\,K - .000004\,P^2 - .000006\,K^2 + .000005\,PK \tag{15.23}$$

$$Y = .490277 + .001431\,P + .002181\,K - .000005\,P^2 - .000012\,K^2 - .000002\,PK \tag{15.24}$$

$$Y = 1.634182 + .008320\,P + .007584\,K - .000032\,P^2 - .000033K^2 - .000005\,PK \tag{15.25}$$

$$Y = 2.124459 + .009751\,P + .009765\,K - .000037\,P^2 - .000045\,K^2 - .000007\,PK \tag{15.26}$$

where the first three are for the first, second, and third cuttings, respectively, while the fourth function is the sum of the first two cuttings, and the fifth is for the sum of all cuttings. Equation 15.25 could have been obtained by adding equations 15.22 and 15.23 and equation 15.26 by adding equations 15.22 through 15.24.

The over-all significance of the regressions was tested by means of the F ratio. For the first cut, the first plus second cut, and the first plus second plus third cut, the F's are all significant at probability levels of less than 5 per cent. For the second and third cuts taken by themselves, the F values fall just outside the 5 per cent level of probability. After examining the standard errors and related statistics, it was decided to retain all of the variables indicated in equations 15.22 through 15.26. The R^2 values for these five equations were respectively .96, .86, .91, .98, and .98.

Figure 15.13 provides three of the production surfaces which can be predicted from the above equations. All three surfaces are of similar configuration and denote negative marginal products, even within the range of experimental observations, for both nutrients on each surface. The differences in heights of the three surfaces represent the addition to total yield due to additional cuttings. The second cutting was greater than the third, as suggested in Figure 15.13 and by equations 15.23 and 15.24.

Hay Isoquants and Isoclines

A set of three isoquants has been derived for each production function in equations 15.22, 15.25, and 15.26. These are presented in

[2] Subsequent interpretation of the data in terms of hay rather than oven dry material means that the regression coefficients would be altered slightly.

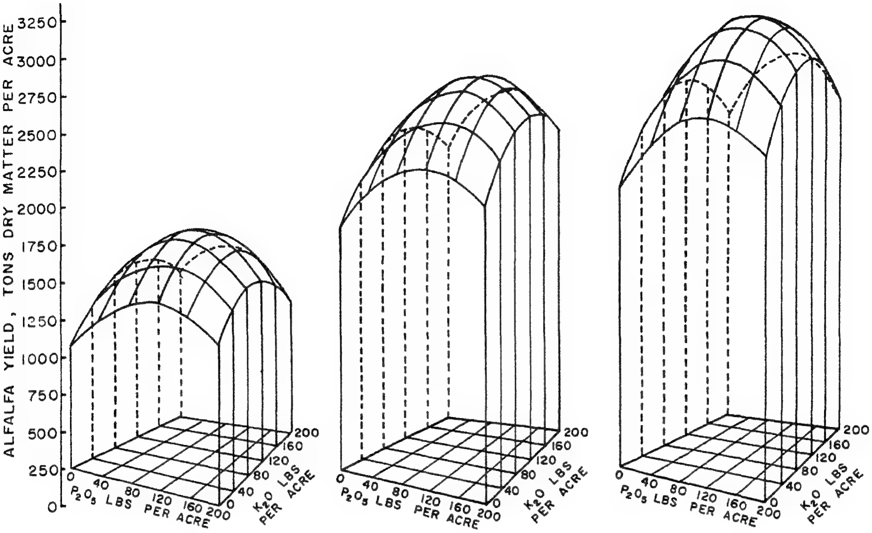

Figure 15.13. Production surfaces for 1st, 1st+2nd, and 1st+2nd+3rd cuttings of alfalfa

Figure 15.14. The isoquants have relatively slight curvature. The change in slope is gradual, indicating that the nutrients are good substitutes, within the range of the experiment. This is true for each set of isoquants.

Yield isoclines (least cost expansion paths) were derived by equating the marginal products of each production function to the nutrient price ratio and solving for one nutrient. Isoclines were worked out for the first, first plus second, and first plus second plus third cuttings, corresponding to production functions in equations 15.22, 15.25, and 15.26. The respective isocline equations corresponding to these functions are presented below where the nutrient price ratio is represented by k.

$$k = \frac{.007887 - .000062\,P - .000011\,K}{.006923 - .000060\,K - .000011\,P} \tag{15.27}$$

$$k = \frac{.009318 - .000072\,P - .000006\,K}{.008494 - .000074\,K - .000006\,P} \tag{15.28}$$

$$k = \frac{.010921 - .000082\,P - .000008\,K}{.010937 - .000100\,K - .000008\,P} \tag{15.29}$$

An isocline family has been drawn for each equation in Figure 15.15. Each isocline represents the least-cost P_2O_5 and K_2O combination for the nutrient price ratio shown. The slopes of the sets of isoclines do

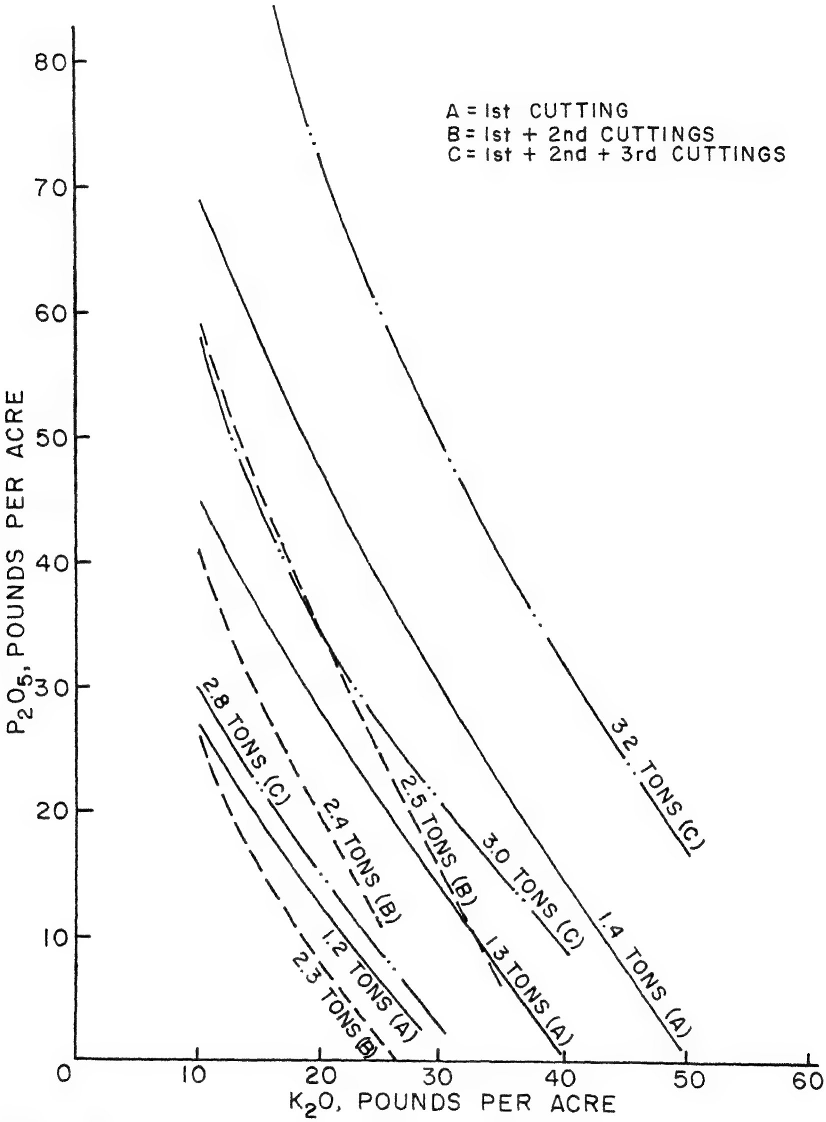

Figure 15.14. Isoquants for 1st, 1st+2nd, and 1st+2nd+3rd cuttings of alfalfa hay

not differ greatly. Since an isocline connects points of equal slope on a family of isoquants, the corresponding sets of isoquants are also similar. The dotted lines again are the ridgelines, beyond which the inputs do not substitute for each other in maintaining a given yield.

The family of isoclines converges to a point of maximum yield

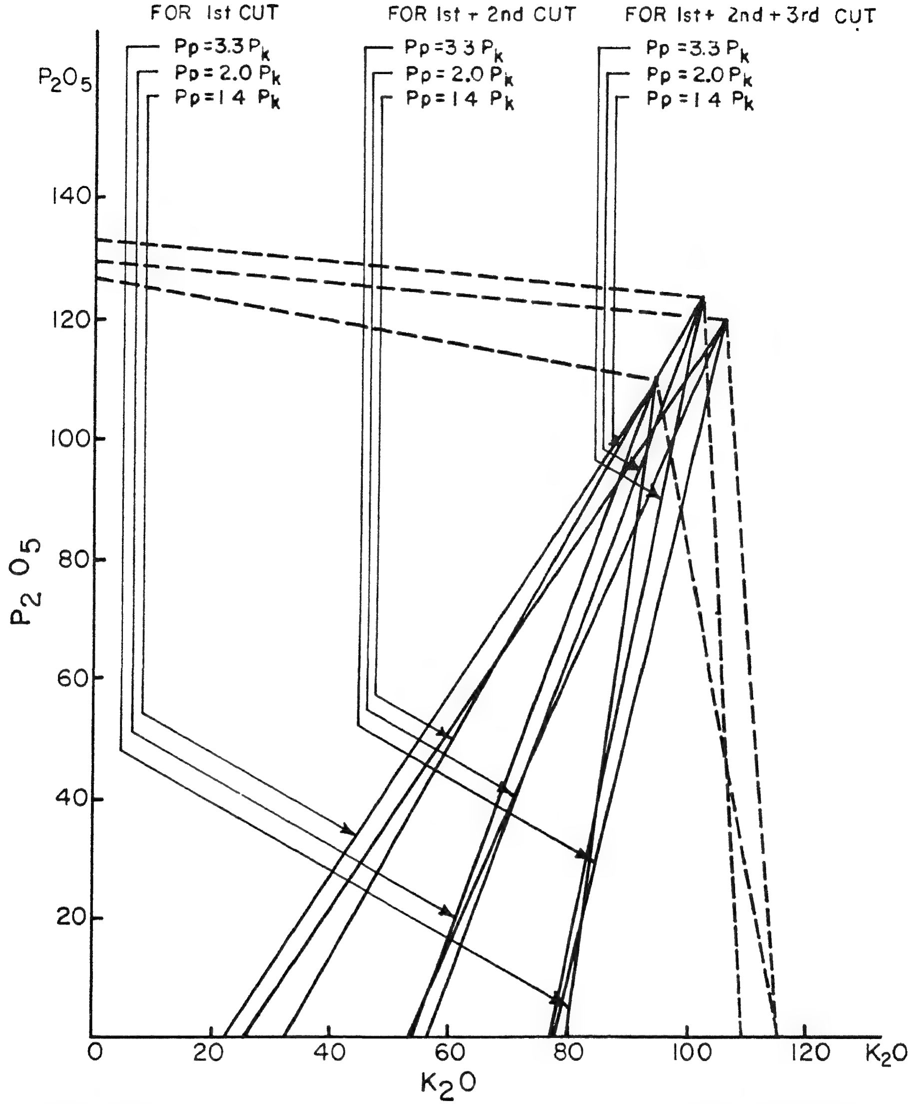

Figure 15.15. Isoclines for 1st, 1st+2nd, and 1st+2nd+3rd cuttings of alfalfa hay

where the partial derivatives of both factors are zero. For one cutting alone, the predicted amounts of P_2O_5 and K_2O required are smaller than for two or three cuttings. Maximum yield for three cuttings is predicted to require more P_2O_5 but less K_2O than for two cuttings, a condition which might appear unrealistic. However, this possibility could easily exist when moisture conditions differ between the growing periods for the three cuttings.

CHAPTER 16

Functions for Fixed Plants and Other Farm Situations

THIS CHAPTER summarizes selected farm production functions derived from cross-sectional samples for farms of given size in acreage. It also reports other selected farm and enterprise production function studies made in Iowa. The production function estimates have the limitations mentioned in earlier chapters for the particular aggregation and specification procedures used. However, the results were useful for diagnostic purposes in analyzing farm resource returns and capital productivity. All studies employ Cobb-Douglas types of equations. The aggregation and sampling methods used may be of particular interest and the studies are reported for these reasons.

DIAGNOSTIC PURPOSES

In contrast to biological functions such as those discussed in Chapters 8-15, firm and industry production functions provide little opportunity for decisions on and recommendations for highly specific types of resources. In preceding chapters, for example, it was possible to derive coefficients which provide the basis for recommendations on use of such specific resources as nitrogen or soybean oilmeal. However, aggregation over resource categories must be much broader for firm production functions. The coefficients which result can have application only to similarly broad categories of inputs. Hence, firm production functions can never be used for recommending quantities of corn feed, tractor fuel, and similar input quantities which are optimum for an individual farmer. Ordinarily, inferences must be restricted to aggregate resource categories such as labor, land, crop capital, etc.

Hence, it is not expected that data from firm production functions will be used to make specific recommendations such as those arising from biological production functions or budgeting and linear programming analyses. Largely, firm functions must be used for more general diagnostic purposes. These uses would include suggestions to farmers whether they are using too much or too little capital, or whether a reallocation of capital from crops to livestock would be profitable. One could never use this empirical tool to tell a farmer whether he should invest in Jersey rather than Ayrshire cows, or that he should invest more in a particular insecticide or treatment.

In this "diagnostic sense," firm production functions also may be used to indicate the "degree of equilibrium" in farming. Coefficients derived from them may give broad indication of whether returns to capital and labor in agriculture are higher than their market prices, or whether great differences exist between different regions in respect to capital productivity.

FIXED PLANTS ON MARSHALL SOIL

The first study involving fixed plant size is for Marshall silt loam.[1] It was completed for the year 1945 and provides estimates of capital and labor productivity when land per farm is fixed.

The sample of 70 farms was drawn in an area of Marshall silt loam and associated soils in southwestern Iowa, and was restricted to 160-acre, owner-operated farms. Farms were included in the sample only if they met certain tests of homogeneity in respect to soils. No farm in the area includes Marshall silt loam alone. Because of the rolling topography of the area and the related factors associated with soil formation, Marshall silt loam is typically found on farms in association with some amount of either bottomland soils in the valleys, Minden silt loam on the level ridges, or Shelby silt loam on the steeper slopes. Several of these soil types can often be found even within a 40-acre field. Although Marshall silt loam is predominant, many farms scattered throughout the entire geographic area have a large acreage of associated soils. In order to obtain a sample of farms reasonably homogeneous in respect to basic soil resources while still retaining a somewhat typical association of soils, upper limits were placed on the acreage of soils other than Marshall silt loam. Soil maps were made of each individual farm indicating the soil types, degree of slope, and degree of erosion for each individual farm as a means of delimiting the farms to be retained in the sample. Farms retained in the original sample contained no more than 25 acres of Shelby soils and between 40 and 75 acres in combination of bottomland soils or Marshall silt loam with a slope of 4 per cent or less. The remainder of the area was thus composed of Marshall silt loam with a slope of more than 4 per cent.

The 70 farms also were selected to fall within certain limits in respect to livestock. (a) No farm was included which had more than five milk cows. (Actually, all of the farms included in the analysis had no less than three and no more than five milk cows.) (b) No farm was included which had a poultry enterprise deviating greatly from a typical farm flock. (c) All farms were excluded which had beef cows or a sheep enterprise of any kind. (d) Farms included in the sample had only the livestock mentioned above plus hogs and feeder cattle. In other words, the sample farms were those with poultry and dairy enterprises

[1] Heady, Earl O. Productivity and income of labor and capital on Marshall silt loam farms in relation to conservation farming. Iowa Agr. Exp. Sta. Bul. 401. Ames. 1953.

of the nature outlined above plus no other livestock, hogs but no feeder cattle, or hogs and feeder cattle.

In the sample and functions derived for this 1945 study, the data allow a transition from an intensive grain farming system on farms where crop inputs are low to intensive forage farming where crop inputs are high; they allow a transition from a small dairy-poultry combination to hogs, then to hogs and feeder cattle, and finally to more feeder cattle as livestock inputs on farms. While the derived productivity estimates do not trace out the function for a single type of product, they represent a typical transition in investment as crop and livestock enterprises are intensified on 160-acre farms in the area, particularly when the transition is in the direction of more forage and a greater degree of conservation farming. Farmers with little capital generally raise only crops, while those with a little additional capital have small milk cow and poultry enterprises. Access to more funds allows a hog enterprise in conjunction with grain production and sales; still greater funds generally bring about a greater forage acreage and some feeder cattle to go with it. Ample funds usually allow larger hog and cattle-feeding enterprises, either from home grown or purchased feeds. The sample upon which this study is based is a random one; the inferences, however, apply only to 160-acre hog-cattle feeding farms, the size and organization of farms most numerous in the area. Because of the stratification procedure used, inputs and output are somewhat greater than for all owner-operated farms in the area. The inferences are representative, however, for farms using different quantities of resources and following the production pattern outlined previously.

Production Functions

Production functions were derived separately for crops and livestock. The variables included in these functions were:

Y_c is the value of crops produced and represents the dependent or output variable in crop production.

Y_l is the value of livestock products produced, including inventory changes and represents the dependent or output variable in livestock production.

L is input of labor, measured in dollars, used on crops or livestock according to the function indicated.

C is input of crop services, measured in dollars, and includes seeds, insecticides, and other supplies attached directly to crop production for the crop function.

M is input of machine services and includes fuel, grease, repairs, depreciation, and a small charge for housing for the crop function.

F is input of fertilizer, lime, and manure and a nominal machine cost, all measured in dollars. Manure has been converted to fertilizer in terms of N, P, and K and priced according to commercial fertilizer. Since fertilizer and manure applications and soil management practices were enumerated for a period of 3 years, the average of the last 2 years was used to obtain residual effects. A small machine cost, calculated to include costs of fertilizer application, was included as part of the value of fertilizer inputs (as a technical

complement of fertilizer itself) and was subtracted from other machine services.

G is input of feed for livestock including grain, protein supplements, hay, pasture, and miscellaneous minerals in the livestock function.

S is the input of livestock capital services measured in dollars in the livestock function. It includes *annual* inputs for livestock (in contrast to the capital stock itself) as follows: For chickens, fattening pigs, and feeder cattle, the beginning value is included as an input. For milk cows, brood sows, and laying hens, depreciation is computed and used as the input. Closing inventory values of growing and fattening stock are used as an output, while beginning inventory figures are used as an input. This input also includes the value of all grain, hay, pasture, and supplemental feeds and building, equipment, and machine services used on livestock. The latter items include depreciation and repairs and other annual inputs (in contrast to the value of the asset itself). No building inputs have been used in the crop function. This procedure has been employed under the assumption that crop storage belongs to a different production process than crop production. (Buildings, aside from facilities for drying corn, mainly contribute storage services for gaining higher seasonal market prices or for later livestock production.)

Thus, in all the derived functions, the inputs are measured in dollars and refer to the flow of services (or expenses) for the year; they are not capital values. Calculation of profitability can be made directly; if the marginal return is greater than $1, the particular dollar of input or expense more than paid for itself. If the input itself costs $1 and 5 per cent interest must be paid on it, a marginal return of $1.20 denotes the addition of $.15 to net returns. This procedure has been followed to ease the conversion of the 1945 prices to other levels.

In examination of the livestock function, it was found that inputs of livestock, feed, and building services were highly correlated. They tend to be technical complements and their quantities are increased together. Most farmers make decisions in this manner. For this reason, the three inputs were aggregated into a single category for the purpose of calculations. Labor was retained as a separate type of input since it was not so highly correlated with other categories of resources.

The regression or elasticity coefficients, derived separately for crop and livestock production, are included in Table 16.1. By coincidence, the sum of elasticities has a value of .935 for both functions. The elasticity coefficients, as averages based on the functional form,

Table 16.1. Regression Coefficients or Elasticities for Crop and Livestock Functions in 1945 Marshall Study

Crop Function		Livestock Function	
Resource	Elasticity	Resource	Elasticity
Labor (L_c)	.333	Labor (L_l)	.251
Crop services (C)	.384	Livestock services (S)	.684
Machine services (M)	.149		
Fertilizer (F)	.069		
Sum of elasticities	.935	Sum of elasticities	.935

are less than 1 for each individual resource indicating diminishing marginal productivity.

The sum of exponents, .935 for both crops and livestock, means that if, as an average, all resources used for each product are increased by 1 per cent, the value of output is increased by only .935 per cent and diminishing returns hold true; each 1-unit increase in resources, on the average, will add a smaller amount to output or return than the previous unit. Neither of these sums of elasticities differed significantly from 1.0. For a 160-acre farm with all acres fully in cultivation, it would be expected that diminishing productivity in crop production resources might be expected as greater inputs are applied to a fixed land area. While increasing returns might hold true where the quantity of resources applied is very small, farms in the sample generally used resources beyond this range, and the entire group had all acres of their units in operation. A small range of increasing returns might also be expected in livestock production. It is likely, however, that most farmers in the sample had pushed investment in livestock resources beyond this point. It also is entirely reasonable that over the range of livestock investment found on these farms, constant returns to scale might well exist even if increasing returns were not realized. Very great livestock inputs might, of course, result in decreasing returns. Decreasing returns in livestock may also result from the fact that the product measured is not homogeneous in terms of enterprise; it includes the "shading" from a little poultry and dairy products to hogs and then to feeder cattle in the manner outlined previously. This change in product, while deviating from a homogeneous enterprise, is realistic in terms of the pattern followed as aggregate livestock output is increased.

The values of t and the correlation coefficients between pairs of resources and products are shown in Table 16.2. All of the regression coefficients, except that for machine services on crops, were significant at a 5 per cent, or higher, level of probability; the regression coefficient for machine services is significant at the 10 per cent level of probability.

FIXED PLANTS ON CLARION-WEBSTER SOIL

This section reports a 1951 study for farms of fixed size in north central Iowa.[2] The farm sample was selected in the Clarion-Webster soil area, to give opportunity of homogeneity in the basic production functions. However, homogeneity of soils does not guarantee that hybrid or mongrel production functions will not result from inter-farm samples where operators use different techniques.

In the analysis which follows, two production functions, one for

[2] Heady, Earl O. Resource productivity and returns on 160-acre farms in north central Iowa. Iowa Agr. Exp. Sta. Bul. 412. Ames. 1954.

Table 16.2. Correlation Coefficients and Values of t

Crop Function				
	Labor	Crop services	Machine services	Fertilizer
Labor (r)	1.00	.49	.16	.62
Crop services (r)		1.00	.32	.12
Machine services (r)			1.00	.10
Fertilizer (r)				1.00
t values	3.14	8.93	1.84	3.06
Multiple correlation coefficient (R)			.86	

Livestock Function		
	Labor	Livestock services
Labor (r)	1.00	.64
Livestock services (r)		1.00
t values	4.45	11.90
Multiple correlation coefficient (R)	.96	

crops and one for livestock, have been derived. The crop function is of the form $Y_c = a_c L_c^l M_c^m C_c^c$ where Y_c refers to the value of crops produced, L_c refers to the quantity of labor measured in months, M_c refers to value of all machinery expenses or inputs of the particular year (including repairs, depreciation, fuel, oil, etc.), and C_c refers to all annual expenses on crops (including seed, fertilizer, lime, seed treatment, etc.); labor inputs are measured in months and machinery, and crop inputs are measured in dollars. The livestock function is of the form $Y_l = a_l L_l^l C_l^c$ where Y_l refers to the value of livestock output during the year (sales + home used + appreciation of young and fattening stock) while L_l is labor in months used on livestock, and C_l is all annual capital inputs, measured in dollars, used on livestock.

Sample

As a further step in introducing homogeneity into the sample, production functions were derived only for 160-acre farms (farms falling in the range of 140 to 169 acres). The random sample included 108 farms. Inferences in this study, therefore, are restricted to 160-acre units; land input is considered as a fixed resource in the production functions presented later.

Elasticity Coefficients and Scale Returns for 160-Acre Farms

The value of the regression coefficients or elasticities for the two production functions are presented in Table 16.3 along with related

Table 16.3. Elasticities and Related Statistics for Basic Functions on Clarion-Webster Soils

Item	Crop Function	Livestock Function
Value of constant a	3.0618	.3580
Value of elasticities*		
1	.0729	.2127
c	.0549	.8735
m	.1798	
Sum of elasticities	.3076	1.0862
Value of t for elasticities*		
1	1.98‡	3.65†
c	1.82§	20.35†
m	3.89†	

*The figure for c refers to crop capital for crops and to all capital for livestock; m refers to machine capital for crops but this category of inputs has been included with all capital for livestock.
Probability level for t's:
†1 > P > 0;
‡5 > P > 1;
§10 > P > 5.

statistics. All but one of the elasticities are significant at the 5 or 1 per cent level of probability; the coefficient for crop service capital was significant at a probability level approaching 5 per cent and is, with recognition of a slightly wider probability range for inferences, accepted for the analysis which follows. Each of the elasticities for crops or livestock is less than 1.0 — indicating that diminishing returns hold true for the particular resource. A 1 per cent increase in input or use of the particular resource results in an increase in value of production by less than 1 per cent; the return per unit of the particular resource will decline to lower magnitudes as more of the resource is used.

For labor used on crops, for example, a 1 per cent increase in labor is predicted to result as an average over the sample, in an increase of only .07 per cent in the value of crop production. For crops, the sum of the elasticity coefficients differs significantly from 1.0 — indicating diminishing returns as more and more of the various resources are used for crops on a given land area. Since all of the crop acres were in cultivation on all farms in the sample but different quantities of capital and labor resources were used on a given acreage, diminishing returns are indicated for each resource and all resources in combination.

On the other hand, the sum of the elasticities for livestock does not differ significantly from 1.0. As an average for farms in the sample, more resources can be put into livestock with returns as great as the mean quantity now being realized. While diminishing returns might be expected on the 160-acre farms if they carried an extreme amount of livestock, few farms approached programs of this intensity.

Resources Used and Factor Productivity

The average value of production, the mean quantity of resources used and the marginal productivity of crop resources at the mean are shown in Table 16.4. The marginal product figures have this meaning: "one more unit" of the particular resource, with its input and that of other resources at the mean of the quantities shown in the top of the table, will add the indicated quantity to total value of production. An increase in crop services will add to total value of production at the rate of $1.08 for each added $1 input of capital services; a $1 input in crop capital services returns itself plus $.08 in profit. The marginal product per month of labor is $77. In other words, the use of "one more month of labor" beyond the mean quantity per farm in the sample, would add this amount to total value of product. Actually, there would be little productive use for labor in crops, beyond the mean quantity used per farm. The computed marginal product of machine capital services is $.93, a marginal return of this amount for each $1 of annual input or expense.

While these return figures may appear low, except for crop services, two things should be remembered. First, the marginal returns are measured with resource inputs at the mean. Because of the diminishing-return nature of the productivity coefficients, marginal returns will be greater than the indicated amounts for labor inputs smaller than 6.2 months, for crop service inputs smaller than $334 and for machine service inputs smaller than $1,260; and, second, the estimates are made from a set of data involving sample variance, and, while probability tests suggest that the elasticity coefficients differ significantly from zero, we may inquire whether they differ significantly from a level necessary to give marginal productivities equal to

Table 16.4. Mean Quantity of Resources Used on Crops, Marginal and Gross Average Productivity of Specified Resources

Item	Amount
Average value of production per farm	$6,452.00
Average input per farm:	
Crop services	$ 334.00
Machine services	$1,260.00
Labor	6.2 months
Marginal products at resource means:	
Crop services ($/$)	$ 1.08
Machine services ($/$)	$ .93
Labor ($/month)	$ 77.87
Gross average products at resource means:	
Crop services ($/$)	$ 19.32
Machine services ($/$)	$ 4.82
Labor ($/month)	$1,041.00
Land ($/acre)	$ 42.00

Table 16.5. Test of Departure Between Marginal Resource Returns and Market Price of Resource

Resource	Value of Elasticity to Give Marginal Product Equal to Market Price of Resource	Value of t
Crop services	.0548	.002*
Machine services	.2070	.587*
Labor	.0729	1.029†

Probability levels:
*P > 50;
†P > 30.

the market price or cost of each of the resources. In other words, does the marginal return of $77.87 per month of labor differ significantly from $132.44, the market wage rate (without board) for labor in northern Iowa; do the capital returns of $1.08 and $.93 differ significantly from the $1.06 cost ($1 principal plus 6 per cent interest) for crop service and machine service capital respectively? As a test of these possibilities, the elasticities of production necessary to give marginal products equal to the market cost of the resources have been computed. Probability tests have been made for the differences between the elasticities derived in the study and those necessary to give marginal products equal to the market price of the resources. The statistics for this test are given in Table 16.5.

Since none of the t values are significant at the 30 per cent level of probability, we cannot say that the marginal returns per month of labor, as an average for farms in the sample, differed significantly from the market wage rate of $132.44 per month for labor or of $1.06 per $1 input for crop or machine capital. While the condition need not hold true for all farms in the sample, inferences based on the above statistics are the following:

Farmers were, on the average, maximizing returns under the particular prices and yields of the year; efficiency in production had been attained in the sense that the cost of resources approached the added return for more of these resources used beyond the per farm mean. These results seem reasonable in view of the fact that farmers in north central Iowa have a capital position about as favorable as those of any other area in the United States. Also, neither of the years included were outstanding in crop yield. Had crop yields been equal to the 10-year average, marginal returns on resources would likely have been significantly greater, given the particular price of the year, than the cost or market price of the resources to the farmers. (Productivity or return of resources would have exceeded the cost of the resources.)

Rates of Substitution Between Machine Services and Labor Used on Crops

The Cobb-Douglas function is not always appropriate for estimating rates of factor substitution because of the algebraic properties outlined in Chapter 3. However, it is used in this section to estimate the mean machine service-labor isoquant for the 160-acre Clarion-Webster sample and for estimating marginal rates of substitution between these two resource categories. It should be remembered that these predictions are simply those which result from the functional form used and the numerical observations of the sample. The limitations discussed earlier still apply, but the estimates may provide some "diagnostic insights" in regard to machine-labor substitution rates.

Using the original crop production function as a basis for estimating, the data of Table 16.6 have been derived. These data indicate (a) the combinations of machinery, measured as annual expenses of power and machinery, and labor which are predicted to produce the mean crop product of $6,545 and (b) marginal rates of replacement between these factors. Because of the asymptotic nature of the isoquants, they undoubtedly overestimate substitution rates for factor combinations extending away from the mean combination. The first two columns suggest that the mean crop product of 1951 could be attained with 10.96 months of labor and a $1,000 annual machinery expense, 6.99 months of labor and $1,200, 4.78 months and $1,400, or any other of the combinations shown. (The upper and lower figures of the two columns are extremes, and, while in the range of observations, they still extend too far for the particular algebraic function, they are included for illustrative purposes.) Column 3 shows the marginal rate at which machine services substitute for labor, predicted by the Cobb-Douglas function used. The figures are computed on the basis of derivatives at exactly the combinations indicated in columns 1 and 2; they are not averages between combinations.

Table 16.6. Mean Isoquant and Marginal Rates of Substitution for Machine Services and Labor in Crop Production, 160-Acre Farms on Clarion-Webster Soils

Isoquant (Combination Machine Service and Labor to Produce Average Crop Output of $6,545)		Marginal Rate of Substitution of Machine Services for Labor (Mos.) if Labor Were Replaced by $100 Machine Service Input	Value of Labor Replaced by $100 in Annual Machine Services at		
Input of machine services	Quantity of labor in months		1951 wage rate	50 per cent greater than wage rate of 1951	50 per cent of wage rate of 1951
($)	(months)	(months)	($)	($)	($)
900	14.22	3.90	516	774	258
1,000	10.96	2.71	359	539	180
1,100	8.66	1.94	257	386	129
1,200	6.99	1.44	191	277	96
1,400	4.78	.84	111	167	56
1,600	3.44	.53	70	105	35
1,800	2.07	.32	41	62	21

These figures show declining rates of substitution of machine services for labor; at larger inputs of machine services, the amount of labor replaced by $100 in machine services becomes less and less. As column 4 shows, $100 in machine services is predicted to replace a $359 value of labor (2.71 months at $132.44 per month) when machine service input is $1,000 and labor input is 10.96 months. Of the combinations shown, it is profitable to substitute machine services through a combination including $1,400 in machine expenses and 4.78 months of labor; costs of the given output of $6,545 can be lessened since the value of labor saved is more than the $100 machine services added. With a 50 per cent increase in wage rates, machine service costs remaining the same, machine expenses could be extended through $1,600; whereas they could be extended only through $1,100 with wage rates equal to one-half those of 1951.

It is of interest that, even with relatively large changes in wage rates (over the range $66 to $198, or from 50 per cent to 150 per cent of $132), the most profitable machine-labor combination would, within the combinations shown, vary only over a $500 range for machine inputs. In other words, a relatively small machine input is extremely profitable, regardless of wage rates. However, once a complement of machines for tilling, cultivating, and certain harvesting operations has been attained, further additions to machine expenses make only limited savings in labor and addition to income.

LONG-RUN PRODUCTION FUNCTIONS FOR TAMA-MUSCATINE SOILS

The production function study reported in this section applies to a sample of farms with Tama-Muscatine soils. The sample was drawn for east central Iowa in the year 1954.[3] In contrast to the two "fixed plant" or short-run production function studies reported above, this study is of long-run production functions. It is long-run in the sense that none of the resource categories are held fixed. All farms, qualifying in respect to Tama-Muscatine soils and falling in the randomly selected sampling units, were included in deriving the production function. Functions were fitted separately for crops and livestock; specification of resource categories were somewhat different than for the two studies reported above.

Crop Function

The crop function, which fitted to the entire sample of 255 farms, took the form of equation 16.1, where:

[3] This section summarizes the study conducted by David W. Brown and Earl O. Heady and reported in Brown, David W. Adjustment of value productivity estimates to changes in price and technical relationships. Unpublished Ph.D. thesis. Iowa State University Library, Ames. 1956.

$$(16.1) \qquad Y_C = 17.9X_1^{.540} X_2^{.390} X_3^{.165} X_4^{.012} X_5^{.073}$$

Y_C denotes the dollar value of estimated gross income from crops. This gross income includes the values of crop products on hand at the end of the year, sold, or used on the farm. Also included are certain miscellaneous receipts: payments for off-farm labor and machine work, government conservation payments, and co-op dividends.

X_1 denotes the acres of cropland used in crop production during the year. Permanent pasture, woodlots, and waste space are excluded.

X_2 denotes the annual labor used for crop production. These quantities are measured in terms of 10-hour days (of labor actually used and not that available).

X_3 denotes the dollar value of annual crop machine services. Included are the values of crop machinery depreciation and repairs, machinery operating expenses, and hired machine work.

X_4 denotes the dollar value of fertilizer and lime applied during the year.

X_5 denotes the dollar value of miscellaneous crop capital services. These include expenses for seed, insecticides, seed treatment, electricity, and telephone service.

The coefficient of multiple determination was .90, indicating that 90 per cent of the variation in crop income was associated with changes in input quantities. The sum of elasticities in equation 16.1 was 1.18, suggesting increasing returns to scale in crop production, assuming all input categories have been included in sufficient manner. All of the individual elasticities were significantly greater than zero at the 5 per cent level of probability.

Table 16.7. Values of t and Significance Levels for Crop Regression Coefficients

Regression Coefficient for	Standard Error of Elasticity	Probability Level for Elasticity Coefficient
Cropland	.065	.001
Crop labor	.065	.001
Crop machine services	.048	.001
Fertilizer and lime	.005	.05
Crop capital services	.028	.01

The mean marginal value productivities of the crop inputs, at their geometric means, are shown in Table 16.8. These show very high returns to land, labor, fertilizer, and crop services. Like the study on Clarion-Webster, they show a low marginal productivity of machine capital at the geometric mean. High productivities of resources are expected in crop production for this soil complex, one of the most productive in the Corn Belt. Linear programming and other types of studies also show, for example, that labor used in corn production has an extremely high return. Even at the lower fiducial limits, to express the probability and sampling characteristics of the estimates, the productivities of labor are still high. Labor used in magnitudes to produce

Table 16.8. Marginal Productivities at Mean Input Levels, Crop Function, 255 Tama-Muscatine Soil Association Farms, 1954

Input Category	Geometric Mean	Marginal Value Product at the Geometric Mean
Cropland	139 acres	\$33.47 per acre
Crop labor	97 days	\$34.64 per day
Crop machine services	\$1,945	\$ .73 per dollar
Fertilizer and lime	\$ 61	\$ 1.69 per dollar
Crop capital services	\$ 359	\$ 1.75 per dollar

crops does have high productivity when combined with the necessary complement of other resources. Increased alone, however, against a fixed mix of other resources, its productivity would decline rapidly for inputs beyond those necessary for "average requirements" in planting, tilling, and harvesting crops. This same statement would apply to other resources used on crops.

Marginal labor productivities for the Tama-Muscatine study are much higher than for the Clarion-Webster study. Two things probably account for most of this difference: (a) the latter study had labor measured in a partial availability manner, the former had labor measured as that actually used on farms; and (b) the Clarion-Webster functions were for smaller farms which have relatively large labor supplies. Without off-farm work, labor inputs have little cost and may be carried to a point where their marginal productivity is quite low.

The low machinery productivity again is partly expected. A considerable part of the machinery investment on these farms can be considered for consumption rather than production purposes. Power units and equipment are found on many of these farms and serve as items of convenience and for lessening the drudgery of hard work, an entirely economic outlay of funds.

Livestock Function

The livestock function, also derived for the 255 farms in the sample, was estimated as equation 16.2 where

$$Y_1 = 1.79Z_1^{.190} Z_2^{.360} Z_3^{.009} Z_4^{.602} \tag{16.2}$$

Y_1 denotes the dollar value of gross income from livestock. Included are the values of nonbreeding stock on hand at the end of the year, sold, or used on the farm, plus the values of all livestock products sold or used on the farm.

Z_1 denotes the dollar value of all feed fed during the year. Both homegrown and purchased feeds are included.

Z_2 denotes the labor used for livestock production (and not the supply available). As in the crop function, labor is measured in terms of 10-hour days.

Z_3 denotes the number of square feet of building space used in livestock production.

Z_4 denotes the dollar value of miscellaneous livestock services. Included are the values of nonbreeding stock on hand at the beginning of the year or purchased during the year; breeding stock depreciation; livestock machinery depreciation, repairs, and operating expenses; supplies and medical expenses; livestock commissions; and electricity and telephone charges.

The coefficient of multiple determination was .89, indicating that this per cent of the variation in livestock income was associated with changes in the livestock inputs in the function. The sum of the elasticities was 1.16 for livestock but did not differ significantly from 1.0. All of the elasticities, except that for the building input category, were significantly greater than zero as indicated in Table 16.9.

Table 16.9. Values of t and Significance Levels for Livestock Regression Coefficients

Regression Coefficient for	Standard Error of Elasticity	Probability Level for Elasticity Coefficient
Feed	.051	.001
Livestock labor	.055	.001
Building space	.056	.50
Livestock capital services	.034	.001

The mean marginal value productivities of the livestock inputs, computed at their geometric means, are shown in Table 16.10.

Again, the mean marginal productivity of labor is high. It bears a ratio to that of labor used on crops in about the magnitude expected by extension specialists. The marginal productivity of livestock capital is considerably greater than the cost of capital. The marginal productivity per dollar of feed for equation 16.2 is low, amounting only to about the average margin over feed costs expected on the basis of farm record summaries. However, a problem of multicollinearity exists, with a correlation coefficient of .91 between feed and livestock capital

Table 16.10. Marginal Productivities at Mean Input Levels, Livestock Function, 255 Tama-Muscatine Soil Association Farms, 1954

Input Category	Geometric Mean	Marginal Value Product at the Geometric Mean
Feed	\$6,839	\$ .32 per dollar
Livestock labor	199 days	\$21.02 per day
Building space	6,776 sq. ft.	\$ .015 per sq. ft.
Livestock capital services	\$4,893	\$ 1.43 per dollar

inputs. Hence, there is empirical basis for the aggregation of livestock inputs used in the study for the Marshall and Clarion-Webster studies described earlier in this chapter. When the feed and livestock inputs from equation 16.2 were aggregated into a single input category with value of livestock production being expressed as a function of labor and this combined category, the coefficient of multiple correlation became .89. Both elasticities were significant at a probability level of .01. The mean marginal productivity for labor became $18.34 per day while the mean marginal productivity for the aggregate capital input became $1.13.

POULTRY ENTERPRISE FUNCTION

The study reported in this section is for a laying flock enterprise.[4] As a poultry production function study, it differs from analyses reported in chapters 10 and 11 in this respect: it is not an experimental short-run function derived for two feed inputs. Instead, the data came from record-keeping farms where all inputs or their costs were recorded for the laying flock enterprise. In contrast to the functions reported in the current chapter, observations are measured for the particular enterprise, although the farms from which the data came also produced other products.

The records from which the observations were obtained were those kept by poultry producers under the guidance of W. R. Whitfield, poultry extension specialist at Iowa State University. Hence, the production functions which follow do not serve as sample predictions for the population of Iowa farmers. Instead, they represent an estimate of the production function for the universe of poultry producers who kept records summarized by Professor Whitfield.

Production Functions

Production functions for egg laying flocks were derived for two breeds: hybrid hens and leghorn hens. First, production functions were computed for each of these breeds separately. Then, the observations for both breeds were pooled and an additional production function was computed. There were 76 observations for hybrids, 64 for leghorns, and 140 for both breeds. The functions of a power farm are presented below where equation 16.3 is for hybrids, equation 16.4 is for leghorns, and equation 16.5 is for the pooled data. The variables are as follows:[5]

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[4] The analysis which follows is based on unpublished statistical results derived by Earl O. Heady and David W. Brown.

[5] In addition to these variables, another was included in initial calculations. It was equipment (waterers, feeders, etc.). However, this variable was highly correlated with hen numbers and the regression coefficient for it was nonsignificant at a 30 per cent level of probability.

$$(16.3) \qquad Y = 85.76X_1^{.4507} X_2^{.1784} X_3^{.0926} X_4^{.1869} X_5^{.0532}$$
$$(4.76)\ (5.54)\ (2.36)\ (2.55)\ (1.30)$$

$$(16.4) \qquad Y = 127.84X_1^{.4723} X_2^{.0204} X_3^{.2504} X_4^{.1264} X_5^{.0742}$$
$$(5.02)\ (.48)\ (5.97)\ (3.08)\ (1.77)$$

$$(16.5) \qquad Y = 134.95X_1^{.3853} X_2^{.1132} X_3^{.2165} X_4^{.1368} X_5^{.0670}$$
$$(5.88)\ (3.95)\ (7.32)\ (3.93)\ (2.12)$$

Y is the number of eggs produced, X_1 is number of hens, X_2 is square feet of housing, X_3 is hours of labor, X_4 is pounds of corn equivalent of grain fed, and X_5 is pounds of 26 per cent protein concentrate equivalent.

The R^2 value was identical for equations 16.3, 16.4, and 16.5, namely, .97, a quantity significant at the .01 probability level. The values of t for the regression coefficients are included in parentheses under the appropriate variable of each function. In equation 16.3, regression coefficients or elasticities were significant at the .01 level of probability for X_1, X_2, and X_4; at the .05 level for X_3; and at the .2 level for X_5. In equation 16.4 they were significant at the .01 level for X_1, X_3, and X_4; at the .1 per cent level for X_5. They were not significant at the .5 level for X_2. In equation 16.5 all coefficients were significant at the .01 probability level except for X_5 which was significant at the .02 probability level. These statements suppose, in somewhat the sense of experimental plots purposely selected by an agronomist or lots of animals especially sorted out by an animal husbandman for their experiments, that the data are for a sample representing a larger population to which inferences may be made. It is true, of course, that the data refer to a particular population of poultry flocks for which records were kept. Interpretations of the derived statistics should be conditioned accordingly.

Analysis was made, in the conventional manner of t, to test whether significant differences exist between each parallel pair of regression coefficients in functions indicated by equations 16.3 and 16.4. None of these pairs of coefficients differed significantly. Hence, the pooled function, equation 16.5, was used for estimates based on this poultry analysis. (Functions based on the resistance formula also were computed but appeared less efficient than those presented.)

The sum of elasticities in equation 16.5 is .9188. When tested against a value of 1.0, this sum of elasticities does not differ significantly from constant returns to scale. Hence, it appears that if housing, hens, labor, and feed are increased in equal proportions, egg output also increases by the same proportion. This statement refers, of course, to the average techniques found on farms in the study. Other types of egg production techniques might display important cost economies, as compared to the techniques of this study.

Estimates From Functions

The pooled function, equation 16.5, was used to derive marginal resource productivities, given the form of function employed and observations from the particular sample explained above. For the detail included in the analysis, predictions can be made for fairly refined input categories such as number of hens, corn equivalent, and protein concentrate equivalent. The data also have been used for estimating the effect of varying the number of hens, relative to a fixed housing capacity, on total egg production. For the 140 farms in the study the mean square feet of housing was 3.4 per bird and mean egg production was 212 per bird for the period of measurement. While they serve in a somewhat different vein than the estimates of biological functions in chapters 10 and 11, certain estimates can be made, over restricted mix ranges, of average marginal rates of substitution between feed categories or other resource pairs.

Mean outputs and inputs are included in Table 16.11, along with mean marginal productivities of the specific resources.

The mean marginal productivities are, of course, all lower than the average product per unit of each resource. (For example, mean egg production per hen was 212.) This holds true because the elasticity coefficient of each resource is less than 1.0 in equation 16.5 and diminishing marginal products are encountered at the outset. The average egg productivity, at the mean of inputs, was 212 per hen, 63 per square foot, and 219 per hour. The marginal productivities show, of course, the predicted addition to egg production if each particular resource were increased by one unit above its mean, the other resources being held fixed at their mean.

The mean marginal productivity of corn feed and protein supplement was, as predicted by the particular functional form in equation 16.5, nearly equal. However, this function would predict different marginal productivities for the two feeds combined in mixes other than that represented by their means. But given such great similarity at the means, it is suggested that, as an average, farmers in the group were not combining feeds in profit-maximizing proportions. This is true because the per pound price of protein concentrate is normally

Table 16.11. Mean Inputs and Mean Marginal Productivities for Function 16.5

Output and Input Categories	Geometric Mean for 140 Farms	Mean Marginal Productivity (Eggs)
Eggs (number)	61,170	--
Hens (number)	288.2	81.72 per hen
Housing (square feet)	979.6	7.05 per square foot
Labor (hours)	275.6	51.79 per hour
Corn (pound)	17,736.2	.47 per pound
Protein supplement (pound)	8,391.3	.49 per pound

considerably higher than for corn. Accordingly, it should be used in proportions which cause its marginal productivity to be similarly higher than for corn. The equation of marginal rate of substitution of protein concentrate for corn is

$$\frac{\delta X_4}{\delta X_5} = -.540 X_4 X_5^{-1} \quad (16.6)$$

The numerical coefficient does not mean that 1 pound of concentrate substitutes only for 1 pound of corn. This rate would hold true in case hens were being fed the same proportion of the two feeds. For the mean quantities in Table 16.11, the marginal rate of substitution of protein concentrate for corn is 1.14, the value of the partial derivative in equation 16.6 when the X_4 and X_5 quantities of Table 16.12 are substituted into it. However, if corn were used in a ratio of 5 pounds to 1 pound of concentrate, the substitution rate would be predicted to be 2.70. It is possible that "samples" of the type analyzed may provide an estimate of only one slice of the production surface and substitution rates may be estimated accordingly. As compared to all Iowa farmers, the particular group providing the observations may feed relatively high on protein. Hence, only the portion of the surface with low substitution rates may be predicted.

Optimum "Crowding" of Hens

One facet of resource mixes which has concerned poultrymen has been the proportion of hens to housing. Recommendations have long been made that a particular amount of housing space should be used per hen; an assumption of limitational factors when recommended apart from the price relatives for hens and housing. Analyses of the type presented here provide a framework for such recommendations, even though other functional forms and coefficients may eventually prove more appropriate. Marginal rates of substitution between these factors provide one basis for analysis of the "optimum degree of crowding."

The "average" marginal rate of substitution of hens, derived from equation 16.5, for housing space is

$$\frac{\delta X_2}{\delta X_1} = -3.407 X_2 X_1^{-1} \quad (16.7)$$

In comparing this rate of substitution of hens for housing space with the price or cost per annum of hens and housing space, the optimum ratio of birds and space might be predicted, given the particular form of the production function used. For the mean quantities of inputs in Table 16.11, the marginal rate of substitution of hens for housing space is 8.25; meaning that if egg production is maintained at 61,170, an added hen can substitute for 8.25 square feet of housing. Based on equation 16.7, the

Table 16.12. Predicted Effect of Increasing Hen Numbers, With Space Fixed at 979.6 Square Feet, on Marginal Hen Productivity and Total Egg Production

Number Hens	Space per Hen	Marginal Productivity of Hens (Eggs)	Total Egg Production	Increase in Total Egg Production
288.2 (mean)	3.40	81.7	61,170	--
308.2	3.18	80.7	64,568	3,398
328.2	2.98	79.7	67,920	3,352
348.2	2.81	78.8	71,239	3,319
368.2	2.66	77.4	74,053	2,814
388.2	2.52	76.7	77,300	3,247

rate of substitution would be 16.50 if the number of hens in the 979.6 square feet of housing space were 144 instead of 288. If the annual discounted cost of a square foot of housing is 8.25 times greater than the annual cost per hen, the mean quantities are, aside from time and uncertainty of building commitments, optimum proportions of hens and housing. If housing costs exceed hen costs by more than this proportion, additional crowding would be profitable (within limitations of predictions by the particular algebraic form).

The Cobb-Douglas function overestimates for purposes of predicting a profit-maximizing optimum. However, the function in equation 16.5 could be used for one set of such predictions. By setting the partial derivative of egg production with respect to each particular resource in equation 16.7 to equal the appropriate egg to resource price ratios, the optimum quantities and ratios of resources would be specified. The optimum of both quantity and ratio of resources will vary with the prices of eggs and individual resources. It does not, even for more appropriate functional forms, turn out to be a specific quantity, such as 3.5 square feet per bird, under all price conditions.

The mean marginal productivity of hen numbers increased relative to housing space can be predicted, on the basis of the particular functional form, from equation 16.5. It is, perhaps, more appropriate to suppose that labor and feed are increased proportionately with hen numbers relative to a given housing space. Hence, column 1 of Table 16.12 shows the number of hens increased relative to a fixed housing space of 979.6 square feet. But at the same time, labor, corn, and protein feed are increased by the same proportion as hens. Only housing space is held constant. Column 3 shows the marginal productivity of hens for each of the resource mixes implied. Column 4 shows the predicted total egg production while column 5 shows incremental egg production for the successive increases in hens, labor, and feed relative to fixed housing space. Because of the algebraic nature of the function, only a few predictions beyond the mean degree of crowding are used.

The optimum degree of crowding in the short run (i.e., before new

housing space can be built) thus is a function of the price of eggs and the price of variable resources. The quantities in column 5 can be compared with the ratio formed by dividing the price of a combined unit of the variable resources by the price of eggs. For a quantity in column 5 greater than this price ratio, the added crowding can be attained to extend profits. Crowding should be extended up to any increment in column 5 which is less than the price ratio.

Because (a) the function used for prediction assumes a constant elasticity of production and (b) the sum of elasticities for variable factors considered in Table 16.12 is quite high, the marginal productivity of hens does not decline rapidly. These data might well overestimate the "degree of crowding" which is profitable under a given price situation. However, the estimates do indicate the principles which are applicable on recommendations to farmers relative to proportioning of resources such as hens and housing.

Of course, a more exact method of specifying the optimum "degree of crowding" is to equate the four partial derivatives (of eggs with respect to hens, labor, corn, and protein) with the individual egg to resource price ratios, and solve for the magnitude of hens relative to the fixed housing space (and for other variable resources as well). In a long-run context, the optimum degree of crowding relative to any set of price ratios would consider the derivative of the housing variable as well. The five partial derivatives below would be equated to the price ratios indicated, and magnitudes of X_1 through X_5 would be solved simultaneously.

$$\frac{\delta Y}{\delta X_1} = .3853YX_1^{-1} = \text{hen cost} \div \text{egg price} \tag{16.8}$$

$$\frac{\delta Y}{\delta X_2} = .1132YX_2^{-1} = \text{space cost} \div \text{egg price} \tag{16.9}$$

$$\frac{\delta Y}{\delta X_3} = .2165YX_3^{-1} = \text{labor cost} \div \text{egg price} \tag{16.10}$$

$$\frac{\delta Y}{\delta X_4} = .1368YX_4^{-1} = \text{corn price} \div \text{egg price} \tag{16.11}$$

$$\frac{\delta Y}{\delta X_5} = .0670YX_5^{-1} = \text{protein price} \div \text{egg price} \tag{16.12}$$

These quantities specify the optimum resource mix or degree of crowding in the long run, subject to the limitations of the power function and the particular sample.

TIME SERIES AND CROSS-SECTIONAL OBSERVATIONS

This section reports production functions derived from time series and cross-sectional data in combination.[6] The empirical method used

[6] The analysis summarized here is from an unpublished study made by Al Egbert, Burton French, and Earl O. Heady.

is relatively simple and alternative approaches might have been used for data of this type. The farms analyzed are those which kept records with the Iowa Agricultural Extension Service over the 18 years, 1937-54. There were 20 such farms, providing 360 total observations. (Observations between years are not necessarily independent and give rise to statistical questions discussed earlier.) These farms with continuous records, but with considerable variations in inputs and outputs, were selected in an attempt to help guarantee a study of a particular production function. Only farms with Clarion-Webster soils in central Iowa were used. All values were converted to a 1950-54 basis, so that inputs and outputs might better represent physical quantities. Crop and livestock production functions computed separately are, respectively:

$$Y_c = 19.22X_1^{.514} X_2^{.297} X_3^{.033} X_4^{.172} \tag{16.13}$$

$$Y_l = 2.18Z_1^{.145} Z_2^{.314} Z_3^{.578} \tag{16.14}$$

The variables are the following: Y_c is crop output measured in constant dollars; X_1 land in crop acres; X_2 is labor used on crops in hours; X_3 is annual expense of machinery used on crops in constant dollars; X_4 is other annual capital (fertilizer, seed, etc.) used on crops in constant dollars; Y_l is value of livestock in constant dollars; Z_1 is labor used on livestock in hours; Z_2 is livestock feed in constant dollars; and Z_3 is annual capital (livestock, buildings, equipment, etc.) used for livestock in constant dollars. In derivation of these functions, each quantity in time and space was used as a separate observation in computing production functions. A time or trend variable was not significant for the particular data. Neither could significant coefficients be obtained for a moisture or rainfall variable.

The coefficient of determination was .79 for the crop function and .95 for the livestock function. All regression coefficients were significant at a .05 level of probability, except for the power and machinery variable for crops, even if the number of degrees of freedom is considered to be equal to the number of farms. The sum of elasticities did not differ significantly from 1.0.

While the magnitudes are somewhat different, the relative relationships among mean marginal productivities are similar to those reported in studies summarized above. As Table 16.13 indicates, the mean marginal productivity of labor was relatively high for both crops and livestock. The marginal productivity of power and machinery, at the geometric mean, was only .17 per $1 of annual expense. The reasons for this small quantity are probably those explained earlier; namely, the tendency for farmers to have overinvested for production purposes. It is likely that this group of record-keeping farms, even more than for farms in a random sample, has invested in machinery for consumption purposes. The land productivity of $26.67 is somewhat higher than the going cash rent of about $19 per acre; but fairly consistent with the average share rent. It is not expected, because of risk and

Table 16.13. Inputs, Marginal Productivities, and Elasticities for Equilibrium

Input	Arithmetic Mean	Mean Marginal Productivity in Dollars	Elasticity Necessary to Give Marginal Productivity Equal to Factor Price
		Crop function	
Land (acres)	271	26.67	.347
Labor (hours)	1,433	2.91	.102
Power and machinery ($)	2,794	.17	.224
Other capital services ($)	1,870	1.29	.141
		Livestock function	
Labor (hours)	2,244	2.59	.056
Feed ($)	13,675	.92	.362
Capital services ($)	19,367	1.19	.513

uncertainty, that farmers would buy or rent land until its marginal productivity is driven to the market price or cash rental level.

Table 16.13 indicates, in column 4, the magnitude of elasticities necessary to give marginal resource productivities, at the geometric mean, equal to the prices of the particular resources. Using the assumptions of random observations for all quantities in time and space, a condition not entirely fulfilled, the significance of difference between the individual elasticities in equations 16.13 and 16.14, as compared to those of column 4 in Table 16.13, was tested. On this basis, the marginal productivities for labor on both crop and livestock, cropland, and machinery and power for crops differed significantly from the market prices for these factors. The marginal productivity of the crop power and machinery input was significantly lower than the per unit price or cost. However, the marginal productivity of crop capital services, livestock capital services, and livestock feed did not differ significantly from the prices of these factors. As an average among farms and over time, it appears that these fairly progressive managers used less than equilibrium quantities, for profit maximization, of land and labor, and more than equilibrium quantities of machinery.

These inferences are consistent with the beliefs of extension personnel servicing these farms. Most of the managers use family labor and may not hire an additional man for family reasons. In general farms are of a size consistent with the total goals of the farm family, including participation in nonfarm activities. Machinery investment by these relatively high income families (average net income over the 18 years of $12,341 in 1950-54 dollars) has been extended into the consumption category to allow greater enjoyment of work and to free time for nonproduction activities. Yet within this family and institutional

setting, conditioning the mix of land and labor inputs, these farmers can and do invest in livestock, feed, fertilizer, and similar resources to the extent that marginal productivities might approach factor prices.

However, while they appear reasonable in terms of knowledge of extension personnel, the above interpretations need qualification. The marginal productivities are means over time and between farms, compared with prices converted to a 1950-54 basis and compared to average prices of this period. Since prices of individual years deviated from this level, farms might well have equated productivities and prices, or failed to do so, even if this condition had not been illustrated under the empirical mechanism used above.

INTERREGIONAL PRODUCTIVITY COMPARISONS

This section reports a production function study which allows interregional comparison of resource productivities.[7] Random samples were used in the four regions included in the analysis. Cobb-Douglas equations were used and inputs and outputs were aggregated similarly in all regions. Separate functions were computed for crops and livestock, with output measured in dollar values. Inputs for crops were cropland in acres, labor in months, and all capital services measured in dollar value of annual input. Inputs for livestock were labor in months and all capital services measured as dollar value of annual input.

Basic Statistics

As background, we present the data in tables 16.14 and 16.15. These are based on the four random samples covering the 1950 production year in the Alabama Piedmont, northern Iowa, southern Iowa, and a dry-land farming area of Montana. Generally, the derived production elasticities, which are presented in Table 16.14, can be accepted at conventional probability levels. These statistics have been used in deriving the marginal quantities of Table 16.15. The different capital to labor ratios lead to the a priori hypothesis that labor productivity will vary greatly in magnitude between the four areas. The ratio of capital to labor is much higher in northern Iowa and Montana than in southern Iowa and Alabama; it is much higher in southern Iowa than in Alabama. Labor productivity for crops is highest in northern Iowa and Montana where the amount of land and capital per worker is greatest; southern Iowa is slightly lower and Alabama is lowest. Probability tests do not allow us to say that differences in marginal labor productivity are

[7] See Heady, Earl O. and Shaw, Russell. Resource returns and productivity coefficients in selected farming areas. Jour. Farm Econ., 36: 243-57. 1954; and Heady, Earl O. and Baker, C. B. Resource adjustments to equate productivities in agriculture. Southern Econ. Jour., 21: 36-52. 1954.

Table 16.14. Regression Coefficients and Related Statistics

Item	Montana	Northern Iowa	Southern Iowa	Alabama
		Crop production		
Value of constant (log)	.595	1.273	.718	.979
Value of elasticities				
Labor	.039	.076†	.088†	.319*
Land	.503*	.912*	.795*	.385*
Capital services	.580*	.165*	.393*	.463*
		Livestock production		
Value of constant (log)	.276	.359	.057	.737
Value of elasticities				
Labor	.084†	.077‡	.117‡	.233*
Capital services	.937*	.907*	.982*	.743*

Significant at:
*1 per cent probability level;
†8 per cent probability level;
‡5 per cent probability level.

significantly greater in the first three areas. However, these orderings are those expected from (a) the capital to labor ratios in the four areas, (b) the knowledge of agricultural workers, and (c) the findings of previous scattered investigations. For capital and land resources used in crops, differences in marginal resource productivity differ significantly between the four areas. Important differences also exist between areas for resources used on livestock. Returns for capital are highest in Montana and southern Iowa where less of this resource is used than in

Table 16.15. Mean Resource Quantities, Marginal Product, and Gross Average Product of Resource Services

	Montana	Northern Iowa	Southern Iowa	Alabama
		Crop function		
Arithmetic mean:				
Product; actual ($)	21,419	8,551	4,777	1,322
Cropland (acre)	975.0	166.6	114.9	23.8
Labor (month)	13.7	9.5	8.7	10.4
Machine-crop capital services ($)	5,207	2,168	1,420	553
Investment per man year	67,866	62,430	32,064	3,255
Marginal product				
Cropland not pasture ($/acre)	11.06	46.83	33.05	21.37
Labor ($/month)	61.80	68.04	48.05	40.57
Machine-crop capital services ($/$)	2.39	.65	1.32	1.15
Average product per month of labor	1,559	905	547	127
		Livestock function		
Arithmetic mean:				
Product; actual ($)	12,084	13,943	9,067	1,336
Labor (month)	8.9	8.2	7.3	3.5
All capital service inputs ($)	8,896	12,543	7,614	1,017
Investment per man year	23,163	18,078	15,207	4,340
Marginal product				
Labor ($/month)	113.92	130.76	144.83	89.09
Capital service inputs ($/$)	1.27	1.06	1.17	.97
Average product per month of labor	1,351	1,694	1,238	378

northern Iowa. They are lowest in Alabama where capital inputs hardly allow livestock on a commercial scale. Since these basic figures are analyzed in detail elsewhere, we proceed with the objectives of our comparisons.

Equilibrium Between Products Within Regions

Having examined the mean marginal products for each resource used in crop and livestock production in each area, we may ask: Are these computed differences between crops and livestock significant? If they are, our first conclusion may be that resources are not allocated between products within a region to either (a) maximize farm profits or (b) give a maximum national product from given resources. Since the marginal return of crop capital in Montana is $2.39 while the return on livestock capital is only $1.27, it appears that capital resources should be shifted from the latter to the former product. On the basis of the predicted marginal figures, capital should be shifted from crops to livestock in northern Iowa and Alabama and in the opposite direction for southern Iowa. Given the productivity figures in Table 16.15 for labor, the other common resource for the two products, it should be allocated from crops to livestock in four areas.

Since, however, the estimates involve sampling errors, our inferences need to be subjected to a probability test. In terms of probability and sampling error, we cannot say that the marginal productivity of labor is higher for crops than livestock in Montana or northern Iowa (Table 16.16). We can more nearly state that the differences expressed for marginal labor productivity on crops and livestock are significant in Iowa and Alabama. Greatest difficulty in a study of this kind arises in the enumeration of labor inputs. While the farmer can quickly enumerate the number of crop acres operated, the amount of fertilizer purchased or expenses for tractor fuel (he usually has the latter recorded with other expenses), it is difficult for him to specify actual labor inputs, except for hired help; he is prone to report the "stock" of operator and family labor available, rather than the amount actually used in production. While detail of our study and questionnaires may have partly eliminated this problem, it still exists. It is our hypothesis that even more detailed labor records might lower the variance associated with enumeration bias and actually unfold more important differences in labor productivity. However, logic may exist for supposing that differences in labor productivity do not exist in fact. The farmer controls the allocation of the family labor, and does not have its use tied to certain resources or products as in the case of capital obtained through the credit market. Accordingly, he may be more nearly able to allocate labor to equate its marginal productivity between products than he is for capital. Still, where capital and labor are technical complements, the banker who specifies that the farmer can have credit for feeder steers but not for seed or fertilizer also is specifying how labor should

Table 16.16. Value of t for Comparing Elasticity Derived in Crop Production with Computed Elasticity Necessary to Give Marginal Productivity for Crops Equal to Marginal Productivity for Livestock*

Resource	Montana	Northern Iowa	Southern Iowa	Alabama
Labor	.39§	.77§	1.43‡	1.87‡
Capital	2.20†	1.31‡	.56§	.62§

*The simple test made here includes the following steps: First, the elasticity of production necessary to give a marginal product in crop production equal to the marginal product of the same resource in livestock production is computed. Our task is to compare β_c, the elasticity actually derived for crops with β'_c, the elasticity necessary to give equal elasticities. It is computed in the manner of (a) and (b):

$$\text{(a)}\quad \frac{d\bar{Y}_c}{d\bar{R}_c} = \beta'_c \frac{\bar{Y}_c}{\bar{R}_c} = \beta_1 \frac{\bar{Y}_1}{\bar{R}_1} \qquad \text{(b)}\quad \beta'_c = \beta_1 \frac{\bar{Y}_1 \bar{R}_c}{\bar{Y}_c \bar{R}_1}$$

The standard error, S_{bp}, for the pooled variances

$$\text{(c)}\quad S_{bp} = \sqrt{S_c^2 + \left(\frac{\bar{Y}_1 \bar{R}_c}{\bar{Y}_c \bar{R}_1}\right)^2 S_1^2}$$

is then computed as in (c) and the value of t is given in (d).

$$\text{(d)}\quad t = \frac{\beta_c - \beta_1 \left(\bar{Y}_1 \bar{R}_c / \bar{Y}_c \bar{R}_1\right)}{\sqrt{S_c^2 + \left(\frac{\bar{Y}_1 \bar{R}_c}{\bar{Y}_c \bar{R}_1}\right)^2 S_1^2}}$$

†$0 < p < 5$;
‡$10 < p < 20$;
§$p < 40$.

be used; labor for feeder steers rather than for crop services may cause its marginal products to diverge even further, rather than to converge between products.

In terms of probability, we may infer that the mean marginal productivity of crop capital in Montana was significantly greater than for livestock capital in 1950. Had Montana farmers known this outcome beforehand, they should have used more of their limited funds for crops and less for livestock. They could not, however, predict the more favorable weather for crops in 1950 and their *ex ante* decisions to operate as they did may not have been in error. Of course, many capital-short farmers produce crops rather than livestock because they do not have to tie up as great an investment per $1 of annual input or output as in beef production. Following the figures of Table 16.15, less capital should have been used for crops and more for livestock in northern Iowa had farmers (1) been attempting to maximize profits and (2) known the actual outcome of yields and prices. Since capital productivity is higher for livestock than for crops in northern Iowa and since the value of t is relatively high, we might infer that too much capital is used on crops and too little on livestock in northern Iowa. While the level of the

computed productivity figures may not serve too accurately in predicting exactly the population parameters, the ordering of productivities and the inference outlined above seems to be in line with the structure of production in the area. About one-half of the farms are rented; many of these produce little or no livestock. Livestock would be profitable on these rented farms but because of uncertain tenure, many tenant operators keep little livestock. On the other hand, tenants tend to invest a relatively large amount of capital in machinery, one of the components of the crop capital services outlined above. It is entirely likely that at the margin, more investment in machinery (and hence in capital services for crops) would give less than $1 in product for every $1 in expense. Northern Iowa, relative to the products produced, is about as highly mechanized for crops as any other United States region. Capital for crops in the form of machinery may have been used, as a matter of convenience and to ease the drudgery of farm work, beyond the input level necessary for equating expected and discounted marginal costs and returns.

Tests of Inter-Area Differences in Marginal Productivity

The regional differentials presented above correspond with the experience of agricultural extension workers and others closely acquainted with the nation's agriculture. It is known that farm families in the central Corn Belt are generally more prosperous than those in the southern Corn Belt; during years of favorable moisture, Great Plains farmers, including those in Montana, have prospered and have given a high out-turn of product for capital and labor; Alabama Piedmont farmers, as an average, are in an area where productivity is known to be low and where the incomes of farm families are meager relative to the national standard. These things are observed facts but since we have samples from each area and for purposes of national policies and guidance to individual farmers, we must ask: "Are the differences in productivity coefficients presented significantly different between regions?" Since a productivity figure of any level involves sampling errors, we must evaluate the differences in terms of the errors attaching to each elasticity coefficient. This step has been taken by means of simple significance tests and the relevant statistics are presented in Table 16.17. The first figure in each cell is the elasticity coefficient which would have been necessary, considering the mean quantity of product and resource in the area of comparison, to give a marginal productivity equal to that computed in the area of contrast. For example, an elasticity of .0435 for labor in Montana (as compared to the Montana sample coefficient of .039 shown in Table 16.14) would have been necessary to give a marginal product of $68.04, the northern Iowa average, in Montana (as compared to the sample prediction of $61.80 for Montana in Table 16.15). In comparing the .0435 elasticity, as a constant, against the actual elasticity of .039 in a null hypothesis

Table 16.17. Comparison of Differences in Marginal Productivities of Resources in Different Areas; Elasticity Coefficient Necessary to Give Marginal Resource Productivity in One State Equal to That in Another State, t Values (in Parentheses) and Significance Level*

Resource and State Against Which Test is Made	State for Which Test is Made			
	Montana	Northern Iowa	Southern Iowa	Alabama
Crop production				
Labor:				
Montana	--	.0684 (.05)§	.1122 (.12)§	.4846 (.19)§
Northern Iowa	.0435 (.05)§	--	.1239 (.387)§	.5352 (.62)§
Southern Iowa	.0307 (.12)§	.0534 (.38)§	--	.3780 (.25)§
Alabama	.0260 (.19)§	.0451 (.62)§	.0739 (.25)§	--
Land:				
Montana	--	.2159 (8.56)†	.2661 (5.63)†	.2007 (1.70)‡
Northern Iowa	2.1266 (8.56)†	--	1.1247 (3.76)†	.8405 (3.85)†
Southern Iowa	1.5029 (5.63)†	.6448 (3.76)†	--	.5994 (1.81)‡
Alabama	.9646 (1.70)‡	.4138 (3.85)†	.5101 (1.81)‡	--
Capital:				
Montana	--	.6053 (3.05)†	.7085 (1.91)‡	.9626 (2.28)†
Northern Iowa	.1579 (3.05)†	--	.1931 (1.78)‡	.2619 (1.55)‡
Southern Iowa	.3214 (1.19)‡	.3352 (1.78)‡	--	.5330 (3.50)†
Alabama	.2790 (2.28)†	.2910 (1.55)‡	1.2327 (3.50)†	--
Livestock production				
Labor:				
Montana	--	.0670 (.17)§	.0917 (.32)§	.2984 (.32)§
Northern Iowa	.0963 (.17)§	--	.1052 (.15)§	.3426 (.57)§
Southern Iowa	.1067 (.32)§	.0852 (.15)§	--	.3749 (.85)§
Alabama	.0656 (.32)§	.0524 (.57)§	.0717 (.85)§	--
Capital:				
Montana	--	1.1450 (3.94)†	1.0688 (4.38)†	.9689 (3.70)†
Northern Iowa	.8063 (3.94)†	--	.8464 (2.94)†	.7672 (.47)§
Southern Iowa	.7229 (4.38)†	.8834 (2.94)†	--	.7475 (2.75)†
Alabama	.7186 (3.70)†	.8782 (.47)§	.8198 (2.75)†	--

*The elasticity coefficient (to give a marginal product in the area of comparison equal to the value considered as a constant) computed as the marginal product in the area of contrast is computed this way: The marginal product, based on the total product, and the input of the particular resource as averages for the area are used. We wish to compare the elasticity, β_a of resource X_a, estimated for region a, with β'_a, the elasticity necessary to give the marginal product of region b, when X_a, and Y_a are those of region a. Hence we have (a) and (b) for estimating β'_a.

$$\text{(a)}\quad \frac{d\bar{Y}_a}{d\bar{X}_a} = \frac{d\bar{Y}_b}{d\bar{X}_b} = \frac{\beta'_a \bar{Y}_a}{\bar{X}_a} = \frac{\beta_b \bar{Y}_b}{\bar{X}_b} \qquad \text{(b)}\quad \beta'_a = \beta_b \frac{\bar{Y}_b \bar{X}_a}{\bar{Y}_a \bar{X}_b}$$

Finally, the value of t has been computed as in the footnote of Table 16.16.

Significance level in probability:
†$0 < p < 5$;
‡$5 < p < 8$;
§$p < 40$.

sense, we obtain a value of only .05; a value which is not significant at an acceptable probability level for t.

We conclude, considering the quantity of resources used, mean marginal labor productivity in Montana is not significantly lower than in northern Iowa. Since northern Iowa uses a different quantity of resources than Montana, we again compute the elasticity coefficient necessary for the marginal productivity of labor in northern Iowa to equal the marginal productivity of labor in Montana. The elasticity of .0684, necessary to give the $61.80 marginal labor productivity of Montana, in northern Iowa, does not differ significantly from the sample elasticity of northern Iowa.

The coefficients of neither crops nor livestock differ significantly among the four areas; however, a test of the average gross products for labor shows significant differences. Thus, at the margin of the mean, labor productivity does not differ significantly; but for parallel months of labor and for smaller quantities, it may do so. This comparability of marginal labor product is in contrast to certain previous inferences on productivity based on extremely gross, residual data for parts of the same regions. From these data, we cannot even say that at the margin of mean labor inputs marginal productivity of labor in Alabama differs significantly from that of other areas, although we know that it differs in average productivity terms. It is entirely possible that the return to "the last month of labor" used "on the average farm" in the four areas is quite similar. However, enumeration of labor inputs is the most difficult part of this type of study. We believe that significant differences in mean marginal productivity might be obtained in such contrasting areas as northern Iowa and Alabama by obtaining the labor records and by lessening variance through samples stratified more closely by soil type and techniques of production.

As would be expected, differences in marginal productivity of land are significant. Not only of interest in farmer investment decisions, these data are also important for certain production policy questions. For example, withdrawal of an acre per farm under a control program would have greater effect in lowering total agricultural production in northern Iowa; and the least in Montana, while the effect in southern Iowa would be greater than for Alabama. Or, during war or other food emergency, it is more important to keep an acre of land in northern Iowa in production than one in Montana, southern Iowa, or Alabama.

Tests of Disequilibrium in Marginal Resource Productivity and Factor Prices

With our four small sectors of the total economy, we can investigate certain conditions of general equilibrium; namely, the extent to which the marginal productivity of resources approach factor prices. (Also, our examination of intra-area and inter-area differences above involved the general equilibrium or stability conditions relating to the inter-firm,

Table 16.18. Comparison of Departure of Marginal Productivities From Resource Prices. Elasticity Coefficients Necessary to Give Marginal Product Equal to Market Price of Resource, Values of t in Parentheses

Resource	Montana	Northern Iowa	Southern Iowa	Alabama
		Crop production		
Labor	1.0234 (13.8)	.1444 (1.41)	.2368 (2.84)	.5900 (4.09)
Land	.2504 (2.66)	.3408 (8.28)	.3505 (5.60)	.2179 (1.66)
Capital	.2577 (2.64)	.2688 (1.52)	.3151 (1.97)	.4518 (.13)
		Livestock production		
Labor	.1178 (.62)	.0765 (.01)	.1047 (.24)	.1965 (.57)
Capital	.7804 (3.76)	.9536 (1.44)	.8901 (2.66)	.8222 (1.48)

inter-product, and inter-area organization of resources and production. It is, of course, true that factor markets are imperfect and have barriers between them; in this situation, unequal marginal returns may not mean market disequilibrium. To make the comparison, we computed the elasticity coefficient necessary for each resource (given the mean input of all resources and the mean product in each area) which would have resulted in a marginal product equal to the factor price. We then tested the significance of the difference between this elasticity coefficient and the one estimated from the sample (Table 16.18). The marginal productivity of labor for crops appears to be significantly lower than the monthly hired wage rate in Montana and the two Iowa areas; the inference for Alabama is that the low marginal product of labor on crops does not differ significantly from the low wage rate for hired workers. Differences in availability of off-farm and part-time work in Alabama may partly explain this difference. While off-farm work during winter months on a Montana wheat farm is not entirely unavailable, it is not plentiful and convenient; the Iowa areas are somewhat similar. Hence, a marginal product lower than the wage rate (weighted particularly by summer crop work) can be consistent with rational choice in areas of few outside opportunities.

Marginal land productivity was significantly greater than rental rates for all areas in 1950. (The rental prices used are share rates; cash rates would be lower and hence t values would be even greater.) Tenant operators or purchasers of land were realizing a short-run returns margin above the rent of land. Some difference is expected, of course, because of uncertainty considerations. Contracts are made into the future and with discounting of anticipated returns for risk, an *ex ante* equilibrium may exist in a subjective sense even if the marginal productivity and price for land services differ in *ex poste* measurement, the manner of the data in this study. Also, weather and lack of pests were favorable in 1950, except perhaps in Alabama; weather was perhaps better than farmers' modal expectations, which extend over several years. Farmers' price expectations in postwar years had centered

on declining prices; rental contracts, made 1-3 years ahead, had not accurately anticipated continuance of inflationary prices.

Capital returns were significantly greater than the cost of capital for crops in Montana and southern Iowa; it appears that mean marginal return of capital was lower than the cost of capital in northern Iowa. (This phenomenon may be a function either of sampling variance or the machine investment situation mentioned earlier; machinery investment often is driven to a point to give ease of work and lessen drudgery on many of these high capital farms.) In terms of probability, we must infer that differences did not exist between the cost and marginal product of capital in Alabama. This statement refers to the capital forms and techniques being used; capital in the form of other techniques and in an amount to allow gains of scale economies might allow much higher returns.

SIMULTANEOUS EQUATIONS

The functions reported in this chapter were estimated by least-squares, single equation methods. Several attempts have been made to use simultaneous equations models in estimating farm production functions. However, none of these have met with success. This was true for such a model applied to the data from record-keeping farms discussed in a preceding section.[8] One difficulty evidently is in the high intercorrelation between the numerous variables necessary in a logical simultaneous equation model. Perhaps another reason is that the basic suppositions of simultaneous models may have no special relevance in terms of whole-farm production functions. While reasonable decision models might be established, there appears to be no real biological or physical basis for supposing that the magnitudes of inputs and outputs are interdependent. Once a farmer has made a decision to use a particular input, the process is purely physical, with input magnitude affecting output magnitude but without the opposite holding true, and should be so reflected in the model. Except for a few minor inter-year exceptions, it appears that inputs can be considered purely as exogeneous variables in their effects on output. In these cases we are concerned with the quantitative effect of input on output, once decision has been made to use a particular magnitude of input, rather than with the decision-making process leading to a particular input level.[9]

[8]Also, as another example, see French, Burton L. Estimation by simultaneous equations of resource productivities from time series and cross-sectional farm observations. Unpublished Ph.D. thesis. Iowa State University Library, Ames. 1952.

[9]But see Klein, L. R. A textbook of econometrics. Row Peterson, Evanston. 1953. pp. 193-6, 226-36.

CHAPTER 17

Comparison of Production Function Estimates From Farm Samples Over the World

IN TODAY'S world problems, both economic and social, resource productivity and the efficiency of resource use play a vital role. International productivity differentials have, of course, always been of major concern to economists; implicit assumptions about productivity coefficients have provided the basis for most of the literature dealing with world exchange and national welfare maximization. Since World War II, however, resource productivity in particular regions of the world and international productivity differentials have taken on additional importance; stirred on the one hand by ideological currents, and on the other by recognition of the poverty in the world and the imminent explosion of world population. Little wonder, therefore, that the need for increasing the productivity of resources and the efficiency of resource use in particular areas of the globe has been deemed especially important, as evidenced by the funds and services that the more advanced countries have invested in an endeavor to assist less developed areas of the world.

In the main, these investments have been concerned with increasing the productivity of resources by the introduction of new technologies. By replacing the existing production functions by new ones that lie higher in the output plane, it is hoped that these investments will lead to the production of more output from the same quantity of inputs, or to the same output from fewer inputs (thereby releasing resources for other economic activities). Only a minor portion of these international development funds have been concerned with increasing the efficiency of resource use within the context of a given production function. The latter, of course, is the more difficult task since its success depends on the availability of empirical information for an array of situations. Still, as is evidenced by the data presented in this chapter, there are significant gains to be made by increasing the efficiency of resource use within given technological environments. The collection of the necessary data is likely to be well worth the cost. This will become increasingly so as the technological lags throughout the world are dissipated and the opportunities for shifting to new production functions become limited. It is, therefore, worthwhile to draw together some of the production function studies for farm-firms that have been carried out in various countries. The primary value of comparing such

analyses lies in the direct indications they offer of economic disequilibria in the use of resources, both inter- and intra-nationally. Such quantitative evidence is essential to any attempt to improve or optimize the allocation of scarce resources either within or between nations. Moreover, such data may serve to indicate in what directions and to what extent it may be desirable to change the value systems of farmers and politicians so as to free the way for greater resource use efficiency. As well, comparison of production function estimates provides empirical evidence of the differentials that exist in resource productivities between and within countries. Such data may enable us to assess the relative need for the introduction of more advanced production processes and to decide, where necessary, between developmental programs oriented to the adoption of new production functions and those programs whose primary aim is to increase resource use efficiency within a given technological environment. Needless to say, when new production processes are substituted for old, every endeavor should be made to ensure that resources are used in optimal fashion under the new methods.

To these ends, and also to provide a résumé of research to date, an array of cross-sectional farm-firm studies from around the world are collated in the present chapter. Without fully searching the fugitive literature, we have endeavored to include representative studies from as many nations as possible. It has, of course, been impossible to include all the production function analyses that have been made; for the United States and Japan it has only been possible to present a selection from among the many available studies. Nor have we been able to locate studies for a number of countries — notably the Communist countries and those of South America. Still, the analyses to be compared cover a large portion of the globe; in particular, they relate to the United States, Canada, New Zealand, Australia, the Union of South Africa, the United Kingdom, Austria, Sweden, Norway, Israel, India, Japan, and Taiwan. Moreover, a number of functions covering different segments of their agricultural production are presented for most of these countries. Within each country, and internationally, the different types of farming studied in these analyses are of varying degrees of importance. As a result, as the implications of each estimated function is assessed, one must bear in mind the relative importance of the production process to which it refers. Both inter- and intra-nationally, for instance, a production function estimate for wool growing in Australia is of far more significance than an estimate for sweet potato production in Japan. Likewise, in evaluating the implications of the analysis for individual resources within each of the production processes examined, account must be taken of the relative importance of each resource. A small measure of inefficiency per input unit in the use of a major resource may have severer repercussions than a large degree of inefficiency per input unit in a factor that is only used in small amounts. Too, for an ideal evaluation, these agricultural studies should be complemented by similar analyses for other regions and industries in the

various countries. However, comparable studies for nonagricultural industries are not available, although quite a number of nonagricultural time series analyses covering some or all of the period since 1900 have been carried out.[1]

Since any over-all comparison of the various studies would be presumptuous without some understanding of the setting of each analysis, we first present the studies on a country by country basis. Before doing so, the following comments are necessary. All of the studies are based on Cobb-Douglas type functions fitted to cross-sectional farm data. Each function refers to a production period of 12 months within the years 1950 to 1958. Studies referring to prior years have been excluded in the hope of maintaining a reasonable degree of homogeneity with respect to time. Country by country, the estimated functions and their related statistics are presented in tabular form. To facilitate comparisons between and within countries, all land, labor, and monetary statistics are listed in the following units: land in acres;[2] labor in man-months;[3] and capital in 1954 United States dollars.[4] Comparisons between and within countries will generally be made as though each of the estimated functions related to the same calendar time period. This assumption is not thought to be overly strong, since all the functions relate to years between 1950 and 1958. Also, in making comparisons, the market prices or opportunity costs prevailing at the time of each study (but converted to 1954 United States dollar values) will be used as benchmarks for estimating resource use efficiency. The assumption is that these prices are not atypical to any major degree; although it is certainly realized that agricultural prices, especially of products, are not as stable as one would desire for such purposes. Assuming no major technological or scale changes to have occurred so that the estimated functions would still be relevant, it would be feasible to evaluate the various functions according to current prices. The lack of the necessary data has prevented the presentation of such an extensive analysis here. Also, although comparisons are made without always mentioning that we are talking about the situation on an average farm, or the situation that prevails on the average in the region relevant to each fitted function, such a proviso is implicit in most of the discussion. Likewise, in discussing resource transfers, we assume that allowance

[1]For a survey of many of these studies, see: Leser, C. E. V. Statistical production functions and economic development. Scottish Journal of Political Economy, 5: 40-49. 1958; Kuhlo, K. C. Zur Auswertung und Ausgest der Produktionsfunktion. IFO-Institut für Wirtschaftsforschung. Munich, 1959.

[2]The conversion factors used were those listed in Webster's New International Dictionary. Merriam-Webster Company. Springfield, Mass. 1958. Pp. 1521-24.

[3]Where necessary, conversion rates from hours and weeks to months of labor were obtained by personal communication with the authors of the original production function estimates, as cited later.

[4]With the exception of Taiwan, the exchange rates used were those recorded by the International Monetary Fund as listed in United Nations Monthly Bulletin of Statistics. 13: 161-67. 1959. For Taiwan, an exchange rate of 24.78 New Taiwan dollars to the United States dollar was used. Adjustment to 1954 United States dollar values was made by way of a composite index based on the cost of living and wholesale price indices listed in the United Nations Statistical Yearbook. 1958. Pp. 408-28.

is made for the natural and institutional restrictions that exist relative to the mobility and quantity of existing resources.

While none of the functions selected for inclusion can be regarded as perfect estimators or representations of the production processes studied, they are all judged to be statistically reasonable. Still, it must be remembered that they are cross-sectional studies; in consequence they suffer in varying degree from the disadvantages peculiar to the cross-sectional approach as outlined in Chapter 6. As well, it should be recognized that some of the functions do not relate to a single product; nor, in some cases, are the input categories aggregated and measured in the most desirable fashion. In this regard, there are lessons to be drawn from the various approaches taken in these studies, bearing in mind the recommended procedures outlined in chapters 4 to 7. Too, all of the analyses are based on single equation models when, in some cases, a multiequation approach may have been more satisfactory; to say nothing of the possibility that a single equation model based on some algebraic model other than the Cobb-Douglas might have been more appropriate in some instances. Still, within these limitations, the usefulness of the fitted functions should not be decried. For many of the countries they constitute the only empirical estimates of resource productivities and measure of resource use efficiency that are available. While they may not be of great use for recommendations to individual farmers, they are of significant importance for the determination of over-all policies dealing with the allocation of land, labor, and capital within countries and for the rational determination of the allocation of scarce capital funds and technical services between nations.

So far as shifts in resource use are concerned, it is true that only capital has any great measure of international mobility. Still, labor has some degree of mobility between countries as evidenced, for instance, by the transfers that have occurred in recent years from Japan to South America, from Europe to Australia, and from China to Tibet. Land, of course, has no international mobility in the usual sense. But significant changes may occur in its ownership — as can be seen by comparing a 1938 map of the world with one for 1958; or by enumerating the international land purchases — Louisiana, Texas, and Alaska — by the United States. Intra-nationally, the allocation of any economic resource is subject to some measure of centralized control, as well as the ordinary market influences. Witness the land reform and consolidation programs that have been carried out recently, or are underway, in many countries; the direct and indirect devices that have been used to induce domestic labor transfers and the centralized institutions that have been developed to control, to varying degrees, the availability of credit to various sectors of most countries' economies.[5]

The more important statistics to be presented for each function are noted below. Since detailed explanations of the derivation and

[5] For a résumé of such national policies as they relate to agricultural resource allocation see: The state of food and agriculture 1958. F.A.O., Rome. 1958.

implications of these statistics have been given in chapters 2, 3, and 4, only brief explanatory comments are made here.

Production Elasticities

For each input resource, these coefficients indicate the expected percentage increase (or decrease) in production that would occur if the amount of the input resource was increased (or decreased) by 1 per cent, other input levels being held constant. Since the models used are of Cobb-Douglas type, these estimates of the elasticities remain unchanged over the range of input levels to which the function is fitted and to which it might be applied. Such constant elasticities are most likely not a true reflection of the real-world situation. They are merely an algebraic attribute of the Cobb-Douglas model. Still, provided other restrictions are also met, the estimated elasticities do validly represent the real-world situation at the geometric mean input levels. Accordingly, in drawing out the implications of the elasticities, it is always assumed that we are talking of incremental changes at the mean input levels. For each function, those elasticities that are significantly different from zero at the 5 per cent probability level are noted. It is recognized that such tests do not strictly apply in those instances where the estimates are based on a nonrandom sample of observations. Also, regardless of these significance tests but with the provisos outlined in Chapter 6, the estimated elasticity is still the most likely value of the true elasticity. Indeed, there is nothing sacred about the 5 per cent probability level. It would be far better to ascertain the probability level of error at which the relevant policy makers desire to operate and to test the estimates accordingly.

Sum of Elasticities

If one is prepared to assume that no relevant input factors have been excluded, this sum provides an indication of returns to scale. On the other hand, if one believes constant returns to scale prevail, the sum may be regarded as indicating the importance of omitted variables if it is less than unity, or as roughly indicating the extent of anomalies in the mode of aggregation if it is greater than unity. Given all the limitations associated with the measurement and interpretation of the over-all production elasticity, as noted in Chapter 7, we do not stress this statistic in discussion. However, for completeness, we have indicated whether or not each sum of elasticities is significantly different from one at the 5 per cent probability level.

Adjusted Coefficient of Multiple Determination, $\bar{R}^2$

This statistic gives the proportion of the variation in the output observations explained by the fitted function, correction having been made

for the size of the sample studied. In every instance, the statistic is significantly different from zero at a probability level of 5 per cent or less.

Sample Means of Outputs and Inputs

Generally, the geometric (rather than arithmetic) mean levels are presented for they are most relevant to the estimates of the marginal products. Since there is probably not a great deal of difference between the geometric and arithmetic sample means, the listed means serve satisfactorily as indicators of the scale of the farm operation to which each function refers. As well, the relative importance of different resources can be judged from their mean levels, thereby enabling comparative assessment of the importance of disequilibria in the various resources. The outputs listed are always gross output per annum, measured in 1954 United States dollars. Land, labor, and capital are always shown in acres per annum, man months per annum, and 1954 United States dollars per annum. The mean total quantity of inputs used is presented in money terms. These were obtained by valuing the land, labor, and other nonmonetary inputs at their opportunity costs, adding this sum to the monetary inputs used. In a few of the functions, buildings, livestock, and machinery have been analyzed in terms of their asset value, rather than indirectly through their annual services as should be done. Where such was the case, these assets have been valued at their annual cost in deriving the mean level of all inputs. In some instances, the opportunity cost method of valuation has probably led to overestimation of the "all inputs" mean. For instance, in many of the situations studied, much of the labor used is family labor which was paid less than hired labor would have cost.

Average Products

The average product of each resource is computed as the mean output divided by the mean input of the resource. The resultant average includes the product returns of all inputs, and not simply the product return attributable to the single resource. Mostly, the average products are presented for the geometric mean levels of output and input. Frequently, due to a lack of marginal productivity data, average products are used as measures of efficiency.[6] We will not discuss them in such fashion, for the obvious reasons that they generally have no strict validity as indicators of resource use efficiency and that we have data in marginal terms. Moreover, if analyses of average productivity were to be made, far better statistical sources are available than the few sets of data we present as byproducts of production function research.

[6]For example, see Strand, E. G. *et al.* Productivity of resources used on commercial farms. USDA Tech. Bul. 1128. Washington, D.C. 1955.

Marginal Products

By necessity, these estimates are given in value terms, physical data being unavailable in most instances. Each marginal product indicates the expected increase in output forthcoming from the use of an additional unit of the relevant input, the level of other inputs remaining unchanged. In general, the marginal productivity of any resource depends on the quantity of it that is already being used and on the levels of the other resources with which it is combined in the production process. For this reason the estimates with widest applicability are those derived at mean input levels, with the proviso that the most accurate estimates from Cobb-Douglas functions are obtained with all inputs held at their geometric mean level. Accordingly, wherever possible, marginal value products are listed for the geometric mean input levels. But in some cases, due to lack of information, it has been necessary to record the marginal products measured at the arithmetic mean input levels. In none of these instances is it likely that great differences exist between the estimates of marginal products as made at the arithmetic means and those estimates that would have been obtained using geometric mean input levels.

Opportunity Costs

For the various resources, these are the market prices that prevailed, on the average, over the twelve months to which each production function refers. The market price of land has always been taken as the annual cost of renting one acre since we are interested in land services rather than land per se. On the assumption that the employment of additional labor would imply the purchase of hired labor, the monthly wage rates quoted refer to hired labor. In some instances this assumption probably overvalues the opportunity cost of labor since additional family labor might be available at a cheaper price. Where necessary, an allowance for food and lodging has been included in the wage rate. The opportunity cost of a dollar of capital has been taken as one dollar plus the relevant annual interest charge. For capital assets, the opportunity cost has been taken as the annual straight-line depreciation associated with the assets plus service charges where these have not been included in other input categories. Following the practice adopted in the original sources, it is assumed throughout the analysis that currently used services of land and labor are purchased out of current funds and thus involve no interest burden. Increases in other input services are assumed to require borrowed capital with its concomitant interest charges. Realistically, each situation should be evaluated according to how the average farmer would finance additional inputs. However, any inconsistencies resulting from the use of the costing procedure adopted are thought to be slight. In listing but a single opportunity cost for each resource in each function, it is assumed that the farm population

relevant to each sample operates under free competition; the individual farmer having no control over the prices he pays or receives. While infringements of this assumption may occur, they must be regarded as minor for all practical purposes.

Marginal Product to Opportunity Cost Ratios

Within the limits of statistical reliability, these ratios provide a measure of the efficiency of resource use prevailing, on the average, throughout the population of farms relevant to each of the samples studied. If the ratio is less (greater) than one, it indicates that too much (too little) of the particular resource is being used under the existing price conditions, given the levels at which other resources are operating. Maximum efficiency in resource use occurs when the revenue from using one additional unit of input is equal to the cost of that additional unit, i.e., when the marginal product to opportunity cost ratio is equal to unity. Under such conditions entrepreneurial profits will be maximized. As well, Pareto optimality with respect to national welfare will also tend to be obtained; nobody could be made better off without somebody being made worse off. Strictly, however, if marginal costs and revenues are equated in agriculture, Pareto optimality will only be attained if marginal costs and revenues are also equated in all other segments of the economy. In the few instances for which appropriate data are available, whether or not the marginal product to opportunity cost ratios are significantly different from unity at an acceptable probability level is noted in the tables. No direct indication is given of the quantitative changes in resource use that would be required in order to drive the marginal product to opportunity cost ratios to unity. This information is obtainable for most of the studies from the original sources cited. In money terms, a greater change in input quantity may be needed to induce equilibrium in the use of a major resource that is already used fairly efficiently than would be needed to induce equilibrium in a factor that is used very inefficiently but in only small amounts. In other words, while the marginal revenue to opportunity cost ratios indicate where the next dollar might best be added to or subtracted from the input mix, they provide only a tentative guide relative to the next thousand dollars. It should also be noted that in those instances where the sum of the production elasticities is equal to or greater than unity, the specification of the fitted function prevents the attainment of a unique equilibrium position relative to all inputs. But in the real world situation it is certain that an equilibrium position either exists within the range of input levels examined but is masked by errors of specification, or could eventually be attained. In the latter case, barring specification errors, the equilibrium position could be approximated by a function fitted to data for input levels beyond those examined in the listed studies to which this comment applies.

With the above preliminaries in mind, we may now consider the

estimated functions on a country by country basis. In large part, the comments on each study are but a précis of those to be found in the original references listed for each country.

UNITED STATES

Relative to other countries, there have been many cross-sectional production function studies of United States farm-firms. We have chosen two of these studies for present purposes, statistics for both being listed in Table 17.1. The first study was designed as a comparative analysis of four areas of United States agriculture.[7] These areas were (a) the Piedmont region of Alabama, which is typical of a sharecropper farm organization built around cotton and a scattering of other crops and livestock products; (b) southern Iowa, which is somewhat typical of the less productive portion of the United States Corn Belt where capital input per farm and soil productivity are lower than in the central Corn Belt; (c) northern Iowa, which is typical of the most productive parts of the Corn Belt, where capital availability is about as great as in any sector of United States agriculture; and (d) a dryland wheat area of Montana, which also includes a considerable amount of cattle ranching under grazing conditions. Between them, these four regions typify a major part of United States agriculture. The actual data analyzed was obtained from a random sample of commercial farms in each area. Over these samples, capital investment per farm was almost 50 per cent greater in Montana than in northern Iowa, which in turn was nearly double total investment per farm in southern Iowa; the latter having nearly eight times as much capital per farm as the Alabama group. Since crop and livestock production are distinct activities on the farms within each area, separate production functions were derived for the crop and livestock activities. Hence, the four-state study involved the estimation of eight production functions, all of which are listed in Table 17.1.

The second United States study presented relates to various farm tenure classes in the Iowa-Illinois area of the Corn Belt. As shown in Table 17.1, separate production functions were estimated for stratified random samples of farm operators who were full owners, livestock-share renters, and crop-share cash-renters.[8] To reduce the confounding influence of age, only owners under 45 years of age were sampled;

[7] Heady, E. O. and Shaw, R. Resource returns and productivity coefficients in selected farming areas. Journal of Farm Economics, 36: 243-57. 1954; and, Resource returns and productivity coefficients in selected farming areas of Iowa, Montana and Alabama. Iowa Agr. Exp. Sta. Res. Bul. 425. Ames. 1955.

[8] Miller, W. G. Comparative efficiency of farm tenure classes in the combination of resources. Agricultural Economics Research, 11: 6-16. 1959; Miller, W. G., *et al.* Relative efficiencies of farm tenure classes in intrafirm resource allocation. Iowa Agr. Exp. Sta. Res. Bul. 461. Ames. 1958. The reader is referred to these sources for exemplification of the various statistical tests pertinent to any detailed analysis of resource use disequilibria based on production function estimates.

they are more comparable to tenants who are generally under 45 years. The functions fitted for these three tenure samples relate to the whole farm operation made up in each case of crop and livestock activities.

Since the stratification of the farms in the tenure study cuts across that of the four-state study, we will first consider the two sets of functions separately.

Crop and Livestock Studies

Between the four regions, the crop functions — as specified in Table 17.1 by their resource elasticity coefficients — are rather dissimilar. Still, as would be expected, the least difference occurs between the southern and northern Iowa functions. Between these areas, climatic and organizational features of the crop enterprise are not as distinctive as they are for the other regions. Over the four samples, the influence of the differences in crop resource elasticities is reflected in predictable fashion in the resource marginal products. Marginal productivity of crop land is ordered as would be expected in terms of soil type, rainfall, and climatic conditions. It is highest in northern Iowa ($46.70 per acre), followed by southern Iowa ($32.32 per acre), Alabama ($19.11 per acre), and Montana ($10.60 per acre). Marginal labor productivity on crops displays a differential to be expected from the capital to labor ratios and resource quantities found on the sample farms. It is also in line with the relative differences known by agricultural workers to exist in the areas. Northern Iowa and Montana have the highest labor marginal value products, $67.07 and $58.17 per month, respectively. Small farms and a smaller quantity of capital per labor unit cause the marginal productivity of labor to be lowered in Alabama and southern Iowa, where marginal labor returns are $39.62 and $43.40 per month, respectively. The largest marginal productivity of crop capital services was found in Montana and southern Iowa. The relatively low returns in northern Iowa are perhaps best explained by the machine component of the capital services. These farms are as highly mechanized as any group in the United States. On the average, added machine investment would likely add less to production on these farms than the annual cost of the machine services.

Comparison of the livestock production functions indicates that they are all rather similar, especially those for Montana, southern and northern Iowa. In every case, capital services are by far the most influential input, with labor playing only a minor — but statistically significant — role. This similarity is explained by the fact that the production of meat from livestock is influenced far less by the climatic environment of the production process than is crop production. However, due to the effects of price differences and variations in the scale of production, resource marginal products in livestock production do show some marked differences between regions. The southern and northern Iowa samples had marginal products of livestock labor

Table 17.1. United States: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for										
	Alabama		Southern Iowa		Northern Iowa		Montana		Iowa and Illinois		
	crops	livestock	crops	livestock	crops	livestock	crops	livestock	full owners	livestock share renters	crop share cash renters
Number of farms	131	131	143	143	142	142	151	151	51	78	75
Year	1950	1950	1950	1950	1950	1950	1950	1950	1954	1954	1954
Production elasticities:											
Land	0.39*		0.79*		0.91*		0.50*		0.09	0.23*	0.29*
Labor	0.32*	0.23*	0.09*	0.12*	0.08	0.08*	0.04	0.08	0.17	0.18	0.25*
Capital services	0.46*	0.74*	0.39*	0.98*	0.16*	0.91*	0.58*	0.94*	0.73*	0.53*	0.48*
Sum of elasticities	1.17†	0.97	1.27†	1.10†	1.15†	0.99	1.12	1.02	0.99	0.95	1.02
$\bar{R}^2$	0.71	0.83	0.82	0.90	0.83	0.90	0.75	0.92	0.74	0.66	0.71
Sample means:‡											
Output ($)	1,042.00	795.00	3,950.00	7,136.00	7,890.00	10,840.00	16,325.00	5,516.00	17,714.00	22,936.00	15,105.00
Land (acres)	21.00		97.60		153.80		774.60		143.00	180.00	184.00
Labor (months)	8.40	2.70	8.00	6.60	8.90	6.90	11.10	6.00	22.70	19.20	19.00
Capital services ($)	453.00	590.00	1,216.00	5,862.00	2,048.00	10,030.00	4,450.00	4,239.00	8,794.00	9,566.00	6,517.00
All inputs ($)	1,418.00	868.00	3,788.00	7,188.00	6,118.00	11,380.00	9,999.00	5,660.00	14,087.00	15,399.00	12,047.00
Average products:‡											
Land ($ per acre)	49.60		40.50		51.30		21.10		123.90	127.40	82.10
Labor ($ per month)	124.00	294.40	493.70	1,081.20	886.50	1,571.00	1,470.70	919.30	780.30	1,194.50	795.00
Capital services ($ per $)	2.30	1.35	3.25	1.22	3.85	1.08	3.67	1.30	2.01	2.40	2.32
Marginal products:‡											
Land ($ per acre)	19.11		32.32		46.70		10.60		11.29	29.53	24.14
Labor ($ per month)	39.62	68.75	43.40	125.06	67.07	120.52	58.17	76.50	134.00	219.16	195.92
Capital services ($ per $)	1.07	1.00	1.28	1.20	0.63	1.03	2.13	1.21	1.48	1.28	1.11
Opportunity costs:											
Land ($ per acre)	4.76		9.89		15.14		3.77		11.52	15.30	13.56
Labor ($ per month)	103.00	103.00	200.85	200.85	195.70	195.70	236.90	236.90	160.00	160.00	160.00
Capital ($ per $)	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.06	1.10	1.10	1.10
Marginal return to opportunity cost ratios:‡											
Land	4.01§		3.27§		3.08§		2.81§		0.98	1.93§	1.78§
Labor	0.38§	0.67§	0.22§	0.62§	0.34§	0.62§	0.25§	0.32§	0.84	1.36	1.23
Capital services	1.01	0.94	1.21§	1.13§	0.59§	0.97	2.00§	1.14§	1.35§	1.16	1.01

*Significantly different from zero at a probability level of ⩽ 5 per cent.
†Significantly different from unity at a probability level of ⩽ 5 per cent.
‡Estimated at the geometric mean input levels.
§Significantly different from unity at a probability level of ⩽ 10 per cent.

($125.06 and $120.52 per month, respectively) that were nearly double those found in Montana ($76.50) and Alabama ($68.75). The low marginal product for livestock labor in Alabama is to be expected because the capital to labor ratio is low, and because of the techniques used and of the forms of capital employed in livestock production. In contrast to the marginal products of livestock labor, differences between the regions were not so great for the marginal products of livestock capital services. They ranged only from about $1.20 in southern Iowa and Montana to $1.00 in northern Iowa and Alabama.

Comparing the crop and livestock functions, the marginal product of labor was higher for livestock than for crops in all areas. In practically all cases, the difference was significant at the 5 per cent probability level. Marginal returns on livestock capital were higher than for crop capital in northern Iowa while they were about equal in southern Iowa and Alabama. Marginal capital productivity was higher for crops than for livestock in Montana. Although production elasticities for both crops and livestock in Montana are affected by the highly variable weather of the area, crop returns were likely favored more by good weather in 1950. Most Montana farmers have beef cow herds of rather fixed size and can make only partial use of above average pasture yields from favorable weather.

From the marginal return to opportunity cost ratios derived in the four-state study, as listed in Table 17.1, the existence of severe resource use disequilibria is apparent. Especially is this true for land and labor. In every region, the average farmer could have increased profits substantially by expanding the area cropped and by reducing the labor input on both crops and livestock. As well, it would have been more efficient to use additional capital services for both livestock and crops in Montana and southern Iowa, in contrast to northern Iowa where slightly excessive capital services were being used in both enterprises. These conclusions are supported by the fact that the resource use efficiency ratios of land and labor for both crops and livestock are significantly different from unity at the 10 per cent probability level in each sample, the same being true for capital services in southern Iowa and Montana. For Alabama, the use of capital services under current production conditions appears to be satisfactory for crops and nearly so for livestock. The general policy implications to be drawn from the crop-livestock study are these: (a) there is too much labor in much of United States agriculture; (b) banks and other credit agencies should recognize that, on the average, farmers would be acting in very logical fashion if they endeavored to increase the size of their farms; (c) the flow of capital should be directed towards Montana and southern Iowa rather than towards northern Iowa and Alabama, with the proviso that capital directed to Alabama for the development of new techniques may be very worthwhile. While Alabama is low in capital productivity it is known, a priori, that returns on this resource could actually be quite large if it were invested in educational resources for farmers; more capital would also allow livestock production on a commerical basis rather than as a semihousehold enterprise.

Supplementing these over-all implications of the study, it might be noted that detailed analysis of the data revealed examples of resource use inefficiencies within each sample just as large as those found between the various samples.[9] Still, these intra-regional disequilibria related to only a small proportion of the farms in each sample. They do not vitiate the over-all interregional policy implications drawn from the four-state data.

Farm Tenure Study

Consider now the Iowa-Illinois tenure study. The basic hypothesis was that full owners, livestock-share renters, and crop-share cash-renters belong to different farm-firm populations having distinctive patterns of resource use and levels of efficiency. Over-all, the analysis indicates that while full owners differ markedly from livestock-share and crop-share renters in terms of the production elasticities of resources and the efficiency of resource use, differences between the two renter groups may not be too important. Moreover, in absolute terms, all three tenure groups displayed roughly the same level of inefficiency as measured by deviations between their actual resource use pattern and the allocation that would minimize the cost of producing their actual average level of production.

Examination of the marginal products and marginal return to opportunity cost ratios for the three tenure classes, as listed in Table 17.1, reveals important differences between the groups. Full owners appear to be using too few capital services and slightly excessive quantities of labor and land. However, if the full owners used additional capital services, the marginal products of their land and labor resources would be increased. Capital rationing, arising from prior commitments to land purchases, is probably the main factor limiting the use of capital services by the full owners. For livestock-share renters, all the marginal returns are greater than resource opportunity costs. The use of all three resources might be extended profitably. Disequilibrium is especially evident for land, the marginal product being nearly double resource cost and substantially above the ratios for labor (1.36) and capital services (1.16). These ratios follow the same order for the crop-share cash-renter group, but are slightly lower in each case. The higher marginal product of labor for both renter classes, compared to full owners, is most likely due to the greater quantity of fixed assets used with their labor.

With a marginal return to opportunity cost ratio of 1.01 for capital services, it would seem that the cash renters were using very near the optimal quantity of capital services. This result does not coincide with theoretical expectations: the alleged nonoptimal sharing of costs and

[9] Heady, E. O. and Baker, C. B. Resource adjustments to equate productivities in agriculture. Southern Economic Journal, 21: 36-52. 1954.

returns under the crop-share system should cause a higher marginal product (relative to full owners) for capital services because of these resources being limitational.[10] Perhaps the usually noted imperfections of crop-share cash-rent leases are balanced by such factors as the sharing of uncertainties and the overcoming of capital shortages by the joint contributions of landlords and tenants. It is also possible that the consistently lower marginal return to opportunity cost ratios of the crop-share tenants is related to superior management. The broad conclusion to be drawn from the tenure study is that the types of resource adjustments needed to optimize production vary according to tenure status. While full owners need additional capital rather than additional land or labor, both tenant groups would contribute more to the general welfare, and their own, if they had more land and labor.

Looking at the over-all picture presented by the two United States studies, it is obvious that extensive disequilibria prevail in the use of resources in United States agriculture. The data of Table 17.1 imply that under the price conditions prevailing, United States farmers should in general (a) increase the size of their farms, and (b) substitute capital services for labor. At the same time, quite large differences exist both between geographic areas and tenure groups, and even between crop and livestock enterprises in the same area, in the extent to which these changes should be made. Moreover, in terms of capital investment, the average farm to which the United States studies refer is quite large relative to most farms in other countries. In "all input" terms, only the Alabama farms (which are of insignificant importance relative to the whole of United States agriculture) used less than $10,000 worth of resources per year. It might be expected, therefore, that the quantities of United States resources that could be transferred fruitfully within agriculture, and between agriculture and industry, would be substantial relative to the shifts that might be required to ensure equilibrium under the smaller and less capital-intensive forms of farm organization found in other countries. Of course, this would not be true if the patterns of farm organization in these other countries were to be changed via the introduction of new larger scale production methods of high capital intensity. The resource transfers required in many of these countries would then be far in excess of those needed to bring resource use in United States agriculture to a reasonable level of efficiency.

CANADA

Table 17.2 presents production functions and their related statistics for samples of mixed livestock-crop farms, cattle ranches, and cattle feedlots, respectively, in the Canadian province of Alberta.[11] In terms

[10] Heady, E. O. Economics of agricultural production and resource use. Prentice-Hall, Inc., New York. 1952. Pp. 587-602.

[11] Darcovich, W. The use of production functions in the study of resource productivity in some beef producing areas of Alberta. Economic Annalist, 28: 85-93. 1958.

Table 17.2. Canada: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics for Selected Production Function Studies

Item	Production Function for Alberta		
	Mixed farms	Cattle ranches	Cattle feedlots
Number of farms	72	36	72
Year	1953-54	1954-55	1953
Production elasticities:			
Land	0.39*	0.20*	
Labor	0.20	0.37*	
Capital services	0.34*	0.39*	0.13*
Livestock inventory			0.84*
Sum of elasticities†	0.93	0.97	0.97
$\overline{R}^2$	0.69	0.81	0.98
Sample means:‡			
Output ($)	12,556.00	22,243.00	17,930.00
Land (acres)	522.00	5,706.00	
Labor (months)	13.8	35.6	
Capital services ($)	4,042.00	5,982.00	3,943.00
Livestock inventory ($)			11,998.00
All inputs ($)	7,999.00	16,496.00	15,941.00
Average products:‡			
Land ($ per acre)	24.00	3.90	
Labor ($ per month)	909.80	624.80	
Capital services ($ per $)	3.10	3.72	4.55
Livestock inventory ($ per $)			1.49
Marginal products:‡			
Land ($ per acre)	9.47	0.81	
Labor ($ per month)	179.08	233.30	
Capital services ($ per $)	1.07	1.46	0.63
Livestock inventory ($ per $)			1.27
Opportunity costs:			
Land ($ per acre)	3.67	0.92	
Labor ($ per month)	147.90	147.90	
Capital ($ per $)	1.06	1.06	1.06
Marginal return to opportunity cost ratios:‡			
Land	2.58§	0.88	
Labor	1.21	1.58§	
Capital services	1.01	1.38	0.59
Livestock inventory			1.20

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at probability level of 5 per cent.
‡Estimated at the arithmetic mean input levels.
§Significantly different from unity at a probability level of ≤ 10 per cent.

of scale of operation and general technological features, these Canadian farms are broadly comparable to the United States farms considered in Table 17.1.

While the cattle ranch and mixed farm functions refer to the overall farm enterprise, the cattle feedlot analysis relates only to the feedlot operation on the farms studied. As noted in Table 17.2, the marginal return to opportunity cost ratios were tested statistically. These tests showed only the ratios for land in mixed farms (2.58) and labor on cattle ranches (1.58) to be different from unity at the 10 per cent probability level. Nonetheless, the listed values for the other inputs still indicate their most likely level of efficiency as currently used. The severe lack of equilibrium in land use on the mixed farms is probably due to relative overcapitalization in machinery. Moreover, none of the marginal return to opportunity cost ratios for the mixed farm sample are less than one. In consequence, no scope exists for attaining equilibrium in resource use by substitution between resources. The general implication is thus that these mixed farms could expand their operations profitably by adding land, labor, and a little capital, in that order. The additional capital would, of course, be necessitated by the effect of additional land and labor in raising the marginal product of capital. In so far as the mixed farm sample was a superior group of farms with high livestock intensity, it is likely that the same disequilibria exist in more severe form on other Alberta mixed farms. Indeed, on these more average farms it is likely that the shortage of capital severely limits the use of all inputs.

For the cattle ranch sample, the marginal return to opportunity cost ratios indicate too much land and too few labor and capital services were being used. Equilibrium would be attained most quickly by substituting labor and capital for land, or if the land input is to remain fixed, by simply using additional quantities of labor and capital services. So far as disequilibria between the mixed farms and cattle ranches is concerned, the only opposing tendencies appear in the land resources: mixed farms were using too little and ranches too much. However, due to the fixity of the land resources, land transfers between the two types of farming are unfeasible. While both types of farm could use more labor and capital, priority in these resources should go to the cattle ranches.

Since the third Canadian function — for cattle feedlots — is on an enterprise basis, the classification of inputs as capital services and livestock inventory differs from the traditional tripartite one used for the mixed farms and cattle ranches. The capital services input is total feed input, while the livestock inventory is the input of feeder cattle at the start of stall feeding. Unlike the United States livestock studies presented in Table 17.1, labor input was not included in this feedlot study. Nonetheless, the fitted function implies, via its marginal return to opportunity cost ratios, that the average feedlot operation was rather inefficient under the price conditions examined; too much feed and too few feeder cattle being used. Since much of the feed is home produced,

the feed input requires little cash outlay and is probably used extravagantly. In contrast, the purchase of feeder cattle requires a large cost outlay. Farmers tend to buy them sparingly. However, given the uncertainty that surrounds fat cattle prices under current marketing procedures, farmers may be behaving quite rationally in overfeeding a small number of cattle. The policy implication to be drawn is that the development of institutions to extend longer-term or more flexible credit would probably be worthwhile. Still, on average, the production function analysis suggests that, even under current credit arrangements, increased profits would be possible if the same quantity of feed were spread over a greater number of animals.

AUSTRALIA AND NEW ZEALAND

For convenience, the only available function for New Zealand has been grouped in Table 17.3 with those estimates available for Australia. Geographically and in terms of farm organization and production, the two countries are rather close, although their climates are somewhat different. The listed production function estimates relate to wool and fat lamb production from sheep, and to dairying. In both New Zealand and Australia, these lines of production are of major agricultural importance. Both are conducted as grazing enterprises with a small measure of supplementary feeding.

The New Zealand study refers to a group of properties producing fat lambs and wool on improved pasture in the light soil area of the south island's Canterbury Plain.[12] For Australia, only a single study is available of wool and fat lamb production.[13] It is based on a random sample of grazing properties in the southwest slope of New South Wales, an area which has been largely converted from natural to improved pasture in recent years. The Australian dairy production function studies listed in Table 17.3 are from various sources. That for New South Wales was conducted conjointly with the sheep study already mentioned. It refers to a random sample of coastal farms producing milk for butter production. The South Australian analysis also refers to a random sample of dairy farms producing for butter production.[14] These farms were located on irrigated land along the Lower Murray Valley. Unlike the other dairying studies, that for Western Australia is based on a random sample of farms producing whole-milk for human consumption.[15] The sample was located in the Perth-Bunbury region.

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[12] Mason, G. Resource productivities from a sample of light plains farms, Canterbury, N.Z. Unpublished M. Agr. Sc. thesis. Canterbury Agricultural College Library, Canterbury, New Zealand. 1958.

[13] Dillon, J. L. Marginal productivities of resources in two farming areas of N.S.W. Economic Society of Australia and N.Z. Economic Monograph 188. Sydney. May, 1956.

[14] Jarrett, F. G. Estimation of resource productivities as illustrated by a survey of the Lower Murray Valley dairying area. Australian Journal of Statistics, 1: 3-11. 1959.

[15] Schapper, H. P. and Mauldon, R. G. A production function from farms in the whole-milk region of Western Australia. Economic Record, 33: 52-59. 1957.

Table 17.3. Australia and New Zealand: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for				
	Sheep		Dairying		
	Canterbury Plain, New Zealand	New South Wales	New South Wales	Western Australia	South Australia
Number of farms	50	52	99	51	48
Year	1955-56	1953-54	1952-53	1954-55	1955-56
Production elasticities:					
Land	0.42*	0.10	0.28*		0.39*
Labor	0.15	0.59*	0.22*	0.23*	0.25*
Capital services		0.55*	0.42*	0.55*	0.32*
Capital assets	0.26*				
Supplementary feed				0.13*	0.18*
Agistment					0.04*
Superphosphate	0.22*			0.07*	
Lime	0.06				
Sum of elasticities†	1.11	1.24	0.92	0.99	1.19
$\bar{R}^2$	0.81	0.76	0.64	0.87	0.62
Sample means:‡					
Output ($)	14,428.00	13,234.00	5,152.00	9,330.00	8,029.00
Land (acres)	698.00	1,516.00	191.00		49,50
Labor (months)	18.10	23.10	21.70	20.20	22.70
Capital services ($)		3,264.00	1,572.00	1,939.00	1,291.00
Capital assets ($)	3,809.00				
Supplementary feed ($)				1,739.00	1,111.00
Agistment ($)					47.80
Superphosphate ($)	838.00			406.00	
Lime ($)	704.00				
All inputs ($)	7,709.00	9,765.00	4,902.00	6,876.00§	8,343.00
Average products:‡					
Land ($ per acre)	20.70	8.70	27.00		162.20
Labor ($ per month)	797.10	572.90	237.40	862.90	353.70
Capital services ($ per $)		4.05	3.28	4.81	6.22
Capital assets ($ per $)	3.79				
Supplementary feed ($ per $)				5.36	7.23
Agistment ($ per $)					168.00
Superphosphate ($ per $)	17.20			23.00	
Lime ($ per $)	20.50				
Marginal products:‡					
Land ($ per acre)	8.59	0.87	6.90		79.83
Labor ($ per month)	122.52	292.70	51.07	107.97	108.66
Capital services ($ per $)		2.07	1.47	2.65	1.34
Capital assets ($ per $)	0.70				
Supplementary feed ($ per $)				0.71	1.80
Agistment ($ per $)					5.20
Superphosphate ($ per $)	4.00			1.56	
Lime ($ per $)	1.16				
Opportunity costs:					
Land ($ per acre)	3.46	2.24	5.60		37.70
Labor ($ per month)	144.14	134.40	104.16	119.39	177.41
Capital ($ per $)	1.05	1.05	1.05	1.06	1.06
Capital assets ($ per $)	0.30				
Marginal return to opportunity cost ratios:‡					
Land	2.48	0.39	1.23		2.12
Labor	0.85	2.18	0.49	0.90	0.61
Capital services		1.97	1.40	2.50	1.26
Capital assets	2.33				
Supplementary feed				0.67	1.70
Agistment					4.91

Table 17.3. (Cont'd)

Item	Production Function for				
	Sheep		Dairying		
	Canterbury Plain, New Zealand	New South Wales	New South Wales	Western Australia	South Australia
Marginal return to opportunity cost ratios:‡					
Superphosphate	3.81			1.47	
Lime	1.10				

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from zero at a probability level of 5 per cent with the exception of the South Australian study for which no significance test is available.
‡Estimated at the geometric mean input levels with the exception of the sample means and average product of the South Australian sample which refer to the arithmetic means.
§Including an allowance of $380 for land services.

As perusal of Table 17.3 indicates, considerable variation exists between these various analyses in the way inputs have been categorized. Labor is the only resource category common to all five studies. While the New South Wales sheep and dairy studies use the broad "land, labor, and capital services" approach, the other analyses generally consider fertilizer and purchased feed inputs separately. Land was excluded from the Western Australian study because it was found to be highly correlated with capital services. The differences between the sample means of output and all inputs across Table 17.3 are noteworthy. They are explained by the fact that in both Australia and New Zealand sheep production has been more profitable than dairying in recent years and that, within dairying, the rationing of whole-milk production has made it far more rewarding than dairying for butter production. Indeed, in Australia, dairying for butter production is a rather depressed industry, as is borne out by the marginal return to opportunity cost data of Table 17.3.

The marginal return to opportunity cost ratios for sheep show considerable differences between the New Zealand and Australian samples. While both were using insufficient capital, the New Zealand farms apparently had too little land and too much labor. The reverse situation is indicated for the New South Wales sheep properties. In consequence, while the New Zealand sheep farms should use more capital and less labor, those in New South Wales should use more labor as well as capital. Such adjustments would in both cases tend to remove some of the disequilibria in land use, especially in the Australian sample, granted that the farmers' preferences or opportunities forestalled the purchase (in New Zealand) or sale (in Australia) of land.

The data of Table 17.3 also gives evidence of quite large differences in resource use efficiency within Australian dairying and between dairying and sheep production. Each of the dairy samples tends to indicate the same pattern of inefficiency: too few capital and land services and an excessive use of labor. Particularly outstanding are the deficiencies of land (and agistment which substitutes for land) in the South Australian sample, and of capital services in the Western Australian whole-milk

study. The greatest overuse of labor appears in the New South Wales dairy sample. This result is not surprising. The area to which the sample refers is patently depressed, having a surfeit of farm population with few nonfarm employment opportunities. Comparing the marginal return to opportunity cost ratios between the Australian sheep and dairying samples, it is apparent that under the price conditions assumed in these studies, policies oriented (a) to the transfer of labor out of dairying into sheep production and (b) to the increased use of capital services in both types of farming would be advantageous. Too, it should be noted that the prices paid for dairy production in Australia have been kept artificially high by government subsidy over the period examined. These subsidies mask the degree of inefficiency that would prevail under free market conditions. With unsubsidized prices for dairy products, the marginal return to opportunity cost ratios would be reduced, thereby further emphasizing the excessive quantity of labor used on Australian dairy farms and, at least initially, tending to restore equilibrium in the use of land and capital services.

UNION OF SOUTH AFRICA

The production function analyses presented in Table 17.4 for the Union of South Africa relate to a one in four sample of cattle ranches in the Eastern Kalahari region.[16] This area includes parts of the Kuruman, Vryburg, and Mafeking districts. The limited and variable rainfall in combination with the sandy soils of the region presents great hazards for cattle ranching, in terms of both grass yields and water supplies. Moreover, grazing is the only major economic activity that is feasible in the region. Data for the analysis were gathered in two surveys covering the 1950-51 and 1951-52 financial years. Only ranchers who had been established for at least two years were included in the samples. Both survey years being rather similar to the recorded average of production years in the region, it is thought that the derived functions may be taken as representative. Still, at the prevailing prices, resource productivities could be considerably higher in periods of above average rainfall and would be much lower in droughty years.

In deriving the production functions, the data were stratified by the two years in which samples were taken, and also by the two systems of ranch organization in the region. To wit: cow-calf herds and growing-fattening herds. As well, a function was fitted to the pooled data for both years and is also presented in Table 17.4. For all five fitted functions, the production elasticities are rather similar; none of the differences for the same resource between the various functions is statistically significant. Hence it is to be expected that ranches with the same

[16] Heady, E. O. and Du Toit, S. Marginal resource productivity for agriculture in selected areas of South Africa and the United States. Journal of Political Economy, 62: 494-505. 1954; Also Du Toit, S. Analysis of cattle ranching in the Eastern Kalahari, Union of South Africa. Unpublished M. Sc. thesis. Iowa State University Library. Ames. 1953.

Table 17.4. Union of South Africa: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for Eastern Kalahari				
	Ranches		Cow - Calf Ranches	Fattening Ranches	Pooled Data
Number of farms	80	82	70	92	162
Year	1950-51	1951-52	1950-52	1950-52	1950-52
Production elasticities:					
Land	0.21*	0.24*	0.19*	0.28*	0.23*
Labor	0.18*	0.13	0.19*	0.13*	0.14*
Capital services	0.54*	0.59*	0.52*	0.55*	0.57*
Sum of elasticities†	0.93	0.96	0.90	0.96	0.94
$\bar{R}^2$	0.86	0.76	0.64	0.85	0.80
Sample means:‡					
Output ($)	9,344.00	8,211.00	6,643.00	10,704.00	8,711.00
Land (acres)	6,576.00	6,790.00	5,891.00	7,358.00	6,684.00
Labor (months)	58.00	63.00	54.00	65.00	60.00
Capital services ($)	2,910.00	2,776.00	1,942.00	3,761.00	2,827.00
All inputs ($)	4,950.00	4,918.00	3,792.00	6,044.00	4,912.00
Average products:‡					
Land ($ per acre)	1.42	1.21	1.13	1.45	1.30
Labor ($ per month)	161.10	130.30	123.00	164.70	145.20
Capital services ($ per $)	3.21	2.96	3.42	2.85	3.08
Marginal products:‡					
Land ($ per acre)	0.30	0.29	0.22	0.41	0.30
Labor ($ per month)	29.01	16.38	23.78	20.99	21.11
Capital services ($ per $)	1.72	1.76	1.77	1.57	1.76
Opportunity costs:					
Land ($ per acre)	0.21	0.21	0.21	0.21	0.21
Labor ($ per month)	11.36	11.36	11.36	11.36	11.36
Capital ($ per $)	1.06	1.06	1.06	1.06	1.06
Marginal return to opportunity cost ratios:					
Land	1.43	1.38	1.05	1.95	1.43
Labor	2.55	1.44	2.09	1.85	1.86
Capital services	1.62	1.66	1.67	1.48	1.66

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at a probability level of 5 per cent.
‡Estimated at the geometric mean input levels.

quantity of resources would have fairly similar marginal resource productivities under each of the fitted functions. However, since the two types of ranch organization — cow-calf and fattening — utilize different amounts of resources, the marginal products vary from one to the other. Discussion is therefore mainly confined to the production functions for these two types of ranch organization and to the "pooled data"

function, the latter analysis being thought to give the best guide to the over-all situation in the region.

As evidenced by the marginal return to opportunity cost ratios, the average sample farms exhibit extreme inefficiencies in resource use. In the pooled data analysis, labor productivity exceeds the wage rate by 86 per cent; capital productivity exceeds its market price by 66 per cent; and land productivity is in excess of cost by 43 per cent. The latter effect is mainly due to the situation on fattening ranches where land productivity is in excess of cost by 95 per cent, the excess being only 5 per cent on cow-calf ranches. The obvious implications are that more labor and capital services should be used on both the fattening and cow-calf ranches. The fattening ranches should also use more land — despite the fact that the average ranch already employs far more land and labor, in quantitative terms although not in value terms, than is used on the average farm in any of the other national studies that we present. Moreover, since the resource use efficiency ratio exceeds unity for every resource in each fitted function, no opportunities exist for approaching optimal efficiency through the substitution of one resource for another. For both the breeding and the fattening farms, the needed reorganization is one of scale. However, these changes in scale may not need to be as large as the data of Table 17.4 imply. The reason is that management was not included as a separate input in the fitted functions due to the lack of a quantitative measure. In fact, management is quite a distinct input on these ranches, being supplied by the rancher himself who makes no manual contribution at all. All the actual labor involved is contributed by a hired work force. Still, the management input is probably partly reflected in the elasticities of the other inputs so that their true elasticities may be somewhat less than those estimated. In consequence, the marginal return to opportunity cost ratios of Table 17.4 probably overestimate the extent of resource use disequilibria. Nonetheless, the exclusion of management is probably not a severe limitation on the analysis, despite the fact that because of the nature of the organization of each ranch its exclusion is of more importance than would otherwise be the case.

The more important factors contributing to the inefficient use of resources are probably the ranchers' lack of knowledge of marginal productivities and the influence of yield, price, and technological and sociological uncertainty. The latter influences are especially important in the area. Historically, major uncertainty has surrounded prices and yields. Because of the low and variable rainfall with its resultant effects on pasture production and livestock water supplies, yield is highly variable and unpredictable from year to year. Technological uncertainty is also high, the techniques of beef production still being in a stage of transition. Indeed, new techniques may have been more successful than most ranchers anticipated, thus contributing to sub-optimal levels of investment. Moreover, uncertainty of a sociological, psychological, and political nature also surrounds the actions of the labor force which generally consists of a fairly large number of African

workers under occidental management. In the face of all these uncertainties, ranchers are reluctant to add more resources by borrowing, despite the comparative cheapness of land and labor. Moreover, good and poor climatic years do not average out because livestock watering facilities limit carrying capacity and the rancher usually keeps cattle numbers down in anticipation of the unfavorable seasons.

UNITED KINGDOM

Three production function estimates, each relating to a sample of United Kingdom dairy farms, are presented in Table 17.5. All are based on material drawn from the continuing Farm Management Survey conducted in England. The first estimate refers to England as a whole.[17] It is based on a sample of 244 farms drawn not at random but arranged purposively to give observations exhibiting a minimum of correlation between input categories. These inputs were labor, purchased feed, and capital services with land services measured as rental value included in the capital services category. As a side effect of the purposive sampling procedure, the function relates only to dairy farms with inventory values of crops, livestock, machinery, and equipment between $2,800 and $11,200. Still, this range covers the largest part of English dairying. Each of the sample farms was located in an officially classified dairy district and all obtained at least half of their gross receipts from milk.

The other two functions listed in Table 17.5 relate only to Bristol Province low-lands farms of between 50 and 200 acres which received at least 75 per cent of their gross income from dairying.[18] The analysis was essentially aimed at assessing the usefulness of production function estimates for individual farm recommendations. And so, in recognition of the short-run fixity of the land resource on British farms, land was eliminated as a variable by considering the other inputs — cows, other large livestock, and capital services — on a per 100 acre basis. On the assumption that resource productivities differed to an important extent between low and high yield farms, separate functions were fitted to these two groups in the over-all farm sample. The average yield classifications used were from 552 to 702 gallons per cow for the low production group, and from 744 to 894 gallons per cow for the high production group.

Both within and between the three United Kingdom samples, quite important differences in the efficiency of resource use are evidenced by the marginal return to opportunity cost ratios of Table 17.5. For the all-England sample, the average picture is a fairly satisfactory one, with some additional capital services and a modicum of labor and

[17] Antill, A. G. Towards a production function for dairy farms. The Farm Economist, 8 (1): 1-11. 1955.

[18] Wragg, S. R. and Godsell, T. E. Production functions for dairy farming and their application. The Farm Economist, 8 (5): 1-6. 1956.

purchased feed being required for maximum efficiency. In contrast, the Bristol Province farms appear to be organized very inefficiently. For both the low and high yield groups, the marginal return to opportunity cost ratios for cows of 3.71 and 3.41, respectively, are very high. Far too few cows were being milked. At the same time, the Bristol samples were running too many large nondairy livestock (especially the low yield farms) and using an excessive quantity of capital services (especially the high yield group). The latter result is in line with the impressions of Bristol Province farm management workers that farms with high quality dairy herds exhibit a degree of conspicuous consumption in their proliferation of capital services, over and above the feed and labor input included in these services. It is not surprising, therefore, that the production elasticity of capital services varies inversely with milk yield. While a 1 per cent increase in capital services would raise production by only 0.05 per cent on high yield farms, it would induce an increase of 0.53 per cent on low yield farms. On the other hand, the production elasticities of cows and other livestock are much the same on both high and low yield farms.

Comparison of the Bristol Province and all-England results points up the fact that, subset by subset, quite large differences may exist within a parent set of farms producing the same product. Bristol Province being part of England, it is to be expected that the disequilibria exhibited in the Bristol sample may be balanced by converse inefficiencies in other sectors of English dairying, leading to the picture of apparently satisfactory resource use found in the all-England sample. This fact emphasizes the desirability of sample homogeneity, even if the resultant estimates are to be used for the broadest of policy decisions.

AUSTRIA

Only two cross-sectional production function estimates are available for Austrian agriculture.[19] Both are listed in Table 17.6. Each of these functions is of a rather macro nature, although the data on which they are based is built up from individual records collected for 784 farms by the Austrian Department of Agriculture. These farms were scattered throughout Austria so that the functions refer to Austrian agriculture as a whole. The first function is based on a stratification of the 784 farms according to their location within the eight geographic types of regions delineated for statistical purposes by the Austrian authorities. The outputs and inputs of record-keeping farms within each of these regions were aggregated and a production function fitted to the resultant eight observations. For the other function, the 784 farms were grouped into 37 strata on the basis of their type of production, the

[19] Tintner, G. Produktionsfunktionen für die österreichische Landwirtschaft. Zeitschrift für Nationalökonomie. 17: 426-42. 1958.

Table 17.5. United Kingdom: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics for Selected Production Function Studies

Item	Production Function for Dairying		
	England	Bristol Province	
		low yield farms	high yield farms
Number of farms	244	24	20
Year	1951-52	1953	1953
Production elasticities:			
Labor	0.29*		
Capital services	0.50*	0.53*	0.05
Cows		0.55*	0.54*
Other livestock		0.00	0.05
Purchased feed	0.33*		
Sum of elasticities	1.12†	1.08	0.64
$\bar{R}^2$	0.84	0.64	0.67
Sample means:‡			
Output ($)	7,582.00	6,815.00	9,646.00
Labor (months)	32.40		
Capital services ($)	2,774.00	5,172.00	6,345.00
Cows (cow equivalents)		24.19	27.50
Other livestock (cow equivalents)		11.58	13.48
Purchased feed ($)	2,164.00		
All inputs ($)	7,012.00	6,674.00	8,640.00
Average products:‡			
Labor ($ per month)	234.00		
Capital services ($ per $)	2.73	1.32	1.52
Cows ($ per cow equivalent)		281.70	350.80
Other livestock ($ per cow equivalent)		588.50	715.60
Purchased feed ($ per $)	3.50		
Marginal products:‡			
Labor ($ per month)	67.59		
Capital services ($ per $)	1.37	0.71	0.08
Cows ($ per cow equivalent)		156.04	190.93
Other livestock ($ per cow equivalent)		0.48	36.74
Purchased feed ($ per $)	1.14		
Opportunity costs:			
Labor ($ per month)	64.02		
Capital ($ per $)	1.05	1.05	1.05
Livestock ($ per cow equivalent)		42.00	56.00
Marginal return to opportunity cost ratios:‡			
Labor	1.06		
Capital services	1.30	0.68	0.08
Cows		3.71	3.41
Other livestock		0.01	0.66
Purchased feed	1.09		

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Significantly different from unity at a probability level of ≤ 5 per cent.
‡Estimated at the geometric mean input levels. For the Bristol province studies, the output and input statistics are on a per 100 acres basis.

Table 17.6. Austria: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Functions for Austrian Agriculture	
Number of observations	8	37
Year	1954-55	1954-55
Production elasticities:		
Land	0.14	0.13*
Labor	0.29*	0.26*
Capital services	0.59*	0.61*
Sum of elasticities†	1.02	1.00
$\bar{R}^2$	0.99	0.99
Marginal products:‡		
Land ($ per acre)	11.87	10.78
Labor ($ per month)	22.53	19.57
Capital services ($ per $)	1.48	1.56
Opportunity costs:		
Land ($ per acre)	11.67	11.67
Labor ($ per month)	36.00	36.00
Capital ($ per $)	1.04	1.04
Marginal return to opportunity cost ratios:‡		
Land	1.02	0.92
Labor	0.63	0.54
Capital services	1.42	1.50

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at a probability level of 5 per cent.
‡Estimated at the geometric mean input levels.

geographic type of region to which they belonged, and the state in which they were located. The outputs and inputs of farms within each of these strata were again considered in aggregate, a production function being fitted to the 37 observations. Because of the aggregative approach used, the fitted functions are of interest solely in terms of their broad policy implications. They are of no significance in relation to resource allocation on specific farms. Indeed, the basic aim of the study was expository with the hope of stimulating econometric research in Austria. Still, the fitted functions are of use for present purposes.

Since both of the functions are based on the same fundamental data, it is gratifying that they exhibit very similar production elasticities. In consequence, the marginal products of resources and the marginal return to opportunity cost ratios are also much the same for both estimates. Too, at the usual probability levels there are no statistically significant differences between corresponding statistics of the two analyses. Before considering the implications of the two studies, it might be noted that the sample means of inputs and output are not listed

in Table 17.6. Those statistics, given the macro nature of the Austrian functions, would be of no value for purposes of international comparisons. Also, the opportunity cost figures quoted for land and labor are strictly average figures. Without doubt, they mask quite marked differences in the price of these resources between the various regions of Austria.

Both functions being essentially the same, their implications in terms of resource use efficiency are similar. From the marginal return to opportunity cost ratios, it appears that Austrian farm land is used in optimal fashion but that too much labor and too few capital services are used. In other words, Austrian farm policy should endeavor to persuade farmers to substitute capital services for labor if farm profits and, within limits, national welfare are to be maximized. Concomitantly, it may be necessary to provide off-farm employment opportunities for the excess labor currently being used in agriculture. It must be emphasized that these conclusions apply in only the broadest "on average" sense. For greater reliability and precision, an array of production function estimates covering the types of farming in the various regions of Austria would be necessary. The form in which the basic data were available prevented such an approach in the present instance.

SWEDEN

Two cross-sectional production function estimates are presented for Sweden. The first, listed with its related statistics in Table 17.7, refers to milk production in southern Sweden.[20] The investigation was based on accounting records of the milk enterprise on a sample of 34 farms with a total of 1,722 cows. All the farms used Swedish low-land dairy cows of fairly uniform quality. As shown in Table 17.7, the estimated production elasticities for labor and capital services are negative, although not significantly different from zero. Only the feed input categories — concentrates, pasture, and roughage — had positive elasticities. In other words, under the average conditions prevailing in the sample, the major determinant of milk output was the input of feed; no significant role being played by the ancillary labor and capital services. It is not to be assumed that the elasticities of labor and capital would have still been negative if they had been used in smaller quantities. Some small amount of both is obviously essential. The difficulty, of course, is that the Cobb-Douglas type function allows only constant elasticities of production. The negative elasticities for labor and capital services presumably reflect the average situation prevailing in the sample. It says nothing of the situation that may prevail at input combinations above or below those found in the sample.

[20] Hjelm, L. Utbytesrelationer i mjolkproduktionen (with an English summary: Input-output relationships in milk production). Institutionem for Lantbrukets Driftsekonomi. Stockholm. 1953.

Table 17.7. Sweden: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Production Function Statistics for Milk Production

Item	Production Function Statistics
Number of farms	34
Year	1951-52
Production elasticities:	
Labor	-0.05
Concentrates	0.28*
Pasture	0.46*
Roughage	0.53*
Capital services	-0.04
Sum of elasticities	1.18†
$\bar{R}^2$	0.88
Sample means:‡	
Concentrates ($ per cow)	60.30
Pasture ($ per cow)	54.00
Roughage ($ per cow)	79.90
Marginal products:‡	
Labor ($ per month per cow)	negative§
Concentrates ($ per $ per cow)	1.30
Pasture ($ per $ per cow)	2.59
Roughage ($ per $ per cow)	2.02
Capital services ($ per $ per cow)	negative§
Opportunity costs:	
Labor ($ per month)	102.29
Capital ($ per $)	1.05
Marginal return to opportunity cost ratios:‡	
Labor	negative§
Concentrates	1.24
Pasture	2.47
Roughage	1.92
Capital services	negative§

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at a probability level of 5 per cent.
‡Estimated at the arithmetic mean input levels.
§Exact figures are not available.

It is apparent from the marginal return to opportunity cost ratios that additional pasture, roughage, and perhaps concentrates could be fed profitably, in that order. To do so, in the actual over-all farm situation, might imply that cow numbers be decreased since production of pasture and roughage may be restricted by the land base of the firm. Whether or not cow numbers should be decreased, however, could only be ascertained directly if cow numbers had been included as a variable. To this extent, the use of a per cow approach has prevented full benefit to be derived from the original data. So far as labor and capital services are concerned, less of these resources should be used. But so long as the function of Table 17.7 is used as guide, labor and capital

Table 17.8a. Production Elasticities and Opportunity Costs of Resources for a Swedish Farm-Firm Production Function

Item	Production Function Statistics
Number of farms	273
Year	1956
Production elasticities:	
Crop land	0.23*
Forest land	0.12*
Labor on crops	0.12*
Labor on livestock	-0.07
Labor on forest	0.00
Labor out of farm	0.00
Machine services	0.11*
Building services	0.05*
Livestock assets	0.18*
Fertilizer	0.18*
Miscellaneous expenses	0.01
Roughage	-0.05*
Grain forage	0.04
Growing timber	-0.06*
Management	0.10*
Sum of elasticities	0.97†
$\bar{R}^2$	0.93
Opportunity costs:	
Crop land ($ per acre)	13.80
Forest land ($ per acre)	9.58
Labor ($ per month)	118.54
Capital ($ per $)	1.05
Livestock assets ($ per $)	0.17
Growing timber ($ per cubic meter)	0.76

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at probability level of 5 per cent.

would always have a negative marginal product. The study, therefore, gives no satisfactory estimate of the degree to which these inputs should be curtailed. To obtain such an estimate it would be necessary to use a new production function covering a sample with labor and capital services at lower levels.

The other production function estimate for Sweden, presented in Table 17.8a, is a preliminary result from a current cross-sectional analysis of record data from 2400 farms.[21] As is typical of Sweden, the majority of the farms are diversified, producing both agricultural and forest products. To give greater reliability, the sample has been stratified in terms of eight passably homogeneous geographic regions. The

[21] This study is being carried out by Eje Sandqvist at the Institute of Agricultural Economics of the Royal Agricultural College of Sweden, Uppsala. We are grateful to Mr. Sandqvist and Professor Hjelm for the opportunity to present the preliminary results listed in the text.

function presented in the table relates to only one of these regions and refers to the over-all farm operation, the various outputs being considered as a single aggregate. Inputs, on the other hand, have been considered in rather disaggregated fashion. In all, there are 15 input categories, including "management." The latter is measured tentatively by the inverse of the farmer's age, on the supposition that education is closely related to management, and the fact that younger farmers generally have a better educational background. The only other exceptional "input" category in this preliminary analysis is "labor out of the farm." This category has a zero production elasticity, as would be expected on the assumption that only excess family labor is transferred out of the farm.

While labor on livestock, roughage, and growing timber have slightly negative production elasticities, only the former two appear anomalous since growing timber might be expected to curtail current production; it is associated with other inputs that might otherwise be used to contribute to current production. Arbitrarily selecting a production elasticity of greater than 0.10 as indicative of a resource that is important, the following resources had an influential role in terms of potential increases in production under prevailing conditions: crop land, livestock assets, fertilizer, forest land, crop labor, and machine services. Within the over-all farm production pattern, other resources — with the possible exception of "management" — appear to be of minor influence although they might be of crucial importance with respect to individual enterprises.

For convenience in presentation, the opportunity costs of resources are also given in Table 17.8a; the sample means, marginal products, and resource use efficiency ratios being presented in Table 17.8b. The latter statistics have been calculated for each group in a fivefold stratification of the original 273 farms, the strata being based on total farm output. As the sample means indicate, there is a consistent increase in the use of all the "input" categories except labor off the farm and management as one moves from smaller to larger farms. Not unexpectedly, labor off the farm decreases as farm size rises. However, since farm output and the various inputs do not change in the same proportions from one group to the next, the marginal product estimates do not exhibit a uniform pattern of change as output increases: those for crop labor, building services, livestock assets, and grain forage rise consistently; for crop land, machine services, and fertilizer the marginal products decline steadily; those for forest land and miscellaneous expenses show no consistent pattern. These various patterns of change with farm size are reflected directly in the marginal return to opportunity cost ratios since the same input prices apply to each size group of farms. Nothing can be said of the marginal products for labor on livestock, roughage, and growing timber except that they are negative. Also, apart from the knowledge that it is positive, no data is available on the marginal product of "management."

For all strata it is evident that it would be profitable to increase the

Table 17.8b. Sweden: Resource Input Means, Marginal Products and Marginal Return to Opportunity Cost Ratios for Various Size Groups of Farms

Item	Size Group				
	I	II	III	IV	V
Sample means:*					
Crop land (acres)	3.44	6.72	10.36	16.19	25.86
Forest land (acres)	3.97	6.07	9.87	9.91	21.73
Labor on crops (months)	6.99	8.70	10.72	14.12	18.27
Labor on livestock (months)	6.47	9.13	10.24	10.74	4.77
Labor on forest (months)	1.36	1.56	2.61	3.05	6.26
Labor out of farm (months)	1.92	0.79	0.68	0.54	0.26
Machine services ($)	415.00	745.00	1,130.00	1,576.00	2,251.00
Building services ($)	284.00	361.00	451.00	582.00	613.00
Livestock assets ($)	1,420.00	2,121.00	2,822.00	3,636.00	4,716.00
Fertilizer ($)	177.00	321.00	504.00	780.00	1,150.00
Operating expenses ($)	114.00	255.00	281.00	462.00	1,017.00
Roughage ($)	729.00	1,075.00	1,583.00	1,778.00	2,448.00
Grain forage ($)	838.00	1,145.00	1,496.00	2,000.00	2,452.00
Growing timber (cubic meters)	990.00	1,515.00	2,635.00	2,548.00	4,779.00
Management index	0.020	0.022	0.022	0.027	0.020
All inputs ($)	5,066.00	7,621.00	10,680.00	13,041.00	18,042.00
Marginal products:*					
Crop land ($ per acre)	34.88	28.21	25.52	22.92	19.77
Forest land ($ per acre)	16.02	16.55	14.18	19.85	12.49
Labor on crops ($ per month)	53.38	67.59	76.95	81.80	86.65
Labor on livestock ($ per month)	negative†	negative	negative	negative	negative
Labor on forest ($ per month)	0.00	0.00	0.00	0.00	0.00
Labor out of farm ($ per month)	0.00	0.00	0.00	0.00	0.00
Machine services ($ per $)	0.80	0.70	0.65	0.65	0.63
Building services ($ per $)	0.62	0.77	0.83	0.93	1.19
Livestock assets ($ per $)	0.40	0.42	0.44	0.48	0.51
Fertilizer ($ per $)	3.23	2.83	2.50	2.26	2.12
Miscellaneous expenses ($ per $)	0.39	0.27	0.35	0.30	0.19
Roughage ($ per $)	negative	negative	negative	negative	negative
Grain forage ($ per $)	0.13	0.15	0.16	0.17	0.19
Growing timber ($ per cubic meter)	negative	negative	negative	negative	negative
Marginal return to opportunity cost ratios:*					
Cropland	2.53	2.04	1.85	1.66	1.43
Forest land	1.67	1.73	1.48	2.07	1.30
Labor on crops	0.45	0.57	0.65	0.69	0.73
Labor on livestock	negative	negative	negative	negative	negative
Labor on forest	0.00	0.00	0.00	0.00	0.00
Labor out of farm	0.00	0.00	0.00	0.00	0.00
Machine services	0.76	0.67	0.62	0.62	0.60
Building services	0.59	0.73	0.79	0.89	1.13
Livestock assets	2.35	2.47	2.59	2.82	3.00
Fertilizer	3.08	2.69	2.38	2.15	2.02
Miscellaneous expenses	0.37	0.26	0.33	0.29	0.18
Roughage	negative	negative	negative	negative	negative
Grain forage	0.12	0.14	0.15	0.16	0.18
Growing timber	negative	negative	negative	negative	negative

*Estimated at the arithmetic mean input levels.
†Exact figures unavailable.

input of crop land, forest land, livestock assets, and fertilizer, and to decrease the input of labor, machine services, miscellaneous expenses, and grain forage. In making such changes, priorities within each resource category should be as follows: crop land and fertilizer increases on smaller farms, livestock asset increases on larger farms, decreases in machine services and miscellaneous expenses on larger farms, and decreases in crop labor and grain forage on smaller farms.

While building services should be curtailed on farms in size groups I to IV, in that order, they probably should be increased in the largest size group. Over-all, however, first priority might best be given to reductions in the use of roughage and of labor on livestock. But for an indication of just how much these inputs should be reduced, it would be necessary to have an estimate of their production elasticities at lower levels of use than those prevailing in the current sample. Also, without additional data on the role of growing timber in the farm organization it would be unwise to draw any conclusions about this resource. Finally, we emphasize again that the estimates presented in tables 17.8a and 17.8b are but the result of an exploratory analysis. The implications drawn must be regarded as somewhat tentative.

NORWAY

Table 17.9 lists four production functions from farm samples in the marine clay area of southeastern Norway. Each analysis relates to 1954; the main purpose of the original study being to examine the shift from dairying to small grains that was occurring on many farms in the region.[22] The actual districts sampled lay in the provinces of Vestfold, Rakkested, and Østfold. Two samples, each containing only farms within the size range 10-35 hectares (24.71-86.48 acres), were drawn. The first consisted of 28 grain farms. These were specified by the requirements that small grains be the major crop and that the farm program should not have included commercial dairying for at least two years, although nondairy livestock production was permitted. The second sample consisted of 30 farms whose main income source was dairying, the farm organization being based on fodder crop production for dairy feed with beef cattle, vegetables, and poultry as sideline enterprises. While the sampling procedure was on an area basis rather than at random, comparison with census data for 10-35 hectare farms in the region indicated that the average farm within each sample was not atypical.

For both the dairy and grain farms samples, separate functions were fitted for crop and livestock enterprises. While cropping and livestock production are independent enterprises on the grain farms, such is not the case on the dairy farms. On the latter, crop production consists mainly of fodder, including pasture, for subsequent feeding to livestock. Hence the allocation of resources for crop production on these farms should depend directly on the market conditions prevailing for dairy products.[23] As well, the crop and livestock enterprises on the dairy farms have, over various ranges, complementary, supplementary, and competitive relationships. However, no direct allowance was made

[22] Sandberg, O. R. Efficiency of resource use in farming in southeastern Norway. Unpublished M. Sc. thesis. Iowa State University Library, Ames. 1956.

[23] See Heady, E. O. The economics of agricultural production and resource use. Prentice-Hall, Inc., New York. 1952. Pp. 260-67; Nelson, M. *et al* The production function and linear programming in valuation of intermediate products. Land Economics, 33: 257-61. 1957.

Table 17.9. Norway: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for Southeastern Norway			
	Dairy farms		Grain farms	
	crops	livestock	crops	livestock
Number of farms	30	30	28	26
Year	1954	1954	1954	1954
Production elasticities:				
Land	0.23		0.47*	
Labor	0.32*	0.18	0.04	0.42*
Machine services	0.18*		0.19	
Other capital services	0.39*	0.80*	0.09	0.79*
Sum of elasticities	1.12	0.98	0.79	1.21†
$\overline{R}^2$	0.87	0.86	0.53	0.94
Sample means:‡				
Output ($)	4,964.00	5,936.00	4,733.00	1,624.00
Land (acres)	52.60		51.30	
Labor (months)	16.20	20.20	9.80	5.70
Machine services ($)	506.00		713.00	
Other capital services ($)	1,132.00	599.00	1,209.00	1,858.00
All inputs ($)	2,922.00	2,080.00	2,734.00	2,276.00
Average products:‡				
Land ($ per acre)	94.40		92.30	
Labor ($ per month)	306.40	293.90	483.00	284.90
Machine services ($ per $)	9.81		6.64	
Other capital services ($ per $)	4.38	9.91	3.91	0.87
Marginal products:‡				
Land ($ per acre)	21.83		43.38	
Labor ($ per month)	99.63	53.81	20.92	119.94
Machine services ($ per $)	1.76		1.29	
Other capital services ($ per $)	1.72	0.79	0.35	0.69
Opportunity costs:				
Land ($ per acre)	1.83		1.83	
Labor ($ per month)	73.32	73.32	73.32	73.32
Capital ($ per $)	1.04	1.04	1.04	1.04
Marginal return to opportunity cost ratios:‡				
Land	11.93		23.66	
Labor	1.36	0.73	0.29	1.64
Machine services	1.69		1.24	
Other capital services	1.65	0.76	0.34	0.66

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Significantly different from unity at a probability level of ≤ 5 per cent.
‡Estimated at the geometric mean input levels.

for such relationships in the present study, the forage output on dairy farms being considered as a final product.

With the exception of labor, the crop enterprises on both grain and dairy farms used rather similar quantities of resources. Still, from comparison of the resource elasticity coefficients it is sure that the two crop enterprises were based on different production functions, reflecting the differences between forage and grain cropping. Likewise, as would be expected from the distinctions between dairy and meat production, the production elasticities and resource use patterns of the livestock enterprises in the two samples were quite dissimilar. Too, it might be noted that relative to the other three estimated functions, the function for cropping on grain farms is not very reliable; it barely explains half of the variation in observed output.

According to the marginal return to opportunity cost ratios of Table 17.9, the average dairy farmer should use more of all resources, especially land, in his forage cropping and less of all resources in his dairy enterprise. In other words, the livestock enterprise on dairy farms should be curtailed and resources shifted to fodder crop production. This result is obviously anomalous since the fodder enterprise is but an intermediate stage in dairy production, there being no off-farm market for fodder. If the dairy enterprise should be reduced, it seems sure that fodder production should also be curtailed since current feeding levels are judged to be satisfactory. The above inferences for dairy farms are, however, rather untrustworthy although they strongly support the action farmers have been taking in shifting from dairy to grain production. Through overestimation of the value of forage, the study probably underestimates the marginal returns on labor and capital services used in milk production. If such is the case, the marginal returns of resources used in forage cropping would be overestimated. Concomitantly, the marginal value products of inputs used in livestock production on the dairy farms would be underestimated. Accordingly, the average dairy farm may be operating at a position closer to the optimum than the marginal return to opportunity cost ratios imply. The real difficulty lies in correctly ascertaining an opportunity cost for the forage, recognizing that no off-farm market exists for this intermediate product. To this end linear programming procedures could be used.

For the average grain farm, the marginal return to opportunity cost ratios suggest that additional land, especially, and perhaps machine services, should be used for cropping with a concomitant reduction in the quantity of labor and other capital services. As well, these farms should use more labor for livestock production. Such labor could be obtained by transferring some of the excess labor out of cropping. Over-all, given the limitations noted with respect to the dairy farm functions, the main implication of this Norwegian study appears to be that the acre size of grain farms should be expanded as a means to the better utilization of labor, machinery, and other capital services. At the same time, more labor and less capital services should be used in the livestock enterprise on grain farms. The surplus livestock capital could be profitably used in grain production.

ISRAEL

The five production function estimates listed for Israel in Table 17.10 are rather unique. Each refers to the same sample of farms but relates to a different year within the period 1953 to 1957.[24] The sample farms are family units averaging about 7.5 acres. They are partially irrigated and produce grain, vegetables, and forage crops together with citrus and vine fruits, having also a few cows and poultry. Such production units, grouped in villages on a co-operative basis, are typical of the bulk of Israel's agriculture.

Table 17.10. Israel: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for 68 Mixed Family Farms				
Year	1953	1954	1955	1956	1957
Elasticities:					
Land	0.04	0.04	0.01	0.02	0.04
Labor	0.26*	0.23*	0.27*	0.29*	0.20*
Operating capital	0.62*	0.69*	0.68*	0.54*	0.71*
Building assets	0.12*	0.08*	0.06*	0.12*	0.10*
Livestock assets	0.08*	0.05*	0.09*	0.06*	-0.01
Sum of elasticities	1.12†	1.09†	1.11†	1.03	1.04
$\overline{R}^2$	0.85	0.94	0.95	0.92	0.92
Marginal products:‡					
Land ($ per acre)	297.25	266.70	15.98	30.54	73.53
Labor ($ per month)	294.51	388.68	108.76	146.36	115.13
Operating capital ($ per $)	1.30	1.45	1.31	0.96	1.28
Building assets ($ per $)	0.22	0.23	0.20	0.37	0.31
Livestock assets ($ per $)	0.14	0.10	0.18	0.12	-0.01
Opportunity costs:					
Labor ($ per month)	293.79	347.03	75.83	92.43	98.80
Capital ($ per $)	1.10	1.10	1.10	1.10	1.10
Building assets ($ per $)	0.10	0.10	0.10	0.10	0.10
Livestock assets ($ per $)	0.20	0.20	0.20	0.20	0.20
Marginal return to opportunity cost ratios:‡					
Labor	1.00	1.12	1.43	1.58	1.16
Operating capital	1.18	1.32	1.19	0.87	1.16
Building assets	2.20	2.30	2.00	3.70	3.10
Livestock assets	0.70	0.50	0.90	0.60	-0.05

*Significantly different from zero at a probability level of ≤ 5 per cent.
† Significantly different from unity at a probability level of 5 per cent.
‡ Estimated at the geometric mean input levels.

[24] Mundlak, Y. Economic structure of established family farms. Falk Project for Economic Research In Israel. Hebrew University of Jerusalem, Rehovot. The results published here, by courtesy of Dr. Mundlak, are preliminary.

Perusal of Table 17.10 indicates that little year to year variation occurred in the production elasticities, the estimates ranging from 0.01 to 0.04 for land; from 0.20 to 0.29 for labor; from 0.54 to 0.71 for operating capital; from 0.06 to 0.12 for building assets; and from - 0.01 to 0.09 for livestock assets. In no case do the estimated elasticities for a particular resource follow a consistent pattern of change over the five years studied. The variations that did occur may therefore be attributed mainly to statistical and climatic variability rather than to changes in the technologies used which are known to have changed little over the period. In other words, it may be said that the production function pertinent to the average mixed farm was, for all practical purposes, invariant over the period 1953 to 1957 and that the five functions are but alternative estimates of the same true production function.

While it has no important influence on the marginal return to opportunity cost ratios, it should be noted that the marginal products and opportunity costs of Table 17.10 reflect the 1955 devaluation of Israel's currency relative to the United States dollar. Also, no opportunity cost or resource use efficiency ratios are given for land. The reason is that no market for farm land exists in Israel, farms being held on long term lease from the government under nominal rentals that are largely symbolic. So far as the other input categories are concerned, the yearly marginal return to opportunity cost ratios exhibit rather uniform patterns. Certainly it would be unwarranted to say that resource use efficiency has changed for the better or the worse over the five year period analyzed. Over-all, it appears that these Israeli farms have been using too many livestock assets and too little of the services of buildings, labor, and operating capital. Given the fixed land base of the farms, resources could be used more efficiently and profits would be raised if the input of buildings, operating capital, and labor were raised in that order. With increases in these resources, it is likely that the marginal product of livestock would also rise so that there might be no need for a reduction in the quantity of livestock used. Also, insofar as the marginal value product of land exceeds the nominal rentals paid for land, it would be economic for the government to raise the rental price of land; provided of course that on the basis of other welfare policies it was not desired to subsidize the farm sector of the economy.

INDIA

Estimates from two production function studies of Indian agriculture are given in Table 17.11. The first study, involving four functions, relates to family farms on loamy soils in the Meerut and Muzaffarnagar districts of the Indo-Gangetic Plain in the north central state of Uttar Pradesh. Agriculturally, this area is one of the most productive and prosperous in India. Rainfall in the region averages about 32 inches; deficiencies in precipitation are of less concern than excesses since nearly three-quarters of the farm land is irrigated. The listed

Table 17.11. India: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics for Selected Production Function Studies

Item	Production Function for						
	Uttar Pradesh				Andhra Pradesh		
	whole farm		planted sugarcane	irrigated wheat	irrigated farms	dry farms	mixed farms
Number of farms	49	63	63	60	18	72	106
Year	1954-55	1955-56	1955-56	1955-56	1953-54	1953-54	1953-54
Production elasticities:							
Land	0.23*	0.22*	0.37*	0.50*	0.57*	0.31*	0.14
Labor	0.58*	0.29*	0.69*	-0.26	0.14	0.04	0.26*
Capital services	0.18*	0.25*	0.30*	0.53*	-0.08	0.07	0.13*
Capital assets	0.05	0.21	-0.27	0.16			
Sum of elasticities	1.04	0.97	1.09	0.93	0.63†	0.42†	0.53†
$\bar{R}^2$	0.80	0.86	0.57	0.81	0.64	0.70	0.76
Sample means:‡							
Output ($)	448.00	378.00	98.00	59.00	12.50	53.00	93.00
Land (acres)	9.70	11.60	1.20	1.60	1.20	9.20	10.20
Labor (months)	13.00	9.30	1.60	1.90	7.00	19.70	25.00
Capital services ($)	42.00	50.50	25.60	3.90	0.80	5.80	9.40
Capital assets ($)	50.60	17.70	2.30	3.90			
All inputs ($)	439.00	367.00	74.00	63.00	82.00	213.00	562.00
Average products:‡							
Land ($ per acre)	46.20	32.60	81.70	36.90	10.40	5.80	9.10
Labor ($ per month)	34.50	40.60	61.20	31.00	1.80	2.70	3.70
Capital services ($ per $)	10.70	7.50	3.80	15.10	15.60	9.10	9.90
Capital assets ($ per $)	8.80	21.30	42.60	15.10			
Marginal products:‡							
Land ($ per acre)	8.82	7.18	29.94	18.44	20.85	3.65	1.94
Labor ($ per month)	19.75	11.71	40.89	-8.04	0.93	0.22	1.47
Capital services ($ per $)	1.94	1.84	1.19	8.21	-1.02	0.28	0.42
Capital assets ($ per $)	0.44	4.39	-11.78	2.43			
Opportunity costs:							
Land ($ per acre)	8.40	8.40	8.40	8.40	26.25	7.35	36.75
Labor ($ per month)	22.97	22.97	22.97	22.97	7.10	7.10	7.10
Capital ($ per $)	1.20	1.20	1.20	1.20	1.20	1.20	1.20
Capital assets ($ per $)	0.33	0.33	0.33	0.33			
Marginal return to opportunity cost ratios:‡							
Land	1.05	0.86	3.60	2.22	0.79	0.50	0.05
Labor	0.86	0.51	1.78	-0.35	0.13	0.03	0.21
Capital services	1.62	1.53	0.99	6.84	-0.85	0.23	0.35
Capital assets	1.33	13.30	-35.70	7.36			

*Significantly different from zero at a probability of ≤ 5 per cent.
†Significantly different from unity at a probability level of 5 per cent.
‡Estimated at the geometric mean input levels.

functions are derived from random subsamples of a much larger group of farms examined in a cost-accounting farm management project.[25] Typically, the farms produce wheat and sugar cane. The latter crop is generally planted biennially, two crops being harvested per planting. The first year's crop is known as "planted sugar cane" and the second

[25] Agrawal, G. D. and Foreman, W. J. Farm resource productivity in west Uttar Pradesh. Indian Jour. Agric. Econ., 14(4): 115-28. 1959; and Agrawal, G. D. Studies in economics of farm management in Uttar Pradesh, 1954-55. Directorate of Economics and Statistics, Ministry of Food and Agriculture, New Delhi. March, 1957.

year's as "ratoon sugar cane." Labor used on the farms is mainly from the farm family, but often there is a more than negligible input of hired labor. The average land base is about 10 acres with a range from one to 25 acres.

The first two functions listed in Table 17.11 refer to the whole farm program involving wheat, planted and ratoon sugar cane. The other two Uttar Pradesh functions are individual enterprise studies. One refers to planted sugar cane and the other to the wheat enterprise. The random sample used in deriving these two functions is the same as that used in the 1955-56 whole farm analysis. It should be noted that not all enterprises on these farms have been examined individually, no analysis being available for ratoon sugar cane. Also, the labor input category refers to both human and bullock labor; the average labor input in months over all farms consisting of approximately seven parts human labor (costing $7.22 per man month) to three parts of bullock labor (costing $1.05 per bullock pair day). These proportions will be assumed to remain fixed in the following discussion of the labor statistics derived from the estimated functions. Needless to say, the assumption is one of necessity, and not of choice.

While the production elasticities of land and capital services in the Uttar Pradesh whole farm analyses do not vary over the two years examined, there are fairly large inter-year variations in the elasticities of labor and capital assets. These differences may be due to sampling effects although climatic variation is probably of more importance. While 1954-55 was a year of average rainfall, 1955-56 was above average with some flood damage occurring. The production elasticities for planted sugar cane and wheat tend to imply that physically excessive quantities of capital assets were being used in the planted sugar cane enterprise while too much labor was also being used in the wheat enterprise. Of course, the elasticity estimates for these resources may be much lower than their true values; despite their relatively large absolute size, they are not significantly different from zero at the 5 per cent probability level. Also, it must be remembered that 1955-56 was a wetter year than is usual.

Consider now the marginal return to opportunity cost ratios found in these Uttar Pradesh studies. From the whole farm analyses, it seems that too few capital services and assets and too much labor were used, the input of land being very near optimal. But these general implications must be modified in terms of the individual enterprise studies. These show quite marked differences to exist between the resource use efficiency ratios of the individual enterprises; the differences being "averaged out" in the whole farm analyses which lump together the production of wheat, and planted and ratoon sugar cane. On an individual crop basis, the statistics strongly suggest that too much labor and too little capital were used in wheat production, while the use of more labor and less capital — especially capital assets — would have increased the profitability of the planted sugar crop. Still, relative to the value of land and labor inputs, capital is only of minor importance. Hence, in

absolute terms, only small changes may be needed to induce optimality in the use of capital assets and services assuming, of course, that their elasticities are not negative over lower stages of the production surface. As noted before, no production function estimates are available for the ratoon sugar crop. Still, a little can be said of this crop from the data for the other two enterprises. Thus, while land use was satisfactory on a whole farm basis and since too little land appears to be used for wheat and planted sugar cane, it may be concluded that too much land was being devoted to the ratoon sugar cane crop.

Data from an area sample of villages in the Telengana district was used to derive the Andhra Pradesh estimates; a function being fitted for each type of farm found in the region: fully irrigated, dry, and mixed.[26] The major activity within each of these groups is cereal cropping of a virtually subsistence character. As the sample means of output and inputs listed in Table 17.11 indicate, Andhra Pradesh farms are of very small scale. Indeed, this southeast state is one of the poorest in India. Hence it is not surprising that at market valuations the mean of all inputs exceeds mean output in each of the samples. However, since the labor input is nearly all family labor paid at a nominal rate, the discrepancy between output and input is probably less than is indicated by the data of Table 17.11. In fact, the quoted opportunity cost for labor is rather unreal. There is virtually no market for hired labor except on large farms, and these constitute only a small proportion of the total. Too, the prices given for land and capital are averages and mask quite large variations from tract to tract in the region.

From the sums of the production elasticities, it is apparent that severe decreasing returns to scale prevail with respect to land, labor, and capital services on these Andhra Pradesh farms. Their existence emphasizes the importance of managerial ability in farming in this area, granted that no other inputs of importance have been omitted from the analysis and that constant returns prevail relative to all inputs. If these assumptions are correct, so that the difference between unity and the sum of each set of elasticities may be taken as approximating the production elasticity of management, then it appears that management has a relatively large influence on production under current conditions. Thus the percentage increase in output from a 1 per cent increase in the input of management would be roughly 0.4, 0.6, and 0.5 per cent, respectively, on irrigated, dry, and mixed farms. These implications, moreover, are supported by the experience of agricultural technicians in the region. Nor are they surprising on a priori grounds, given the primitive nature of agricultural production in the area.

In terms of their marginal return to opportunity cost ratios, all three types of farm appear to be grossly inefficient. Under the assumed opportunity costs, each sample used too much land, labor, and capital.

[26] Suryanarayana, K. S. Resource returns in Telengana farms: a production function study. Indian Journal of Agricultural Economics, 13(2): 20-26. 1958. A noteworthy feature of this study is that land acreage was corrected for soil heterogeneity by a soil fertility index.

Moreover, these conclusions remain true even if the monthly wage rate of labor is lowered from $7.10 to as little as $1.50 so as to correspond to the minimum payments received by family labor. Indeed, were it not for the subsistence element in this agriculture, the data of Table 17.11 would imply that farming in the area could never be profitable under current conditions. The difficulty, of course, is that there are extremely few other means of earning a livelihood. Alternatively, farming in the area might be reorganized by the provision of nonfarm employment opportunities and the introduction of new production techniques of a sufficiently productive nature to make commercial agriculture possible.

Comparison of the two India studies indicates that farming in the two states has few common features. The semicommercial farms of Uttar Pradesh are large and profitable relative to the subsistence units of Andhra Pradesh. While Andhra Pradesh farms exhibit severe decreasing returns to land, labor, and capital, constant returns appear to prevail in more prosperous Uttar Pradesh, perhaps indicating a more efficient type of farm organization in the latter state. Distinct differences are also noticeable in the patterns of resource use efficiency: Uttar Pradesh farms used too little capital and too much labor, while Andhra Pradesh farms had a surfeit of all resource services, with the probable exception of management.

JAPAN

Relative to most other countries, a large number of cross-sectional production function studies are available for Japanese agriculture.[27] Results of two of these studies, involving seven functions, are presented in Table 17.12. The first study is based on cost data from an area sample of mixed farms in the Shizuoka Prefecture of Honshu, separate functions being estimated for each type of enterprise found in the area.[28] Accordingly, the listed sample means of input and output for sweet potato, tea, and paddy rice production on these Honshu farms give only a partial indication of the over-all scale of farm operation. The other four functions of Table 17.12 refer to various types of rice producing farms in Hokkaido.[29] Data for the two 1955 functions were taken from a random sample of farms keeping records under the supervision of the Ministry of Agriculture. The sample, consisting of 24 small and

[27] See Kamiya, K. On productivity of labor. Journal of Rural Economics, 17: 22-44. 1941; Okawa, I. The theory and measurement of food economy. Hitosubashi College of Commerce and Economics. 1945. Ch. 8; Watanabe, T. Theory of production functions. Journal of Rural Economics, 21. 1945; Yuwata, Y. Production functions for rice and barley. Quarterly Journal of Agricultural Economy, 7. 1953.

[28] Tsuchiya, K. Production functions of agriculture in Japan. Quarterly Journal of Agricultural Economy, 9. 1955.

[29] Takayama, T. A study on the Cobb-Douglas production function — with an application to rice production in Hokkaido. Review of the Society of Agricultural Economics of Hokkaido University, 15: 1-24. 1959.

Table 17.12. Japan: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for						
	Honshu			Hokkaido			
	sweet potatoes	tea	rice	rice			
Number of farms	11	30	69	24	36	17	42
Year	1951	1951	1951	1955	1955	1956	1958
Production elasticities:							
Land	0.85*	0.29*	0.56*	0.80*	0.55*	0.87*	0.78*
Labor	0.29*	0.30*	0.29*	0.10	0.29*	0.12	0.21*
Capital services				0.07	0.00	0.00	0.00
Capital assets	0.00	0.46*	0.15*	0.02	0.18*		0.00
Sum of elasticities†	1.14	1.05	1.00	0.99	1.02	0.99	0.99
$\overline{R}^2$	0.82	0.85	0.77	0.70	0.61	0.81	0.60
Sample means:‡							
Output ($)				936.00	2,039.00	1,877.00	2,036.00
Land (acres)				3.60	7.00	7.10	6.30
Labor (months)				10.90	19.70	22.20	13.30
Capital services ($)				287.20	551.20	214.50	125.30
Capital assets ($)				394.50	672.10		115.60
All inputs ($)				1,084.00	2,026.00	1,108.00	1,043.00
Average products:‡							
Land ($ per acre)				260.00	291.30	264.40	328.40
Labor ($ per month)				85.90	103.50	84.50	153.10
Capital services ($ per $)				3.30	3.70	8.70	16.30
Capital assets ($ per $)				2.40	3.00		17.60
Marginal products:‡							
Land ($ per acre)	212.53	135.03	97.46	208.70	160.21	242.87	229.66
Labor ($ per month)	25.29	21.57	8.18	10.43	7.56	15.57	28.29
Capital services ($ per $)				0.24	0.00	0.00	0.00
Capital assets ($ per $)	0.00	0.51	0.41	0.03	0.56		0.00
Opportunity costs:							
Land ($ per acre)	6.77	6.97	6.83	89.94	89.94	77.12	83.64
Labor ($ per month)	16.89	16.89	16.89	36.11	36.11	35.39	40.30
Capital ($ per $)				1.11	1.11	1.11	1.11
Capital assets ($ per $)	0.20	0.20	0.20	0.20	0.20		0.20
Marginal return to opportunity cost ratios:‡							
Land	31.39	19.37	14.27	2.32	1.78	3.15	2.75
Labor	1.50	1.28	0.48	0.29	0.21	0.44	0.70
Capital services				0.22	0.00	0.00	0.00
Capital assets	0.00	2.55	2.05	0.15	2.80		0.00

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at a probability level of 5 per cent.
‡Estimated at the arithmetic mean input levels in the Honshu studies and at the geometric means in the Hokkaido studies.

36 large farms, covered most of the rice producing areas of Hokkaido. On a priori grounds, farmers in the sample were thought to be of above average managerial capacity. Main objective of the study was to ascertain if there were structural differences between large and small units in their production process. Both the 1956 and the 1958 analyses refer to well-established farms an alluvial soil in the Kuriyama region of Hokkaido. Like the 1955 study, they relate to the over-all activity of the

farm. While the 1956 function is based on an area sample, a random sampling procedure was used in the 1958 study.

A pattern is discernible among the resource production elasticities listed in Table 17.12. Excluding the function for tea, capital has a low elasticity with a range from zero to 0.18. The land coefficients are relatively large, ranging from 0.55 to 0.87. Labor appears to be of intermediate importance with elasticities from 0.10 to 0.30. The explanation, of course, is that land is used very intensively in the production of paddy rice and sweet potatoes. Relative to other inputs, the greatest production response is to be expected from an expansion of the land base with other factors held fixed; rather than from an increase in the use of other resources with land held constant. Indeed, under the conditions prevailing in the sweet potato and some of the rice samples, an incremental increase in the use of capital assets (mainly machinery) or services (mainly fertilizer) would have no influence on production. Tea production does not fit such a picture because of the distinctive features of its production process, reflecting the specialized nature of tea cropping relative to other types of crop production.

The marginal return to opportunity cost ratios of Table 17.12 evidence rather severe inefficiencies in Japanese resource use. Especially striking are the exceedingly high ratios for land of 31.39, 19.37, and 14.27, respectively, in sweet potato, tea, and rice production in Honshu. They result not so much from inefficiencies on the part of individual farmers, but from the exceedingly low land rents fixed by government action under the Land Reform Law. This law also limits the possibility of renting additional land. In a normative economic sense, the land rents are obviously too low. Still, they may be justifiable on welfare grounds. We do not know.

Disequilibria in land use is not so great on the Hokkaido farms, although it is apparent that substantial increases in the land input would be profitable. With the exception of the 1955 study which implies more capital assets should be used on larger farms, Hokkaido farmers appear to have used too much labor and capital. A decrease in the input of labor and capital, as well as tending to optimize the use of these resources, would do much towards reducing inefficiency in the use of land. The only common feature of the three Honshu enterprise analyses is the indication of too little land being used. In common with the Hokkaido rice samples, Honshu rice farmers use too much labor. However, additional labor should be used in Honshu sweet potato and tea production, with additional capital being used in the tea and rice enterprises rather than in sweet potato production. More of all resources should be used in tea production, especially land and capital. Land and labor could be profitably substituted for capital in the sweet potato enterprise, while land and capital should be substituted for labor in the rice enterprise.

Recognizing the general importance of rice in Japanese agriculture, the above analyses tend to substantiate the opinions of agricultural authorities that these Japanese farmers generally use more than enough

fertilizer, are overcapitalized in farm machinery, and have a surfeit of labor. Either the land input should be increased or nonland resource inputs curtailed, if farmers are to maximize their profits. Concomitantly, it would seem necessary for the sake of national welfare to provide alternative employment opportunities for surplus farm labor. Too, insofar as the inefficiencies shown to exist are caused by the fairly stringent price and production controls applied in Japanese agriculture and not by lack of knowledge on the part of farmers, strong welfare grounds would be necessary to justify the controls.

TAIWAN

The only two production function studies available for Taiwan are listed in Table 17.13. One relates to sugar cane and the other to a cropping system that is competitive with sugar cane. Both types of production are important in Taiwan's agricultural economy. The data on which the functions are based relate to a random sample of 100 farms in the Tainan area of southern Taiwan.[30] The observations analyzed, however, do not refer to the farms themselves but to a field of sugar cane and a field of competitive crop on each farm. Hence the sample means of output and inputs listed in Table 17.13 for each crop give no indication, when added together, of the over-all scale of operation of the average sample farm. As well, the consideration of fields, rather than the over-all cropping enterprise on each farm, implies that the fitted functions may not be extrapolated to the over-all farm situation without some danger of error. Still, since the relative proportions of the various inputs are most likely the same for a whole farm as for a single field, the study provides an estimate of resource use disequilibria within and between the two cropping enterprises on a whole farm basis. Also, given the competitive relationship between the two cropping systems, the estimated functions are of direct relevance as decision guides between the two alternatives. Thus it is reassuring that the marginal productivity estimates for the two cropping systems diverge in the fashion expected from comparison of the labor to land and capital to land ratios in the two systems.

While an equilibrium amount of capital services appears to have been used for both cropping systems, proportionately too much land and too little labor were being used. The greatest disequilibria exist in the excessive proportion of land used in sugar cane production and the restrictive quantity of labor utilized relative to other inputs in the competitive cropping system. To rectify these inefficiencies, the best procedure would seem to be the use of additional labor. Increasing the labor to land ratio would increase the marginal productivity of land and

[30] Wang, Y. Resource returns and productivity coefficients for selected crop systems in Tainan area. Proceedings of Agricultural Economics Seminar, Sept. 16-20, 1958. National Taiwan University, Taipei. 1959. Pp. 90-98.

Table 17.13. Taiwan: Production Elasticities, Marginal Products and Opportunity Costs of Resources, Marginal Return to Opportunity Cost Ratios, and Related Statistics as Derived for Selected Production Function Studies

Item	Production Function for Tainan Region	
	Sugar cane	Other crops
Number of fields	100	100
Year	1957	1957
Production elasticities:		
Land	0.36*	0.44*
Labor	0.25	0.33*
Capital services	0.34	0.31*
Sum of elasticities†	0.95	1.08
$\overline{R}^2$	0.60	0.78
Sample means:‡		
Output ($)	318.70	287.10
Land (acres)	1.20	0.87
Labor (months)	3.50	2.40
Capital services ($)	78.00	67.50
All inputs ($)	324.00	244.00
Average products:‡		
Land ($ per acre)	265.60	330.00
Labor ($ per month)	91.10	119.60
Capital services ($ per $)	4.09	4.25
Marginal products:‡		
Land ($ per acre)	96.44	142.41
Labor ($ per month)	22.88	39.25
Capital services ($ per $)	1.36	1.30
Opportunity costs:		
Land ($ per acre)	246.67	246.67
Labor ($ per month)	13.80	13.80
Capital ($ per $)	1.33	1.33
Marginal return to opportunity cost ratios:‡		
Land	0.39	0.58
Labor	1.66	2.84
Capital services	1.02	0.99

*Significantly different from zero at a probability level of ≤ 5 per cent.
†Not significantly different from unity at a probability level of 5 per cent.
‡Estimated at the arithmetic mean input levels.

decrease the marginal productivity of labor. Both of these changes would tend to reduce the disequilibria in land and labor use. Concomitantly, an increase in labor inputs would tend to raise the marginal productivity of capital so that the marginal return to opportunity cost ratio for capital would be greater than unity. In consequence, it would also be desirable to use a little more capital — assuming the land input to be held fixed — despite the fact that the resource use efficiency ratios for capital services of Table 17.13 are very close to unity. Given the

marginal return to opportunity cost ratios prevailing in the two samples, priority in the use of additional labor should be given to the competitive crop enterprise.

Having considered the collected functions on a country by country basis, we may now examine their implications within an international framework. This will be done in two stages. First, a comparison of the production elasticities will be made. As outlined in Chapter 2, these elasticities of production play an important role in the determination of resource marginal productivity. International comparisons of the efficiency of resource use in terms of profit maximization will then be made, using the estimated marginal return to opportunity cost ratios as a guide to the efficiency of resource use.

INTERNATIONAL COMPARISON OF RESOURCE PRODUCTION ELASTICITIES

For the purpose of making international comparisons of input elasticities, resources will be considered under three categories: land, labor, and other services. The other services category consists of many particular types of inputs such as livestock, miscellaneous capital services, fertilizer, purchased feed, building and machinery services, etc. It is a catch-all category, necessitated by the fact that land and labor are the only inputs considered individually in a majority of the estimated functions presented above. Certainly it would have been desirable to consider capital services as a separate category. But to do so is unfeasible. There is insufficient correspondence between the inputs designated as capital services in the various national studies. For instance, while the United States, Canadian, South African, Austrian, and Taiwan studies use the traditional threefold resource classification of land, labor, and capital services, the Australian, New Zealand, United Kingdom, Swedish, Norwegian, Indian, and Japanese studies generally include a number of capital input categories. Accordingly, in these latter studies we have added the nonland and nonlabor resource elasticities together to give a single elasticity coefficient for these other resource services. This aggregate figure may be taken as corresponding — admittedly roughly, but closely enough for current purposes — to the capital input elasticity derived under the tripartite classification using land, labor, and capital. Still, the elasticity of "other services" needs to be interpreted with caution. Strictly, it is the percentage increase in output that is expected to result from a 1 per cent increase in the input of each of the resources included in the other services category, the input of land and labor being held constant.

The estimated elasticities and their sums are presented in Table 17.14, the location and specification of each parent function noted on the left. These functions are the various ones already discussed on a country by country basis, with the exception that only one estimate is included in those cases where a number of studies are available for a

Table 17.14. Production Elasticities of Land, Labor, and Other Resource Services as Estimated in Forty-one Cross-Sectional Cobb-Douglas Type Production Function Studies

Location of Sample	Function for	Elasticity of Production			Sum of Elasticities
		Land services	Labor services	Other services	
United States, northern Iowa	corn	0.91	0.08 (27)	0.16 (25)	1.15
Japan, Honshu	sweet potatoes	0.85	0.29 (9)	0.00 (31)	1.14
United States, southern Iowa	corn	0.79	0.09 (26)	0.39 (18)	1.27
Japan, Hokkaido*	rice	0.75	0.18 (20)	0.07 (28)	1.00
India, Andhra Pradesh	irrigated farms	0.57	0.14 (24)	-0.08 (32)	0.63
Japan, Honshu	rice	0.56	0.29 (10)	0.15 (26)	1.00
United States, Montana	wheat	0.50	0.04 (29)	0.58 (5)	1.12
India, Uttar Pradesh	wheat	0.50	-0.26 (32)	0.69 (3)	0.93
Norway, southeast	cereals	0.47	0.04 (30)	0.28 (24)	0.79
Taiwan, Tainan	cereals	0.44	0.33 (5)	0.31 (23)	1.08
New Zealand, Canterbury	sheep	0.42	C.15 (23)	0.54 (10)	1.11
United States, Alabama	crops	0.39	0.32 (6)	0.46 (15)	1.17
South Australia	dairy	0.39	0.25 (13)	0.54 (11)	1.19
Canada, Alberta	wheat, beef	0.39	0.20 (18)	0.34 (21)	0.93
India, Uttar Pradesh	sugar cane	0.37	0.69 (1)	0.03 (30)	1.09
Taiwan, Tainan	sugar cane	0.36	0.25 (14)	0.34 (22)	0.95
Sweden	mixed farms	0.35	0.05 (28)	0.57 (6)	0.97
India, Andhra Pradesh	dry farms	0.31	0.04 (31)	0.07 (29)	0.42
Japan, Honshu	tea	0.29	0.30 (8)	0.46 (16)	1.05
United States, Iowa-Illinois	crop-share	0.29	0.25 (15)	0.48 (14)	1.02
Australia, New South Wales	dairy	0.28	0.22 (17)	0.42 (17)	0.92
South Africa, Kalihari	cattle fattening	0.28	0.13 (25)	0.55 (8)	0.96
India, Uttar Pradesh †	wheat, sugar cane	0.23	0.43 (3)	0.35 (20)	1.01
Norway, southeast	fodder	0.23	0.32 (7)	0.57 (7)	1.12
United States, Iowa-Illinois	livestock-share	0.23	0.18 (21)	0.53 (12)	0.95
Canada, Alberta	cattle ranches	0.20	0.37 (4)	0.39 (19)	0.97
South Africa, Kalihari	cow-calf ranches	0.19	0.19 (19)	0.52 (13)	0.90
India, Andhra Pradesh	mixed farms	0.14	0.26 (11)	0.13 (27)	0.53
Austria‡	mixed farming	0.13	0.26 (12)	0.61 (4)	1.00
Australia, New South Wales	sheep	0.10	0.59 (2)	0.55 (9)	1.24
United States, Iowa-Illinois	owners	0.09	0.17 (22)	0.73 (2)	0.99
Israel§	mixed farms	0.03	0.25 (16)	0.80 (1)	1.08
Norway, southeast	beef cattle		0.42	0.79	1.21
United Kingdom, England	dairy		0.29	0.83‖	1.12
Western Australia	dairy		0.23	0.76	0.99
United States, Alabama	livestock		0.23	0.74	0.97
Norway, southeast	dairy		0.18	0.80	0.98
United States, southern Iowa	hogs, cattle		0.12	0.98	1.10
United States, Montana	cattle		0.08	0.94	1.02
United States, northern Iowa	hogs, cattle		0.08	0.91	0.99
Sweden	dairy		-0.05	1.23	1.18

*Average of coefficients in 1955, 1956, and 1958 analyses.
†Average of 1954-55 and 1955-56 analyses.
‡Study of 37 strata.
§Average of analyses for 1953, 1954, 1955, 1956, and 1957.
‖Includes land services.

given area and type of product. At the top of the table are listed those analyses in which land was a relevant input. These analyses relate to crops and grazing livestock. Listed at the bottom of the table are the analyses in which, with the exception of the United Kingdom study, land was not a relevant input variable. The latter studies relate to livestock production, generally under nongrazing conditions. The exceptions are the United Kingdom analysis wherein land services are included in other services, and the Swedish milk production study in which pasture intake substitutes for land services and is included in the other services

category. The only national analyses not included in Table 17.14 are the United Kingdom Bristol Province dairy study and the Canadian beef feedlot analysis. Both of these include only inputs in the other services group. The following discussion will be confined to those functions which include land, labor, and other services. In all, there are 32 such functions. For present purposes, they are implicitly assumed to constitute a random sample from the super-population of all agricultural production function studies that might have been carried out over the globe during the period 1950 to 1958. We will not discuss the set of livestock functions which do not include land. This sample is too small to warrant detailed examination.

Those analyses which include land, labor, and other services are ordered in Table 17.14 in terms of the size of the production elasticity of land. Such a basis of ordering is used because land is the most fixed of the three resource categories; labor and other services being more mobile, in that order. For convenience, the ordering of the labor and other services coefficients is also noted in the table within brackets beside the estimated coefficients. The order, from larger to smaller, ranges from (1) to (32).

Relative to the size of the land coefficient, it is obvious that no pattern of ordering prevails between countries. But a strongly identifiable sequence does exist in terms of the type of production. For cropping, the production elasticity of land is generally higher than it is for grazing livestock or mixed crop-livestock production. Thus while all of the livestock studies have production elasticities for land of less than 0.42, a majority of the crop functions have an elasticity for land between 0.44 and 0.91. This result is not unexpected on theoretical grounds. Land is used far more intensively in crop production than in livestock production. Consequently one would expect an expansion of the land input to have a greater influence on crop output than on livestock output. Nor is it surprising that corn production in Iowa and Japanese rice production have the largest production elasticities for land. In both these regions, the intensity of land use exceeds that in any of the other regions examined.

Although the pattern is not so pronounced as for the land coefficient, the data of Table 17.14 indicates that the elasticity of other services tends to be lower for crops than for livestock and mixed livestock-crop activities. The justification, of course, is the converse of that which explains the high elasticity of land for cropping relative to livestock production. The more intensively the land is used, the less the response to additional inputs of other services is likely to be. Within the crop enterprise, it also appears that the elasticity of other services tends to be higher on average for the non-Asian samples than for the Asian. Concomitantly, the coefficient for crop labor is lower on average among the non-Asian samples than in the Asian. This result is surprising since Asian countries are very plentifully supplied with labor. They might be expected to be using labor at such intensive levels that increased inputs of this resource would yield smaller

additions to output than would occur in non-Asian crop production. Still, it is explainable in that other services tend to be of greater importance in the non-Asian crop production samples due to the more mechanized nature of their production processes. Unlike the elasticities for land and other services, those for labor exhibit no pronounced pattern relative to cropping as against livestock production.

Looking at the over-all picture presented by the elasticities of Table 17.14, the most striking feature is the variation present within each input category. Obviously there can be little logic associated with the selection of any particular triplet of figures as *the* elasticities of land, labor, and capital in macro studies of an international or national type involving specification of a production function for the agricultural sector of the economy. This point is again obvious from the frequency distributions of the elasticities, and the standard deviations of their means, presented in Table 17.15. With a mean value of 0.39 over the sample of 32 studies, the other services category tends to be the most important in terms of output response, closely followed by land with an average elasticity of 0.38. As would be expected, given the tendency to a surfeit of labor that generally prevails in agriculture, the average labor elasticity of 0.21 is the smallest of the three elasticities. In terms of relative variability, the labor coefficient is the most variable, although its variance is somewhat smaller than those of land and other services which have approximately equal standard deviations.

Without implying that the mean elasticities of Table 17.15 have any broad usefulness of an empirical nature, it is interesting to note that they differ quite markedly from the elasticities hypothesized for agricultural production in the literature on international trade and development. For instance, Tinbergen and Polak used elasticities of 0.1, 0.7,

Table 17.15. Frequencies, Means, Standard Deviations, and Coefficients of Variation of the Production Elasticities of Land, Labor, and Other Services in a Sample of Thirty-two Cross-sectional Cobb-Douglas Type Production Function Studies

Statistic	Land Services	Labor Services	Other Services
Frequencies:			
< 0		1	1
0.00-0.19	7	13	7
0.20-0.39	14	14	7
0.40-0.59	7	3	13
0.60-0.79	2	1	4
0.80-0.99	2		
Mean	0.38	0.21	0.39
Standard deviation	0.22	0.17	0.23
Coefficient of variation	0.58	0.80	0.59

and 0.2 for land, labor, and capital, respectively;[31] Palvia has hypothesized an elasticity of 0.5 for labor and a figure of 0.5 for land and capital combined;[32] and Belshaw very tentatively hypothesizes values of 0.75 for labor and 0.25 for capital, including land.[33] In contrast, the mean values listed in Table 17.15 are not too dissimilar from the elasticities of 0.39, 0.28, and 0.33 for land, labor, and other services estimated in a cross-sectional Cobb-Douglas study of world agriculture based on national statistics for the year 1949.[34] Still, the variation shown among the elasticities of Table 17.14 bears witness to the dangers associated with the use of any such global production function.

INTERNATIONAL COMPARISONS OF RESOURCE USE EFFICIENCIES

For production under a given technological environment with both output and input variable, the ideal measure of the efficiency of resource use is provided by the marginal return to opportunity cost ratios of the various resources. Concomitantly, these ratios indicate the direction of the changes that should be made in resource allocation if profits are to be maximized. Both within and between nations such data is essential to any thorough analysis of resource allocation and trade, and to any endeavor to improve the pattern of resource use.

Accordingly, having already considered the intra-national implications of the available agricultural production function estimates from around the globe, we now proceed to examine their implications for resource use within an international framework. Of course, such an analysis cannot be complete. We only have estimates from a few scattered locations. In contrast, even assuming the problems of statistical and economic specification to be negligible, a full analysis would require estimates covering all the various regions and types of production found in the world. Only then would it be possible to specify a model from which the optimal allocation of world resources under the given distribution of technologies might be deduced for various price regimes. With insertion of the appropriate restrictions, such a model might take into account the size of national factor endowments as well as the natural and institutional fixity of some resources, and the cost of resource transfers. Even so, such an extensive model would be far from ideal. By necessity, it could only be based on the production functions currently being used throughout the world. As we all know, the greatest gains in efficiency and welfare in many regions of the globe are to be

[31] Tinbergen, J. and Polak, J. J. The dynamics of business cycles; a study in economic fluctuations. Routledge, London. 1950. P. 122.

[32] Palvia, C. M. An econometric model for development planning. Institute of Social Studies, The Hague. 1953.

[33] Belshaw, H. Population growth and levels of consumption with special reference to countries in Asia. Allen and Unwin, Ltd., London. 1956.

[34] Bhattacharjee, J. P. Resource use and productivity in world agriculture. Journal of Farm Economics. 37: 57-71. 1955. See also Econ. Bul. Asia and Far East, 9(1): 17-31. 1958.

made through the introduction of more efficient production functions. In short, the optimal use of world resources does not depend solely on shifts in resource use within the context of a given technology. Changes in technology are just as important, if not more so. Moreover, since technological shifts are inherently uncertain, their specification in an empirical model would be no easy task. Too, any analysis of world resource allocation that omits to take account of peoples' aspirations, needs, potentialities, cultural level, and national sentiments must be less than complete. Hence, granted the magnitude and difficulty of any full appraisal of world resource allocation, the humble nature of the material we present in readily apparent. To know that it encompasses nearly all the available data is but small consolation. Still, even these few comparisons of resource-use efficiency are of interest. They illuminate some important features of current resource allocation in world agriculture. As well, they introduce a quantitative element into an area of discussion that is prone to be shrouded in platitudes and ideological clichés.

Throughout, we assume both output and input are variable. In consequence, provided decreasing economic returns to scale eventually prevail (as we believe they must), the optimal pattern of resource use occurs when all the marginal return to opportunity cost ratios are equal to unity. Should output be regarded as fixed, the best pattern of resource use is that at which all the marginal return to opportunity cost ratios are equal, but not necessarily equal to unity. Such a resource distribution minimizes the cost of producing the given output and is easily calculated from the estimated production function. Moreover, should output be fixed, the discrepancy between the current pattern of resource allocation and the least-cost pattern provides the appropriate measure of resource use efficiency.[35] We do not present such data; the context of our discussion precludes the assumption of fixed output.

The marginal return to opportunity cost ratios estimated for the services of land, labor, and capital in the various studies are listed in tables 17.16, 17.17, and 17.18, respectively. In drawing up the tables, only an average estimate has been included for those countries where a number of studies are available for a given region, type of farm organization, and type of product. The capital services category corresponds to the other services input of the previous section; it is only the same as the capital services item of the original functions in those cases where the classical resource categories of land, labor, and capital were used. Moreover, in deriving the marginal return to opportunity cost ratios for capital services, the ratios for other services in the original functions were weighted according to the proportion each of these services was of the total input of nonland and nonlabor services. Thus the capital services ratios indicate the relative discrepancy between the revenue and cost associated with an input of one more unit of capital

[35] See, for example, Miller, W. G. *et al.* Relative efficiencies of farm tenure classes in intrafirm resource allocation. Iowa Agr. Exp. Sta. Bul. 461. Ames. 1958. Also Tramel, T. Using production functions for making recommendations. Jour. Farm Econ., 39: 790-93. 1957.

services composed of the various nonland and nonlabor services in the proportions currently used.

In assessing the marginal return to opportunity cost ratios, it must be remembered they are based on the prices prevailing, on the average, over the twelve months to which each of the production functions refers. We assume these prices are comparable over the eight years spanned by the estimates. As well, account needs to be taken of the relative importance of the production process to which each estimate relates. In total welfare terms, a small degree of inefficiency for each of the members of a large population of entrepreneurs may be of more import than the existence of severe disequilibrium in a small population. Another restrictive feature of the data is that it only relates to resource changes of an incremental nature. Although the marginal return to opportunity cost ratios tell us the relative effect of using one more unit of each resource, they provide only a tentative indication of the possible influence of large scale changes in resource use. However, such information may be derived from the estimated production functions and is already available in some of the original sources cited for the various studies. We do not present such data here. Given the small sample of estimates with which we are concerned, the additional information would be of little worth beyond that given by the resource use efficiency ratios at the current margin of resource use, especially since what is needed in many of the less developed areas of the globe is not the attainment of optimality under current methods of production but the introduction of new technologies. In many instances current methods, even if used in profit maximizing fashion, are quite inadequate to sustain a reasonable level of welfare.

The marginal return to opportunity cost ratios for land services are presented in Table 17.16. Of the 33 estimates, 10 imply the use of too much land while 23 indicate additional land could have been used profitably. Over-all, the ratios range from 31.39 for sweet potato cropping in Japan to 0.05 for mixed crop farms in Andhra Pradesh, India. The general tendency appears to be for farmers to use too little land. In fact, half of the estimates are greater than 2.00, indicating quite severe deficiencies in land input. Moreover, a chi square test indicates there is less than one chance in twenty that the distribution of the ratios between those above and below unity is due to chance. Such a result is as would be expected, given the relative immobility of land and the pressure upon its use as populations increase. Apart from the extremely severe disequilibria in Honshu cropping, caused by the Japanese Land Reform Law, and the prevalence of too many land services in the Austrian, Taiwan, Australian sheep, and Indian Andhra Pradesh studies, the data exhibits no pronounced pattern. Differences within countries in the efficiency of land use appear to be just as great as those between nations. Still, it might be noted that all of the estimates for Japanese cropping, Australian dairying, Norway, New Zealand, Sweden, and South Africa imply the use of too few land services. As well, only one of the seven United States studies indicates an excessive

Table 17.16. Marginal Return to Opportunity Cost Ratios of Land Services as Estimated From Selected Cross-sectional Production Function Studies of Farm-firms

Location of Sample	Function for	Marginal Return to Opportunity Cost Ratio of Land Services
Japan, Honshu	sweet potatoes	31.39
Norway, southeast	cereals	23.66
Japan, Honshu	tea	19.37
Japan, Honshu	rice	14.27
Norway, southeast	fodder	11.93
United States, Alabama	crops	4.01
India, Uttar Pradesh	planted sugar cane	3.60
United States, southern Iowa	corn	3.27
United States, northern Iowa	corn	3.08
Japan, Hokkaido	rice	2.95
United States, Montana	wheat	2.81
Canada, Alberta	wheat, beef	2.58
New Zealand, Canterbury Plain	sheep	2.48
Japan, Hokkaido, small farms	rice	2.32
India, Uttar Pradesh	irrigated wheat	2.22
South Australia	dairy	2.12
South Africa, Kalihari	cattle fattening	1.95
United States, Iowa-Illinois	livestock-share	1.93
United States, Iowa-Illinois	crop-share	1.78
Sweden	mixed farms	1.78
Japan, Hokkaido, large farms	rice	1.78
Australia, New South Wales	dairy	1.23
South Africa, Kalihari	cow-calf ranches	1.05
United States, Iowa-Illinois	full owners	0.98
India, Uttar Pradesh	wheat, sugar cane	0.95
Austria	mixed farming	0.92
Canada, Alberta	cattle ranches	0.88
India, Andhra Pradesh	irrigated farms	0.79
Taiwan, Tainan	cereals	0.58
India, Andhra Pradesh	dry farms	0.50
Australia, New South Wales	sheep	0.39
Taiwan, Tainan	sugar cane	0.39
India, Andhra Pradesh	mixed farms	0.05

input of land, and it has a ratio of 0.98 implying near optimality. The other six United States estimates range from 1.78 to 4.01 so that the use of too few land services per farm also appears to be a common feature of United States agriculture.

In terms of type of farming, 19 of the 33 ratios for land refer to cropping alone and 14 to production involving livestock, generally under grazing conditions. Nine of the 14 livestock estimates indicate the use of too little land. Of the 19 crop estimates, 14 imply an insufficient input of land and there is less than one chance in 20 that this division is due to sampling variation. Indeed, 11 of these 14 ratios are larger than

Table 17.17. Marginal Return to Opportunity Cost Ratios of Labor Services as Estimated From Selected Cross-sectional Production Function Studies of Farm-firms

Location of Sample	Function for	Marginal Return to Opportunity Cost Ratio of Labor Services
Taiwan, Tainan	cereals	2.84
Australia, New South Wales	sheep	2.18
South Africa, Kalihari	cow-calf ranches	2.09
South Africa, Kalihari	cattle fattening	1.85
India, Uttar Pradesh	planted sugar cane	1.78
Taiwan, Tainan	sugar cane	1.66
Norway, southeast	beef cattle	1.64
Canada, Alberta	cattle ranches	1.58
Japan, Honshu	sweet potatoes	1.50
Norway, southeast	fodder	1.36
United States, Iowa-Illinois	livestock-share	1.36
Japan, Honshu	tea	1.28
Israel	mixed farms	1.26
United States, Iowa-Illinois	crop-share	1.23
Canada, Alberta	wheat, beef	1.21
United Kingdom, England	dairy	1.06
Western Australia	dairy	0.90
New Zealand, Canterbury Plain	sheep	0.85
United States, Iowa-Illinois	full owners	0.84
Norway, southeast	dairy	0.73
India, Uttar Pradesh	wheat, sugar cane	0.68
United States, Alabama	livestock	0.67
United States, southern Iowa	hogs, cattle	0.62
United States, northern Iowa	hogs, cattle	0.62
South Australia	dairy	0.61
Japan, Hokkaido	rice	0.57
Austria	mixed farming	0.54
Australia, New South Wales	dairy	0.49
Japan, Honshu	rice	0.48
United States, Alabama	crops	0.38
United States, northern Iowa	corn	0.34
United States, Montana	cattle	0.32
Sweden	mixed farms	0.31
Norway, southeast	cereals	0.29
Japan, Hokkaido, small farms	rice	0.29
United States, Montana	wheat	0.25
United States, southern Iowa	corn	0.22
India, Andhra Pradesh	mixed farms	0.21
Japan, Hokkaido, large farms	rice	0.21
India, Andhra Pradesh	irrigated farms	0.13
India, Andhra Pradesh	dry farms	0.03
India, Uttar Pradesh	irrigated wheat	-0.35
Sweden	dairy	<0.00

the highest ratio for land in livestock or mixed crop-livestock production. Thus the data suggest crop production is more prone to disequilibrium in the use of land than livestock production, and that cropping is most frequently carried out with suboptimal inputs of land services. Justification for these tendencies is found in the inelastic nature of the supply of land for cropping relative to grazing and the greater intensity of land use in crop production. As the intensity of land use rises, small variations in input have relatively greater effects on output so that the attainment of optimality is a more difficult task in crop production than it is in more extensive activities involving grazing livestock. Moreover, these influences are accentuated by the quantitative and locational fixity of the land resource.

Except in terms of ownership changes, land services are immobile. However, changes in ownership between nations are unlikely under normal circumstances. Accordingly, we will not discuss the data of Table 17.16 in terms of land transfers. To do so would be unreal.

For the labor resource, Table 17.17 lists marginal return to opportunity cost ratios as estimated from 43 analyses. Of these, 27 (63 per cent) imply the use of too much labor. Although this distribution is not significant at usual probability levels, it corresponds with the theoretical expectation of surplus labor in agriculture arising from frictions in the adjustment to economic progress.[36] But compared to the efficiency ratios for land, which range from 0.05 to 31.39 with a mean of 4.5, the labor ratios display a much smaller degree of disequilibria, ranging from -0.35 to 2.84 with a mean of approximately 0.9.[37] The reason for greater efficiency in the use of labor is probably the relative mobility and freedom of supply of labor within nations.

As for land services, no pronounced ordering is discernible between countries in terms of their efficiency of labor use, except for the grouping of the Indian cereal studies at the bottom of the table. Still, the excessive use of labor on these Indian peasant farms is only slightly worse than prevails in commercial corn and wheat production in the United States samples for Montana and southern Iowa. More detailed perusal of the data shows too much labor was invariably used in the samples for Australian dairying, the Japanese island of Hokkaido, the Indian state of Andhra Pradesh, New Zealand, Austria, Sweden, and the United States states of Iowa, Montana, and Alabama. In contrast, all the analyses for Taiwan, South Africa, Israel, Canada, and the United Kingdom imply additional labor inputs would be profitable. Provided the costs of transfer did not exceed the benefits, transfer of labor from the former to the latter group of countries would be worthwhile. Nevertheless, it is probable that more feasible opportunities for beneficial

[36] See Ojala, E. M. Agriculture and economic progress. Oxford University Press, New York. 1952; also Fisher, A. G. B. Economic progress and social security. Macmillan, New York. 1946.

[37] No great significance should be attached to the average values presented. If the relevant information were available it would be preferable to consider the means of the estimates weighted according to the size of the farm population to which they refer.

Table 17.18. Marginal Return to Opportunity Cost Ratios of Capital Services as Estimated From Selected Cross-sectional Production Function Studies of Farm-firms

Location of Sample	Function for	Marginal Return to Opportunity Cost Ratio of Capital Services
India, Uttar Pradesh	irrigated wheat	6.97
Japan, Honshu	tea	2.55
New Zealand, Canterbury Plain	sheep	2.47
India, Uttar Pradesh	wheat, sugar cane	2.13
Japan, Honshu	rice	2.05
United States, Montana	wheat	2.00
Australia, New South Wales	sheep	1.97
South Africa, Kalihari	cow-calf ranches	1.67
Norway, southeast	fodder	1.66
Western Australia	dairy	1.61
Japan, Hokkaido, Large farms	rice	1.54
South Australia	dairy	1.53
Austria	mixed farming	1.50
South Africa, Kalihari	cattle fattening	1.48
Sweden	dairy	1.41
Australia, New South Wales	dairy	1.40
Canada, Alberta	cattle ranches	1.38
United States, Iowa-Illinois	full owners	1.35
United Kingdom, England	dairy	1.21
United States, southern Iowa	corn	1.21
United States, Iowa-Illinois	livestock-share	1.16
United States, Montana	cattle	1.14
United States, southern Iowa	hogs, cattle	1.13
United Kingdom, Bristol Province	low-yield dairy	1.09
Israel	mixed farms	1.09
Canada, Alberta	cattle feedlots	1.05
Taiwan, Tainan	sugar cane	1.02
United States, Iowa-Illinois	crop-share	1.01
United States, Alabama	crops	1.01
Canada, Alberta	wheat, beef	1.01
Taiwan, Tainan	cereals	0.99
United States, southern Iowa	hogs, cattle	0.97
United States, Alabama	livestock	0.94
Norway, southeast	dairy	0.76
United Kingdom, Bristol Province	high-yield dairy	0.73
Norway, southeast	cereals	0.67
Norway, southeast	beef cattle	0.66
Sweden	mixed farms	0.64
United States, northern Iowa	corn	0.59
India, Andhra Pradesh	mixed farms	0.35
India, Andhra Pradesh	dry farms	0.23
Japan, Hokkaido, small farms	rice	0.18
Japan, Honshu	sweet potatoes	0.00
Japan, Hokkaido	rice	0.00
India, Uttar Pradesh	planted sugar cane	-0.11
India, Andhra Pradesh	irrigated farms	-0.85

transfers of labor exist within these various countries. Obviously, such domestic adjustments should normally precede international shifts.

Nineteen of the 43 labor-efficiency estimates of Table 17.17 refer to cropping alone and 24 to mixed crop-livestock production. Of the 19 crop estimates, slightly more than two-thirds are greater than unity, indicating the use of too much labor. The corresponding proportion is not quite so great for production involving livestock. If anything, there may be a greater tendency for a surfeit of labor in crop than in livestock production. Certainly such a tendency would be expected for the world as a whole, given the prevalence of subsistence agriculture which is necessarily based on cropping.

Table 17.18 lists the resource use efficiency ratios for capital services as estimated from 46 studies. The general tendency is towards the use of insufficient capital. Thirty (65 per cent) of the estimates are greater than unity while their average value is 1.2. Moreover, assuming the 46 studies to be a random sample from the superpopulation of all possible studies, there is only 5 per cent probability that this distribution is due to chance. The range of capital service ratios, from -0.85 to 6.97, is slightly larger than that for the labor estimates, but still far less than the range for land services. Relative to land, the mobility of capital probably explains its smaller degree of inefficiency. In terms of type of production, 19 of the estimated ratios refer to cropping and 27 to production involving livestock. Among the latter, 71 per cent imply the use of too few capital services. In contrast, the 19 crop estimates divide rather evenly between excessive and deficient capital services. The data strongly suggest, therefore, that the over-all tendency to the use of too few capital services is mainly due to the lack of sufficient capital in production involving livestock.

While all the New Zealand, Australian, South African, Austrian, Canadian, and Israeli studies indicate that more capital services should be used, for no country do the various estimates invariably imply that less capital should be used. Hence, even from the few scattered estimates we have been able to assemble, it is apparent that intra-national chances for capital transfers abound and that international shifts in capital can only be made in the face of favorable domestic opportunities. If the unreal assumption is made that prevailing technologies are immutable, it is obvious that the data presents no strong argument for allocating additional capital to the underdeveloped countries rather than to the more advanced nations. In order of priority relative to the benefits to be derived from the use of additional capital, the various studies for each country are dispersed throughout the listing of Table 17.18.

So far we have been considering land, labor, and capital services separately. Such an approach is not entirely adequate. Opportunities exist for substitution between land, labor, and capital; changes in the input of one influence the efficiency of use of the other resources. Accordingly, we need to consider the over-all picture of resource use

Table 17.19. Frequency Distributions, Means, Standard Deviations and Coefficients of Variation of the Marginal Return to Opportunity Cost Ratios for Land, Labor, and Capital Services as Estimated From Selected Cross-sectional Production Function Studies of Farm-firms

Statistic	Land Services	Labor Services	Capital Services
Frequencies:			
$\leqslant 0$		2	4
0.01-0.20	1	2	1
0.21-0.40	2	10	2
0.41-0.60	2	4	1
0.61-0.80	1	6	5
0.81-1.00	4	3	3
1.01-1.20	1	1	10
1.21-1.40	1	6	5
1.41-1.60		2	5
1.61-1.80	3	3	3
1.81-2.00	2	1	2
2.01-3.00	7	3	4
$\geqslant 3.01$	9		1
Mean	4.5	0.9	1.2
Standard deviation	7.3	0.7	0.7
Coefficient of variation	1.62	0.78	0.58

efficiency. The broad picture is given by Table 17.19 which lists the frequency distributions, means, standard deviations, and coefficients of variation of the marginal return to opportunity cost ratios for the various resources. The wide dispersal of the ratios is readily apparent from these statistics. As expected from the pervasive influence of frictions in the adjustment to economic progress and population growth, the general tendency is for farms to have too few land and capital services, and a surfeit of labor services. Differences in the coefficients of variation of the land, labor, and capital ratios are noteworthy: the variability for capital is slightly less than for labor, while the ratios for land are more than twice as variable as those for capital and labor. Also, the relatively flat distribution for land contrasts with the peaked distributions for labor and capital services. Too, while nearly half of the land estimates are greater than 2.00, half of the labor and capital ratios lie, respectively, in the ranges 0.21 to 0.80 and 1.01 to 1.60. As already noted, the tendency for more efficient use of labor and capital services is probably due to their relative mobility compared to land.

From the statistics of Table 17.19, the broadest recommendation that might be made for world agriculture is that farmers should substitute land and capital services for labor. Such a proposal, of course, implies a net inflow of capital into agriculture (perhaps obtainable via a reduction in the nonproductive portion of the flow of capital out of

agriculture), as well as a reduction in the number of farms since the supply of land is highly inelastic. Concomitantly, nonfarm employment opportunities would need to be provided for labor transferred out of agriculture. Still, little weight can be attached to such a broad recommendation which, we might note, could have been made on a priori grounds anyway. Indeed, the whole rationale of the data we have presented is that, if available in sufficient quantity, it would enable far more specific recommendations to be made.

Nevertheless, it is apparent that no simple ordering of priorities could be made on a country by country basis. The marginal revenue to opportunity cost ratios of the various studies within each country are too dissimilar. For instance, for all resources examined, the United States studies have resource use efficiency ratios that are scattered widely over the range of the marginal return to opportunity cost ratios. The same statement applies to the various studies from the less advanced countries. Further evidence of this intra-national diversity is provided by a categorization of the studies among the eight possible combinations of too many or too few inputs of land, labor, and capital services. These eight combinations may be paired in terms of converses. For example, using capital letters to denote too few inputs and small letters excessive inputs of the relevant resources, the combination (LAND, LABOR, capital) is the converse of (land, labor, CAPITAL). In such form, the studies involving all three inputs are distributed as follows:

LAND, LABOR, CAPITAL	land, labor, capital
Canada, mixed farms	India, Andhra Pradesh, dry farms
Israel	India, Andhra Pradesh, irrigated farms
Japan, Honshu, tea	India, Andhra Pradesh, mixed farms
Norway, fodder	
South Africa, cattle fattening	
South Africa, cow-calf	
United States, livestock-share	
United States, crop-share	

LAND, LABOR, capital	land, labor, CAPITAL
India, Uttar Pradesh, sugar cane	Austria
	India, Uttar Pradesh, wheat, sugar cane
	United States, full owners

LAND, labor, capital	land, LABOR, CAPITAL
Japan, Hokkaido, rice	Australia, sheep
Norway, cereals	Canada, cattle ranches
Sweden, mixed farms	Taiwan, sugar cane
United States, northern Iowa, corn	

LAND, labor, CAPITAL	land, LABOR, capital
Australia, New South Wales, dairy	Taiwan, cereals
Australia, South Australia, dairy	
India, Uttar Pradesh, wheat	
Japan, Hokkaido, large farms	
Japan, Honshu, rice	
New Zealand, sheep	
United States, Alabama, crops	
United States, Montana, wheat	
United States, southern Iowa, corn	

Such an arrangement of the studies brings out the predominance of deficiencies in the input of land and capital services. As well, it again emphasizes the existence of similar patterns of inefficiency in both the advanced and less advanced nations. Hence, if the technological environment is assumed fixed, there is no economic justification for a blanket policy of capital transfers from the more advanced to the less advanced countries. These transfers could only be justified on humanitarian grounds. On the other hand, such transfers are justifiable in terms of positive economics when technological change is admitted. However, the estimated marginal return to opportunity cost ratios tell us nothing of the situation that would exist over the globe if the most advanced methods of production were used in a satisfactory manner. Still, provided that necessary ancillary adjustments were made to these newer technologies, they might amend much of the poverty in the less developed areas of the world. Accordingly, accepting current scale of enterprise, net income, and return on investment as guides to the need for new technologies and using the more advanced countries as benchmarks, priorities could be attached to the need for using more advanced technologies relative to the gains that might be made from improving the efficiency of resource use within the context of the current technological pattern. Certainly, the best approach to improving the efficiency of world agriculture must involve both methods. Too, investments in these activities would probably be most beneficial if carried out simultaneously in a number of nations.

Finally, the data of tables 17.16 and 17.17 strongly evidence disequilibria in the use of land and labor that are just as great as the inefficiencies found for capital services. Moreover, land and labor are virtually immobile in an international context. It can be concluded intra-national resource transfers must play the dominant role in any initial shift towards greater efficiency in world agriculture. Still, capital service transfers between countries should not be judged unimportant, especially in relation to the implementation of more efficient production processes.

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